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POPOVICIU TYPE INEQUALITIES VIA GREEN FUNCTION AND GENERALIZED MONTGOMERY IDENTITY

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Abstract. We obtained useful identities via generalized Montgomery identity, by which the inequality of Popoviciu for convex functions is generalized for higher order convex functions. We investigate the bounds for the identities related to the generalization of the Popoviciu inequality using inequalities for the Čebyšev functional. Some results relating to the Grüss and Ostrowski type inequalities are constructed. Further, we also construct new families of exponentially convex functions and Cauchy-type means by looking at linear functionals associated with the obtained inequalities.

1. Introduction and preliminary results

The theory developed under the theme of convex functions, arising from intuitive geometrical observations, may be readily applied to topics in real analysis and economics. In modern Era, their occurs a rapid development in the theory of convex functions. Their are serval reasons behind it: firstly, so many areas in modern analysis directly or indirectly involve the application of convex functions; secondly, convex functions are closely related to the theory of inequalities and many important inequalities are consequences of the applications of convex functions (see [13]).

A characterization of convex function established by T. Popoviciu [14] is studied by many people (see [15, 13] and references with in). For recent work, we refer [4, 7, 8, 9, 10]. The following form of Popoviciu's inequality is by Vasić and Stanković in [15] (see also page 173 [13]):

THEOREM 1. *Let* $m,k \in \mathbb{N}$, $m \geqslant 3$, $2 \leqslant k \leqslant m-1$, $[\alpha,\beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1,\dots,x_m) \in [\alpha,\beta]^m$, $\mathbf{p} = (p_1,\dots,p_m)$ *be a positive* m*-tuple such that* $\sum_{i=1}^{m} p_i = 1$. *Also let* $f : [\alpha,\beta] \to \mathbb{R}$ *be a convex function. Then*

$$p_{k,m}(\mathbf{x},\mathbf{p};f) \leqslant \frac{m-k}{m-1} p_{1,m}(\mathbf{x},\mathbf{p};f) + \frac{k-1}{m-1} p_{m,m}(\mathbf{x},\mathbf{p};f), \tag{1}$$

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where

$$p_{k,m}(\mathbf{x},\mathbf{p};f) = p_{k,m}(\mathbf{x},\mathbf{p};f(x)) := \frac{1}{\binom{m-1}{k-1}} \sum_{1\leqslant i_1<\ldots<i_k\leqslant m} \left(\sum_{j=1}^{k} p_{i_j}\right) f\left(\frac{\sum_{j=1}^{k} p_{i_j}x_{i_j}}{\sum_{j=1}^{k} p_{i_j}}\right)$$

is the linear functional with respect to f.

By inequality (1), we write

$$\mathbf{P}(\mathbf{x},\mathbf{p};f) := \frac{m-k}{m-1}p_{1,m}(\mathbf{x},\mathbf{p};f) + \frac{k-1}{m-1}p_{m,m}(\mathbf{x},\mathbf{p};f) - p_{k,m}(\mathbf{x},\mathbf{p};f). \tag{2}$$

REMARK 1. It is important to note that under the assumptions of Theorem 1, if the function f is convex then $\mathbf{P}(\mathbf{x},\mathbf{p};f) \geqslant 0$ and $\mathbf{P}(\mathbf{x},\mathbf{p};f) = 0$ for $f(x) = x$ or f is a constant function.

The mean value theorems and exponential convexity of the linear functional $\mathbf{P}(\mathbf{x},\mathbf{p};f)$ are given in [7] for a positive m-tuple $\mathbf{p}$. Some special classes of convex functions are considered to construct the exponential convexity of $\mathbf{P}(\mathbf{x},\mathbf{p};f)$ in [7]. Consider the Green function $G : [\alpha,\beta]\times[\alpha,\beta] \to \mathbb{R}$ defined as

$$G(t,s) = \begin{cases} \frac{(t-\beta)(s-\alpha)}{\beta-\alpha} & \alpha \leqslant s \leqslant t; \\ \frac{(s-\beta)(t-\alpha)}{\beta-\alpha}, & t \leqslant s \leqslant \beta. \end{cases} \tag{3}$$

The function G is convex and continuous w.r.t s and due to symmetry also w.r.t t.

For any function $\psi : [\alpha,\beta] \to \mathbb{R}$, $\psi \in C^2([\alpha,\beta])$, we have

$$\psi(x) = \frac{\beta-x}{\beta-\alpha}\psi(\alpha) + \frac{x-\alpha}{\beta-\alpha}\psi(\beta) + \int_{\alpha}^{\beta} G(x,s)\psi''(s)ds, \tag{4}$$

where the function G is defined in (3) (see [16]).

In Theorem 1 we have that p_i $(i = 1,\ldots,n)$ are positive real numbers. In [8] (see also [4]), the results related to $\mathbf{P}(\mathbf{x},\mathbf{p};f)$ are generalized with help of Green function for real values of p_i $(i = 1,\ldots,n)$ with $\sum_{i=1}^{n} p_i = 1$ in the following theorem:

THEOREM 2. *Let* $m,k \in \mathbb{N}$, $m \geqslant 3$, $2 \leqslant k \leqslant m-1$, $[\alpha,\beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1,\ldots,x_m) \in [\alpha,\beta]^m$, $\mathbf{p} = (p_1,\ldots,p_m)$ *be a real* m*-tuple such that* $\sum_{j=1}^{k} p_{i_j} \neq 0$ *for any* $1 \leqslant i_1 < \ldots < i_k \leqslant m$ *and* $\sum_{i=1}^{m} p_i = 1$. *Also let* $\frac{\sum_{j=1}^{k} p_{i_j}x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} \in [\alpha,\beta]$ *for any* $1 \leqslant i_1 < \ldots < i_k \leqslant m$. *Then the following statements are equivalent:*

(i) For every continuous convex function $f : [\alpha,\beta] \to \mathbb{R}$

$$f_{k,m}(\mathbf{x},\mathbf{p}) \leqslant \frac{m-k}{m-1}f_{1,m}(\mathbf{x},\mathbf{p}) + \frac{k-1}{m-1}f_{m,m}(\mathbf{x},\mathbf{p}), \tag{5}$$

where

$$f_{k,n}(\mathbf{x},\mathbf{p}) := \frac{1}{\binom{m-1}{k-1}} \sum_{1\leqslant i_1<\ldots<i_k\leqslant m} \left(\sum_{j=1}^{k} p_{i_j}\right) f\left(\frac{\sum_{j=1}^{k} p_{i_j}x_{i_j}}{\sum_{j=1}^{k} p_{i_j}}\right).$$

(ii) For all $s \in [\alpha,\beta]$

$$G_{k,m}(\mathbf{x},s;\mathbf{p}) \leqslant \frac{m-k}{m-1} G_{1,m}(\mathbf{x},s;\mathbf{p}) + \frac{k-1}{m-1} G_{m,m}(\mathbf{x},s;\mathbf{p}), \tag{6}$$

where

$$G_{k,m}(\mathbf{x},s;\mathbf{p}) := \frac{1}{\binom{m-1}{k-1}} \sum_{1\leqslant i_1<\ldots<i_k\leqslant m} \left(\sum_{j=1}^{k} p_{i_j}\right) G\left(\frac{\sum_{j=1}^{k} p_{i_j}x_{i_j}}{\sum_{j=1}^{k} p_{i_j}}, s\right); \quad 1\leqslant k\leqslant m,$$

for the function $G : [\alpha,\beta]\times[\alpha,\beta] \to \mathbb{R}$ *defined in (3).*

Moreover, the statements (i) and (ii) are also equivalent if we change the sign of inequality in both (5) and (6).

In order to obtain our main results in the present paper, we use the generalized Montgomery identity via Taylor's formula given in paper [1].

THEOREM 3. *Let* $n\in\mathbb{N}$, $\psi : I \to \mathbb{R}$ *be such that* $\psi^{(n-1)}$ *is absolutely continuous,* $I\subset\mathbb{R}$ *an open interval,* $\alpha,\beta\in I$, $\alpha<\beta$. *Then the following identity holds*

$$\psi(x) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \psi(t)\,dt + \sum_{l=0}^{n-2} \frac{\psi^{(l+1)}(\alpha)}{l!\,(l+2)} \frac{(x-\alpha)^{l+2}}{\beta-\alpha} - \sum_{l=0}^{n-2} \frac{\psi^{(l+1)}(\beta)}{l!\,(l+2)} \frac{(x-\beta)^{l+2}}{\beta-\alpha}$$
$$+ \frac{1}{(n-1)!}\int_\alpha^\beta R_n(x,s)\,\psi^{(n)}(s)\,ds \tag{7}$$

where

$$R_n(x,s) = \begin{cases} -\frac{(x-s)^n}{n(\beta-\alpha)} + \frac{x-\alpha}{\beta-\alpha}(x-s)^{n-1}, & \alpha\leqslant s\leqslant x, \\ -\frac{(x-s)^n}{n(\beta-\alpha)} + \frac{x-\beta}{\beta-\alpha}(x-s)^{n-1}, & x<s\leqslant\beta. \end{cases} \tag{8}$$

In case $n=1$ the sum $\sum_{l=0}^{n-2}\cdots$ is empty, so identity (7) reduces to well-known Montgomery identity (see for instance [11])

$$\psi(x) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \psi(t)\,dt + \int_\alpha^\beta P(x,s)\,\psi'(s)\,ds$$

where $P(x,s)$ is the Peano kernel, defined by

$$P(x,s) = \begin{cases} \frac{s-\alpha}{\beta-\alpha}, & \alpha\leqslant s\leqslant x, \\ \frac{s-\beta}{\beta-\alpha}, & x<s\leqslant\beta. \end{cases}$$

The organization of the paper follows the following pattern: In Section 2, we generalize weighted Popoviciu's inequality by using Green function and generalized Montgomery identity for higher order convex functions. In Section 3, we use the classical Čebyšev functional and obtain results related to Grüss-type inequalities and Ostrowski-type inequalities. In Section 4, we study the functional defined as the difference between the R.H.S. and the L.H.S. of the generalized inequality and our goal is to investigate the n-exponential and logarithmic convexity of the obtained functional. Furthermore, we prove monotonicity property of the generalized Cauchy means obtained via this functional. Finally, we conclude our paper by giving several examples of the families of functions for which the obtained results can be applied.

2. Popoviciu's inequality by Green function and extension of Montgomery identity via Taylor's formula

Motivated by identity (2), we construct the following identity with the help of (4) and the generalized Montgomery identity.

THEOREM 4. *Let all the assumptions of Theorem 3 be valid with $n > 2$ and let $m,k \in \mathbb{N},\ m \geqslant 3,\ 2 \leqslant k \leqslant m-1,\ [\alpha,\beta] \subset \mathbb{R},\ \mathbf{x} = (x_1,\ldots,x_m) \in [\alpha,\beta]^m,\ \mathbf{p} = (p_1,\ldots,p_m)$ be a real m-tuple such that $\sum_{j=1}^{k} p_{i_j} \neq 0$ for any $1 \leqslant i_1 < \ldots < i_k \leqslant m$ and $\sum_{i=1}^{m} p_i = 1$. Also let $\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} \in [\alpha,\beta]$ for any $1 \leqslant i_1 < \ldots < i_k \leqslant m$ with $G,\ R_n$ be the same as defined in (3), (8) respectively. Then we have the following two identities:*

$$\begin{aligned}
&\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x)) \\
&= \frac{\psi'(\alpha)-\psi'(\beta)}{\beta-\alpha}\int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))ds \\
&\quad + \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))\left(\sum_{l=2}^{n-1}\frac{l}{(l-1)!}\left(\psi^{(l)}(\alpha)(s-\alpha)^{l-1}-\psi^{(l)}(\beta)(s-\beta)^{l-1}\right)\right)ds \\
&\quad + \frac{1}{(n-3)!}\int_\alpha^\beta \psi^{(n)}(v)\left(\int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))\tilde{R}_{n-2}(s,v)\,ds\right)dv,
\end{aligned} \tag{9}$$

where

$$\tilde{R}_{n-2}(s,v) = \begin{cases} \frac{1}{\beta-\alpha}\left[\frac{(s-v)^{n-2}}{(n-2)} + (s-\alpha)(s-v)^{n-3}\right], & \alpha \leqslant v \leqslant s, \\ \frac{1}{\beta-\alpha}\left[\frac{(s-v)^{n-2}}{(n-2)} + (s-\beta)(s-v)^{n-3}\right], & s < v \leqslant \beta, \end{cases}$$

and

$$\begin{aligned}
&\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x)) \\
&= \left(\frac{\psi'(\beta)-\psi'(\alpha)}{\beta-\alpha}\right)\int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))ds \\
&\quad + \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))\left(\sum_{l=3}^{n-1}\frac{\psi^{(l)}(\alpha)(s-\alpha)^{l-1}-\psi^{(l)}(\beta)(s-\beta)^{l-1}}{(l-3)!(l-1)}\right)ds \\
&\quad + \frac{1}{(n-3)!}\int_\alpha^\beta \psi^{(n)}(v)\left(\int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))R_{n-2}(s,v)ds\right)dv. \qquad (10)
\end{aligned}$$

Proof. Using (4) in (2) and following the linearity of $\mathbf{P}(\mathbf{x},\mathbf{p};\cdot)$, we have

$$\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x)) = \int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))\,\psi''(s)ds. \qquad (11)$$

Differentiating (7), twice with respect to the first variable, we have

$$\begin{aligned}
\psi''(s) &= \frac{\psi'(\alpha)-\psi'(\beta)}{\beta-\alpha} \\
&\quad + \sum_{l=2}^{n-1}\left(\frac{l}{(l-1)!}\right)\left(\frac{\psi^{(l)}(\alpha)(s-\alpha)^{l-1}-\psi^{(l)}(\beta)(s-\beta)^{l-1}}{\beta-\alpha}\right) \\
&\quad + \frac{1}{(n-3)!}\int_\alpha^\beta \tilde{R}_{n-2}(s,v)\,\psi^{(n)}(v)\,dv. \qquad (12)
\end{aligned}$$

Using (12) in (11), we get

$$\begin{aligned}
&\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x)) \\
&= \frac{\psi'(\alpha)-\psi'(\beta)}{\beta-\alpha}\int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))ds \\
&\quad + \sum_{l=2}^{n-1}\frac{l}{(l-1)!}\int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))\left(\frac{\psi^{(l)}(\alpha)(s-\alpha)^{l-1}-\psi^{(l)}(\beta)(s-\beta)^{l-1}}{\beta-\alpha}\right)ds \\
&\quad + \frac{1}{(n-3)!}\int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))\left(\int_\alpha^\beta \tilde{R}_{n-2}(s,v)\,\psi^{(n)}(v)\,dv\right)ds.
\end{aligned}$$

By applying Fubini's Theorem in the last term, we have (9).

Next, using formula (7) on the function ψ'', replacing n by $n-2$ $(n \geqslant 3)$ and rearranging the indices, we have

$$\begin{aligned}
\psi''(s) &= \left(\frac{\psi'(\beta)-\psi'(\alpha)}{\beta-\alpha}\right) \\
&\quad + \sum_{l=3}^{n-1}\left(\frac{1}{(l-3)!(l-1)}\right)\left(\frac{\psi^{(l)}(\alpha)(s-\alpha)^{l-1}-\psi^{(l)}(\beta)(s-\beta)^{l-1}}{\beta-\alpha}\right) \\
&\quad + \frac{1}{(n-3)!}\int_\alpha^\beta R_{n-2}(s,v)\psi^{(n)}(v)\,dv. \qquad (13)
\end{aligned}$$

Similarly, using (13) in (11) and applying Fubini's Theorem, we get (10). □

For n-convex functions, we can give the following form of new identities (9) and (10).

THEOREM 5. *Let all the assumptions of Theorem 4 be satisfied and $n \geqslant 3$. Also let ψ be n-convex function such that $\psi^{(n-1)}$ is absolutely continuous. Then we have the following two results:*

If

$$\int_{\alpha}^{\beta} \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))\tilde{R}_{n-2}(s,v)\,ds \geqslant 0, \quad v \in [\alpha,\beta] \tag{14}$$

then

$$\begin{aligned}
&\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x))\\
&\geqslant \frac{\psi'(\alpha)-\psi'(\beta)}{\beta-\alpha}\int_{\alpha}^{\beta}\mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))ds\\
&+\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))\left(\sum_{l=2}^{n-1}\frac{l}{(l-1)!}\left(\psi^{(l)}(\alpha)(s-\alpha)^{l-1}-\psi^{(l)}(\beta)(s-\beta)^{l-1}\right)\right)ds,
\end{aligned} \tag{15}$$

and if

$$\int_{\alpha}^{\beta} \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))R_{n-2}(s,v)\,ds \geqslant 0, \quad v \in [\alpha,\beta] \tag{16}$$

then

$$\begin{aligned}
&\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x))\\
&\geqslant \left(\frac{\psi'(\beta)-\psi'(\alpha)}{\beta-\alpha}\right)\int_{\alpha}^{\beta}\mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))ds\\
&+\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))\left(\sum_{l=3}^{n-1}\frac{\psi^{(l)}(\alpha)(s-\alpha)^{l-1}-\psi^{(l)}(\beta)(s-\beta)^{l-1}}{(l-3)!(l-1)}\right)ds.
\end{aligned} \tag{17}$$

Proof. Since $\psi^{(n-1)}$ is absolutely continuous on $[\alpha,\beta]$, $\psi^{(n)}$ exists almost everywhere. As ψ is n-convex, so $\psi^{(n)}(x) \geqslant 0$ for all $x \in [\alpha,\beta]$ (see [13], p. 16). Hence we can apply Theorem 4 to obtain (15) and (17) respectively. □

REMARK 2. The inequalities (15) and (17) hold in reverse directions if the inequalities in (14) and (16) are reversed.

Now we give generalization of Popoviciu's inequality for n-convex functions.

THEOREM 6. *Let all the assumptions of Theorem 4 be satisfied in addition with the condition that $\mathbf{p} = (p_1,\ldots,p_m)$ be a positive m-tuple such that $\sum_{i=1}^{m} p_i = 1$ and consider $\psi : [\alpha,\beta] \to \mathbb{R}$ is n-convex function.*

(i) If n be even and $n \geqslant 4$, then (15) and (17) holds.

(ii) Let the inequality (15) be satisfied and

$$\sum_{l=1}^{n-1} \frac{l}{(l-1)!}\left(\psi^{(l)}(\alpha)(s-\alpha)^{l-1} - \psi^{(l)}(\beta)(s-\beta)^{l-1}\right) \geqslant 0; \quad \forall s \in [\alpha,\beta], \tag{18}$$

or (17) be satisfied and

$$\psi'(\beta) - \psi'(\alpha) + \sum_{l=3}^{n-1} \frac{\psi^{(l)}(\alpha)(s-\alpha)^{l-1} - \psi^{(l)}(\beta)(s-\beta)^{l-1}}{(l-3)!(l-1)} \geqslant 0; \quad \forall s \in [\alpha,\beta]. \tag{19}$$

Then we have

$$\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x)) \geqslant 0. \tag{20}$$

Proof. Since Green's function G is convex and the weights are positive, $\mathbf{P}(\mathbf{x},\mathbf{p}; G(x,s)) \geqslant 0$ by virtue of Remark 1.

(i) $\tilde{R}_{n-2}(s,v) \geqslant 0$ and $R_{n-2}(s,v) \geqslant 0$ for $n = 4,6,\ldots$, so (14) and (16) holds. As ψ is n-convex, hence by following Theorem 5, we obtain (15) and (17).

(ii) Using (18) in (15) and (19) in (17), we have (20). □

3. Bounds for identities related to generalization of Popoviciu's inequality

In this section we present some interesting results by using Čebyšev functional and Grüss type inequalities. For two Lebesgue integrable functions $f,h : [\alpha,\beta] \to \mathbb{R}$, we consider the Čebyšev functional

$$\Delta(f,h) = \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} f(t)h(t)dt - \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} f(t)dt.\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} h(t)dt.$$

The following Grüss type inequalities are given in [3].

THEOREM 7. *Let $f : [\alpha,\beta] \to \mathbb{R}$ be a Lebesgue integrable function and $h : [\alpha,\beta] \to \mathbb{R}$ be an absolutely continuous function with $(.-\alpha)(\beta-.)[h']^2 \in L[\alpha,\beta]$. Then we have the inequality*

$$|\Delta(f,h)| \leqslant \frac{1}{\sqrt{2}}[\Delta(f,f)]^{\frac{1}{2}} \frac{1}{\sqrt{\beta-\alpha}}\left(\int_{\alpha}^{\beta}(x-\alpha)(\beta-x)[h'(x)]^2 dx\right)^{\frac{1}{2}}. \tag{21}$$

The constant $\frac{1}{\sqrt{2}}$ in (21) is the best possible.

THEOREM 8. *Assume that* $h:[\alpha,\beta]\to\mathbb{R}$ *is monotonic nondecreasing on* $[\alpha,\beta]$ *and* $f:[\alpha,\beta]\to\mathbb{R}$ *be an absolutely continuous with* $f'\in L_\infty[\alpha,\beta]$. *Then we have the inequality*

$$|\Delta(f,h)|\leqslant\frac{1}{2(\beta-\alpha)}||f'||_\infty\int_\alpha^\beta(x-\alpha)(\beta-x)[h'(x)]^2dh(x). \tag{22}$$

The constant $\frac{1}{2}$ *in* (22) *is the best possible.*

In the sequel, we consider above theorems to derive generalizations of the results proved in the previous section. In order to avoid many notions let us denote

$$\tilde{\mathfrak{H}}(v)=\int_\alpha^\beta\mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))\tilde{R}_{n-2}(s,v)\,ds\geqslant 0,\quad v\in[\alpha,\beta], \tag{23}$$

and

$$\mathfrak{O}(v)=\int_\alpha^\beta\mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))R_{n-2}(s,v)\,ds\geqslant 0,\quad v\in[\alpha,\beta]. \tag{24}$$

Consider the Čebyšev functional $\Delta(\tilde{\mathfrak{H}},\tilde{\mathfrak{H}})$ and $\Delta(\mathfrak{O},\mathfrak{O})$ given as:

$$\Delta(\tilde{\mathfrak{H}},\tilde{\mathfrak{H}})=\frac{1}{\beta-\alpha}\int_\alpha^\beta\tilde{\mathfrak{H}}^2(v)dv-\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta\tilde{\mathfrak{H}}(v)dv\right)^2,$$

and

$$\Delta(\mathfrak{O},\mathfrak{O})=\frac{1}{\beta-\alpha}\int_\alpha^\beta\mathfrak{O}^2(v)dv-\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta\mathfrak{O}(v)dv\right)^2,$$

respectively.

THEOREM 9. *Let* $\psi:[\alpha,\beta]\to\mathbb{R}$ *be such that for* $n\geqslant 4$, $\psi^{(n)}$ *is absolutely continuous with* $(.-\alpha)(\beta-.)[\psi^{(n+1)}]^2\in L[\alpha,\beta]$. *Let* $m,k\in\mathbb{N}$, $m\geqslant 3$, $2\leqslant k\leqslant m-1$, $[\alpha,\beta]\subset\mathbb{R}$, $\mathbf{x}=(x_1,\ldots,x_m)\in[\alpha,\beta]^m$, $\mathbf{p}=(p_1,\ldots,p_m)$ *be a real* m*-tuple such that* $\sum_{j=1}^k p_{i_j}\neq 0$ *for any* $1\leqslant i_1<\ldots<i_k\leqslant m$ *and* $\sum_{i=1}^m p_i=1$. *Also let* $\frac{\sum_{j=1}^k p_{i_j}x_{i_j}}{\sum_{j=1}^k p_{i_j}}\in[\alpha,\beta]$ *for any* $1\leqslant i_1<\ldots<i_k\leqslant m$ *with* $\tilde{\mathfrak{H}}$ *and* $\mathfrak{O}$ *defined in* (23) *and* (24) *respectively. Then*

$$\begin{aligned}&\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x))\\&=\frac{\psi'(\alpha)-\psi'(\beta)}{\beta-\alpha}\int_\alpha^\beta\mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))ds\\&\quad+\frac{1}{\beta-\alpha}\int_\alpha^\beta\mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))\left(\sum_{l=2}^{n-1}\frac{l}{(l-1)!}\left(\psi^{(l)}(\alpha)(s-\alpha)^{l-1}-\psi^{(l)}(\beta)(s-\beta)^{l-1}\right)\right)ds\\&\quad+\frac{\psi^{(n-1)}(\beta)-\psi^{(n-1)}(\alpha)}{(\beta-\alpha)(n-3)!}\int_\alpha^\beta\tilde{\mathfrak{H}}(v)dv+\tilde{\mathfrak{K}}_n(\alpha,\beta;\psi),\end{aligned} \tag{25}$$

and

$$\begin{aligned}
&\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x))\\
&=\frac{\psi'(\beta)-\psi'(\alpha)}{\beta-\alpha}\int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))ds\\
&\quad+\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))\left(\sum_{l=3}^{n-1}\frac{\psi^{(l)}(\alpha)(s-\alpha)^{l-1}-\psi^{(l)}(\beta)(s-\beta)^{l-1}}{(l-3)!(l-1)}\right)ds\\
&\quad+\frac{\psi^{(n-1)}(\beta)-\psi^{(n-1)}(\alpha)}{(\beta-\alpha)(n-3)!}\int_\alpha^\beta \mathfrak{O}(v)dv+\mathfrak{K}_n(\alpha,\beta;\psi),
\end{aligned}\tag{26}$$

the remainders $\tilde{\mathfrak{K}}_n(\alpha,\beta;\psi)$ *and* $\mathfrak{K}_n(\alpha,\beta;\psi)$ *satisfy the bounds*

$$|\tilde{\mathfrak{K}}_n(\alpha,\beta;\psi)|\leqslant\frac{1}{(n-3)!}[\Delta(\tilde{\mathfrak{O}},\tilde{\mathfrak{O}})]^{\frac{1}{2}}\sqrt{\frac{\beta-\alpha}{2}}\left|\int_\alpha^\beta (v-\alpha)(\beta-v)[\psi^{(n+1)}(v)]^2dv\right|^{\frac{1}{2}},\tag{27}$$

$$|\mathfrak{K}_n(\alpha,\beta;\psi)|\leqslant\frac{1}{(n-3)!}[\Delta(\mathfrak{O},\mathfrak{O})]^{\frac{1}{2}}\sqrt{\frac{\beta-\alpha}{2}}\left|\int_\alpha^\beta (v-\alpha)(\beta-v)[\psi^{(n+1)}(v)]^2dv\right|^{\frac{1}{2}},\tag{28}$$

respectively.

Proof. If we apply Theorem 7 for $f\mapsto\tilde{\mathfrak{O}}$ and $h\mapsto\psi^{(n)}$, we get

$$\begin{aligned}
&\left|\frac{1}{(\beta-\alpha)}\int_\alpha^\beta \tilde{\mathfrak{O}}(v)\psi^{(n)}(v)dv-\frac{1}{(\beta-\alpha)}\int_\alpha^\beta \tilde{\mathfrak{O}}(v)dv.\frac{1}{(\beta-\alpha)}\int_\alpha^\beta \psi^{(n)}(v)dv\right|\\
&\leqslant\frac{1}{\sqrt{2}}[\Delta(\tilde{\mathfrak{O}},\tilde{\mathfrak{O}})]^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left|\int_\alpha^\beta (v-\alpha)(\beta-v)[\psi^{(n+1)}(v)]^2dv\right|^{\frac{1}{2}}.
\end{aligned}\tag{29}$$

Divide both sides of (29) by $(n-3)!$ and multiplying by $(\beta-\alpha)$, we have

$$\begin{aligned}
&\left|\frac{1}{(n-3)!}\int_\alpha^\beta \tilde{\mathfrak{O}}(v)\psi^{(n)}(v)dv-\frac{1}{(n-3)!}\int_\alpha^\beta \tilde{\mathfrak{O}}(v)dv.\frac{\psi^{(n-1)}(\beta)-\psi^{(n-1)}(\alpha)}{(\beta-\alpha)}\right|\\
&\leqslant\frac{1}{(n-3)!}[\Delta(\tilde{\mathfrak{O}},\tilde{\mathfrak{O}})]^{\frac{1}{2}}\sqrt{\frac{\beta-\alpha}{2}}\left|\int_\alpha^\beta (v-\alpha)(\beta-v)[\psi^{(n+1)}(v)]^2dv\right|^{\frac{1}{2}}.
\end{aligned}\tag{30}$$

By denoting

$$\tilde{\mathfrak{K}}_n(\alpha,\beta;\psi)=\frac{1}{(n-3)!}\int_\alpha^\beta \tilde{\mathfrak{O}}(v)\psi^{(n)}(v)dv-\frac{1}{(n-3)!}\int_\alpha^\beta \tilde{\mathfrak{O}}(v)dv.\frac{\psi^{(n-1)}(\beta)-\psi^{(n-1)}(\alpha)}{(\beta-\alpha)}.$$

in (30), we have (27). Hence, we have

$$\frac{1}{(n-3)!}\int_\alpha^\beta \tilde{\mathfrak{O}}(v)\psi^{(n)}(v)dv=\frac{\psi^{(n-1)}(\beta)-\psi^{(n-1)}(\alpha)}{(\beta-\alpha)(n-3)!}\int_\alpha^\beta \tilde{\mathfrak{O}}(v)dv+\tilde{\mathfrak{K}}_n(\alpha,\beta;\psi),$$

where the remainder $\tilde{\mathfrak{K}}_n(\alpha,\beta;\psi)$ satisfies the estimation (27). Now from identity (9), we obtain (25).

Similarly, from identity (10), we obtain (26). □

The following Grüss type inequalities can be obtained by using Theorem 8

THEOREM 10. *Let $\psi:[\alpha,\beta]\to\mathbb{R}$ be such that for $n\geqslant 2$, $\psi^{(n)}$ is absolutely continuous and let $\psi^{(n+1)}\geqslant 0$ on $[\alpha,\beta]$ with $\tilde{\mathfrak{O}}$ and $\mathfrak{O}$ defined in (23) and (24) respectively. Then the representation (25) and the remainder $\tilde{\mathfrak{K}}_n(\alpha,\beta;\psi)$ satisfies the estimation*

$$|\tilde{\mathfrak{K}}_n(\alpha,\beta;\psi)|\leqslant\frac{(\beta-\alpha)||\tilde{\mathfrak{O}}'||_\infty}{(n-3)!}\left[\frac{\psi^{(n-1)}(\beta)+\psi^{(n-1)}(\alpha)}{2}-\frac{\psi^{(n-2)}(\beta)-\psi^{(n-2)}(\alpha)}{(\beta-\alpha)}\right], \tag{31}$$

whereas the representation (26) and the remainder $\mathfrak{K}_n(\alpha,\beta;\psi)$ satisfies the estimation

$$|\mathfrak{K}_n(\alpha,\beta;\psi)|\leqslant\frac{(\beta-\alpha)||\mathfrak{O}'||_\infty}{(n-3)!}\left[\frac{\psi^{(n-1)}(\beta)+\psi^{(n-1)}(\alpha)}{2}-\frac{\psi^{(n-2)}(\beta)-\psi^{(n-2)}(\alpha)}{(\beta-\alpha)}\right]. \tag{32}$$

Proof. Applying Theorem 8 for $f\mapsto\tilde{\mathfrak{O}}$ and $h\mapsto\psi^{(n)}$, we get

$$\begin{aligned}&\left|\frac{1}{(\beta-\alpha)}\int_\alpha^\beta\tilde{\mathfrak{O}}(v)\psi^{(n)}(v)dv-\frac{1}{(\beta-\alpha)}\int_\alpha^\beta\tilde{\mathfrak{O}}(v)dv.\frac{1}{(\beta-\alpha)}\int_\alpha^\beta\psi^{(n)}(v)dv\right|\\ &\leqslant\frac{1}{2(\beta-\alpha)}||\tilde{\mathfrak{O}}'||_\infty\int_\alpha^\beta(v-\alpha)(\beta-v)\psi^{(n+1)}(v)dv.\end{aligned} \tag{33}$$

Multiplying both sides of (33) by $(\beta-\alpha)$, we get

$$\begin{aligned}&\left|\int_\alpha^\beta\tilde{\mathfrak{O}}(v)\psi^{(n)}(v)dv-\int_\alpha^\beta\tilde{\mathfrak{O}}(v)dv.\frac{1}{(\beta-\alpha)}\int_\alpha^\beta\psi^{(n)}(v)dv\right|\\ &\leqslant\frac{1}{2}||\tilde{\mathfrak{O}}'||_\infty\int_\alpha^\beta(v-\alpha)(\beta-v)\psi^{(n+1)}(v)dv.\end{aligned} \tag{34}$$

Since

$$\begin{aligned}&\int_\alpha^\beta(v-\alpha)(\beta-v)\psi^{(n+1)}(v)dv\\ &=\int_\alpha^\beta[2v-(\alpha+\beta)]\psi^{(n)}(v)dv\\ &=(\beta-\alpha)\left[\psi^{(n-1)}(\beta)+\psi^{(n-1)}(\alpha)\right]-2\left(\psi^{(n-2)}(\beta)-\psi^{(n-2)}(\alpha)\right).\end{aligned}$$

Therefore, using identity (9) and the inequality (34), we deduce (31).

Similarly, using (10) instead of (9), we have (32). □

Now we intend to give the Ostrowski type inequalities related to generalizations of Popoviciu's inequality.

THEOREM 11. *Suppose all the assumptions of Theorem 4 be satisfied. Moreover, assume (p,q) is a pair of conjugate exponents, that is $p,q \in [1,\infty]$ such that $1/p+1/q=1$. Let $|\psi^{(n)}|^p : [\alpha,\beta] \to \mathbb{R}$ be a R-integrable function for some $n \geqslant 4$. Then, we have*

$$\left| \begin{array}{c} \mathbf{P}(\mathbf{x},\mathbf{p};\psi(x)) - \frac{\psi'(\alpha)-\psi'(\beta)}{\beta-\alpha} \int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))ds \\ -\frac{1}{\beta-\alpha} \int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s)) \left(\sum_{l=2}^{n-1} \frac{l}{(l-1)!} \left(\psi^{(l)}(\alpha)(s-\alpha)^{l-1} - \psi^{(l)}(\beta)(s-\beta)^{l-1} \right) \right) ds \end{array} \right|$$
$$\leqslant \frac{1}{(n-3)!} ||\psi^{(n)}||_p \left(\int_\alpha^\beta \left| \int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s)) \tilde{R}_{n-2}(s,v)\, ds \right|^q dv \right)^{1/q}, \tag{35}$$

$$\left| \begin{array}{l} \mathbf{P}(\mathbf{x},\mathbf{p};\psi(x)) - \frac{\psi'(\beta)-\psi'(\alpha)}{\beta-\alpha} \int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))ds \\ -\frac{1}{\beta-\alpha} \int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s)) \left(\sum_{l=3}^{n-1} \frac{\psi^{(l)}(\alpha)(s-\alpha)^{l-1} - \psi^{(l)}(\beta)(s-\beta)^{l-1}}{(l-3)!(l-1)} \right) ds \end{array} \right|$$
$$\leqslant \frac{1}{(n-3)!} ||\psi^{(n)}||_p \left(\int_\alpha^\beta \left| \int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s)) R_{n-2}(s,v)\, ds \right|^q dv \right)^{1/q}. \tag{36}$$

The constants on the R.H.S. of (35) and (36) are sharp for $1 < p \leqslant \infty$ and the best possible for $p = 1$, respectively.

Proof. Denote

$$\mathfrak{I} = \frac{1}{(n-3)!} \left(\int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s)) \tilde{R}_{n-2}(s,v)\, ds \right), \quad v \in [\alpha,\beta].$$

Using identity (9), we obtain

$$\left| \begin{array}{c} \mathbf{P}(\mathbf{x},\mathbf{p};\psi(x)) - \frac{\psi'(\alpha)-\psi'(\beta)}{\beta-\alpha} \int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))ds \\ -\frac{1}{\beta-\alpha} \int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s)) \left(\sum_{l=2}^{n-1} \frac{l}{(l-1)!} \left(\psi^{(l)}(\alpha)(s-\alpha)^{l-1} - \psi^{(l)}(\beta)(s-\beta)^{l-1} \right) \right) ds \end{array} \right|$$
$$= \left| \int_\alpha^\beta \mathfrak{I}(v) \psi^{(n)}(v) dv \right|. \tag{37}$$

Apply Hölder's inequality for integrals on the right hand side of (37), we have

$$\left| \int_\alpha^\beta \mathfrak{I}(v) \psi^{(n)}(v) dv \right| \leqslant \left(\int_\alpha^\beta \left| \psi^{(n)}(v) \right|^p dv \right)^{\frac{1}{p}} \left(\int_\alpha^\beta |\mathfrak{I}(v)|^q dv \right)^{\frac{1}{q}},$$

which combine together with (37) gives (35).

For the proof of the sharpness of the constant $\left(\int_\alpha^\beta |\mathfrak{I}(v)|^q dv \right)^{1/q}$, let us define the function ψ for which the equality in (35) is obtained.

For $1 < p \leqslant \infty$ take ψ to be such that

$$\psi^{(n)}(v) = \operatorname{sgn}\Im(v)|\Im(v)|^{\frac{1}{p-1}}.$$

For $p = \infty$ take $\psi^{(n)}(v) = \operatorname{sgn}\Im(v)$.

For $p = 1$, we prove that

$$\left|\int_{\alpha}^{\beta} \Im(v)\psi^{(n)}(v)dv\right| \leqslant \max_{v\in[\alpha,\beta]} |\Im(v)| \left(\int_{\alpha}^{\beta} \psi^{(n)}(v)dv\right) \tag{38}$$

is the best possible inequality. Suppose that $|\Im(v)|$ attains its maximum at $v_0 \in [\alpha, \beta]$. To start with first we assume that $\Im(v_0) > 0$. For δ small enough we define $\psi_\delta(v)$ by

$$\psi_\delta(v) = \begin{cases} 0\,, & \alpha \leqslant v \leqslant t_0\,, \\ \frac{1}{\delta n!}(v-v_0)^n\,, & v_o \leqslant v \leqslant v_0+\delta\,, \\ \frac{1}{n!}(v-v_0)^{n-1}\,, & v_0+\delta \leqslant v \leqslant \beta\,. \end{cases}$$

Then for δ small enough

$$\left|\int_{\alpha}^{\beta} \Im(v)\psi^{(n)}(v)dv\right| = \left|\int_{v_0}^{v_0+\delta} \Im(v)\frac{1}{\delta}dv\right| = \frac{1}{\delta}\int_{v_0}^{v_0+\delta} \Im(v)dv.$$

Now from inequality (38), we have

$$\frac{1}{\delta}\int_{v_0}^{v_0+\delta} \Im(v)dv \leqslant \Im(v_0)\int_{v_0}^{v_0+\delta} \frac{1}{\delta}dv = \Im(v_0).$$

Since

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_{v_0}^{v_0+\delta} \Im(v)dv = \Im(v_0),$$

the statement follows. The case when $\Im(v_0) < 0$, we define $\psi_\delta(v)$ by

$$\psi_\delta(v) = \begin{cases} \frac{1}{n!}(v-v_0-\delta)^{n-1}\,, & \alpha \leqslant v \leqslant v_0\,, \\ \frac{-1}{\delta n!}(v-v_0-\delta)^n\,, & v_o \leqslant v \leqslant v_0+\delta\,, \\ 0\,, & v_0+\delta \leqslant v \leqslant \beta\,, \end{cases}$$

and rest of the proof is the same as above.

The proof of (36) is also similar, but we use (10) instead of (9). □

4. Mean value theorems and n-exponential convexity

We recall some definitions and basic results from [2], [5] and [12] which are required in sequel.

DEFINITION 1. A function $\psi : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} \xi_i \xi_j \psi\left(\frac{x_i + x_j}{2}\right) \geqslant 0,$$

hold for all choices $\xi_1, \dots, \xi_n \in \mathbb{R}$ and all choices $x_1, \dots, x_n \in I$. A function $\psi : I \to \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

DEFINITION 2. A function $\psi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

A function $\psi : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

PROPOSITION 1. *If $\psi : I \to \mathbb{R}$ is an n-exponentially convex in the Jensen sense, then the matrix $\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{m}$ is a positive semi-definite matrix for all $m \in \mathbb{N}, m \leqslant n$. Particularly,*

$$\det\left[\psi\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{m} \geqslant 0$$

for all $m \in \mathbb{N}$, $m = 1, 2, \dots, n$.

REMARK 3. It is known that $\psi : I \to \mathbb{R}$ is a log-convex in the Jensen sense if and only if

$$\alpha^2 \psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2\psi(y) \geqslant 0,$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.

A positive function is log-convex if and only if it is 2-exponentially convex.

REMARK 4. By the virtue of Theorem 5, we define the positive linear functionals with respect to n-convex function ψ as follows

$$\begin{aligned}\Omega_1(\psi) := \mathbf{P}(\mathbf{x}, \mathbf{p}; \psi(x)) &- \frac{\psi'(\alpha) - \psi'(\beta)}{\beta - \alpha} \int_{\alpha}^{\beta} \mathbf{P}(\mathbf{x}, \mathbf{p}; G(x,s)) ds \\ &- \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mathbf{P}(\mathbf{x}, \mathbf{p}; G(x,s)) \\ &\times \left(\sum_{l=2}^{n-1} \frac{l}{(l-1)!} \left(\psi^{(l)}(\alpha)(s-\alpha)^{l-1} - \psi^{(l)}(\beta)(s-\beta)^{l-1} \right) \right) ds \geqslant 0, \quad (39)\end{aligned}$$

and

$$\Omega_2(\psi) := \mathbf{P}(\mathbf{x},\mathbf{p};\psi(x)) - \frac{\psi'(\beta)-\psi'(\alpha)}{\beta-\alpha}\int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))ds$$
$$-\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G(x,s))$$
$$\times\left(\sum_{l=3}^{n-1}\frac{\psi^{(l)}(\alpha)(s-\alpha)^{l-1}-\psi^{(l)}(\beta)(s-\beta)^{l-1}}{(l-3)!(l-1)}\right)ds \geqslant 0. \quad (40)$$

Lagrange and Cauchy type mean value theorems related to defined functional is given in the following theorems.

THEOREM 12. *Let $\psi:[\alpha,\beta]\to\mathbb{R}$ be such that $\psi\in C^n[\alpha,\beta]$. If the inequalities in (14) and (16) are valid, then there exist $\xi_i\in[\alpha,\beta]$ such that*

$$\Omega_i(\psi)=\psi^{(n)}(\xi)\Omega_i(\varphi);\quad i=1,2,$$

where $\varphi(x)=\frac{x^n}{n!}$ and $\Omega_i(\cdot)$ are defined in Remark 4.

Proof. Similar to the proof of Theorem 4.1 in [6]. □

THEOREM 13. *Let $\psi,\lambda:[\alpha,\beta]\to\mathbb{R}$ be such that $\psi,\lambda\in C^n[\alpha,\beta]$. If the inequalities in (14) and (16) are valid, then there exist $\xi_i\in[\alpha,\beta]$ such that*

$$\frac{\Omega_i(\psi)}{\Omega_i(\lambda)}=\frac{\psi^{(n)}(\xi)}{\lambda^{(n)}(\xi)};\quad i=1,2,$$

provided that the denominators are non-zero and $\Omega_i(\cdot)$ are defined in Remark 4.

Proof. Similar to the proof of Corollary 4.2 in [6]. □

Theorem 13 enables us to define Cauchy means, because if

$$\xi_i=\left(\frac{\psi^{(n)}}{\lambda^{(n)}}\right)^{-1}\left(\frac{\Omega_i(\psi)}{\Omega_i(\lambda)}\right),$$

which show that ξ_i $(i=1,2)$ are means of α, β for given functions ψ and λ.

Next we construct the non trivial examples of n-exponentially and exponentially convex functions from positive linear functionals $\Omega_i(\cdot)$ $(i=1,2)$. We use the idea given in [12]. In the sequel I and J are intervals in $\mathbb{R}$.

THEOREM 14. *Let $\Theta=\{\psi_t : t\in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$ such that the function $t\mapsto[x_0,\dots,x_n;\psi_t]$ is n-exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0,\dots,x_n\in I$. Then for the linear functionals $\Omega_i(\psi_t)$ $(i=1,2)$ as defined in Remark 4, the following statements are valid for each $i=1,2$:*

(i) *The function* $t \to \Omega_i(\psi_t)$ *is* n*-exponentially convex in the Jensen sense on* J *and the matrix* $[\Omega_i(\psi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m$ *is a positive semi-definite for all* $m \in \mathbb{N}$, $m \leqslant n$, $t_1,..,t_m \in J$. *Particularly,*

$$\det[\Omega_i(\psi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m \geqslant 0 \quad \textit{for all} \quad m \in \mathbb{N},\ m = 1,2,\ldots,n.$$

(ii) *If the function* $t \to \Omega_i(\psi_t)$ *is continuous on* J, *then it is* n*-exponentially convex on* J.

Proof. Fix $i = 1,2$.

(i) For $\xi_j \in \mathbb{R}$ and $t_j \in J$, $j = 1,\ldots,n$, we define the function

$$h(x) = \sum_{j,l=1}^{n} \xi_j \xi_l \psi_{\frac{t_j+t_l}{2}}(x).$$

Using the assumption that the function $t \mapsto [x_0,\ldots,x_n;\psi_t]$ is n-exponentially convex in the Jensen sense, we have

$$[x_0,\ldots,x_n,h] = \sum_{j,l=1}^{n} \xi_j \xi_l [x_0,\ldots,x_n;\psi_{\frac{t_j+t_l}{2}}] \geqslant 0,$$

which in turn implies that h is a n-convex function on J, therefore from Remark 4 we have $\Omega_i(h) \geqslant 0$. The linearity of $\Omega_i(\cdot)$ gives

$$\sum_{j,l=1}^{n} \xi_j \xi_l \Omega_i(\psi_{\frac{t_j+t_l}{2}}) \geqslant 0.$$

We conclude that the function $t \mapsto \Omega_i(\psi_t)$ is n-exponentially convex on J in the Jensen sense.

The remaining part follows from Proposition 1.

(ii) If the function $t \to \Omega_i(\psi_t)$ is continuous on J, then it is n-exponentially convex on J by definition. □

The following corollary is an immediate consequence of the above theorem

COROLLARY 1. *Let* $\Theta = \{\psi_t : t \in J\}$, *where* J *is an interval in* $\mathbb{R}$, *be a family of functions defined on an interval* I *in* $\mathbb{R}$, *such that the function* $t \mapsto [x_0,\ldots,x_n;\psi_t]$ *is exponentially convex in the Jensen sense on* J *for every* $(n+1)$ *mutually different points* $x_0,\ldots,x_n \in I$. *Then for the linear functional* $\Omega_i(\psi_t)$ $(i = 1,2)$, *the following statements hold:*

(i) *The function* $t \to \Omega_i(\psi_t)$ *is exponentially convex in the Jensen sense on* J *and the matrix* $[\Omega_i(\psi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m$ *is a positive semi-definite for all* $m \in \mathbb{N}$, $m \leqslant n$, $t_1,..,t_m \in J$. *Particularly,*

$$\det[\Omega_i(\psi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m \geqslant 0 \quad \textit{for all} \quad m \in \mathbb{N},\ m = 1,2,\ldots,n.$$

(ii) If the function $t \to \Omega_i(\psi_t)$ is continuous on J, then it is exponentially convex on J.

COROLLARY 2. *Let $\Theta = \{\psi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $t \mapsto [x_0, \ldots, x_n; \psi_t]$ is 2-exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \ldots, x_n \in I$. Let $\Omega_i(\cdot)$ $(i = 1,2)$ be linear functionals, then the following statements hold:*

(i) If the function $t \mapsto \Omega_i(\psi_t)$ is continuous on J, then it is 2-exponentially convex function on J. If $t \mapsto \Omega_i(\psi_t)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true:

$$[\Omega_i(\psi_s)]^{t-r} \leqslant [\Omega_i(\psi_r)]^{t-s} [\Omega_i(\psi_t)]^{s-r},$$

for every choice $r,s,t \in J$, such that $r < s < t$.

(ii) If the function $t \mapsto \Omega_i(\psi_t)$ is strictly positive and differentiable on J, then for every $p,q,u,v \in J$, such that $p \leqslant u$ and $q \leqslant v$, we have

$$\mu_{p,q}(\Omega_i, \Theta) \leqslant \mu_{u,v}(\Omega_i, \Theta), \tag{41}$$

where

$$\mu_{p,q}(\Omega_i, \Theta) = \begin{cases} \left(\frac{\Omega_i(\psi_p)}{\Omega_i(\psi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\frac{d}{dp}\Omega_i(\psi_p)}{\Omega_i(\psi_p)}\right), & p = q, \end{cases} \tag{42}$$

for $\psi_p, \psi_q \in \Theta$.

Proof. Fix $i = 1,2$.

(i) This is an immediate consequence of Theorem 14 and Remark 3.

(ii) Since $p \mapsto \Omega_i(\psi_t)$ is positive and continuous, by (i) we have that $t \mapsto \Omega_i(\psi_t)$ is log-convex on J, that is, the function $t \mapsto \log \Omega_i(\psi_t)$ is convex on J. Hence we get

$$\frac{\log \Omega_i(\psi_p) - \log \Omega_i(\psi_q)}{p-q} \leqslant \frac{\log \Omega_i(\psi_u) - \log \Omega_i(\psi_v)}{u-v}, \tag{43}$$

for $p \leqslant u$, $q \leqslant v$, $p \neq q$, $u \neq v$. So, we conclude that

$$\mu_{p,q}(\Omega_i, \Theta) \leqslant \mu_{u,v}(\Omega_i, \Theta).$$

Cases $p = q$ and $u = v$ follow from (43) as limit cases. □

5. Applications to Cauchy means

In this section, we present some families of functions which fulfil the conditions of Theorem 14, Corollary 1 and Corollary 2. This enables us to construct a large families of functions which are exponentially convex. Explicit form of this functions is obtained after we calculate explicit action of functionals on a given family.

EXAMPLE 1. Let us consider a family of functions

$$\Theta_1 = \{\psi_t : \mathbb{R} \to \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$\psi_t(x) = \begin{cases} \frac{e^{tx}}{t^n}, \ t \neq 0, \\ \frac{x^n}{n!}, \ t = 0. \end{cases}$$

Since $\frac{d^n \psi_t}{dx^n}(x) = e^{tx} > 0$, the function ψ_t is n-convex on $\mathbb{R}$ for every $t \in \mathbb{R}$ and $t \mapsto \frac{d^n \psi_t}{dx^n}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 14 we also have that $t \mapsto [x_0, \ldots, x_n; \psi_t]$ is exponentially convex (and so exponentially convex in the Jensen sense). Now, using Corollary 1 we conclude that $t \mapsto \Omega_i(\psi_t)$ $(i = 1, 2)$ are exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although the mapping $t \mapsto \psi_t$ is not continuous for $t = 0$), so it is exponentially convex. For this family of functions, $\mu_{t,q}(\Omega_i, \Theta_1)$ from (42), becomes

$$\mu_{t,q}(\Omega_i, \Theta_1) = \begin{cases} \left(\frac{\Omega_i(\psi_t)}{\Omega_i(\psi_q)}\right)^{\frac{1}{t-q}}, & t \neq q, \\ \exp\left(\frac{\Omega_i(id \cdot \psi_t)}{\Omega_i(\psi_t)} - \frac{n}{t}\right), & t = q \neq 0, \\ \exp\left(\frac{1}{n+1}\frac{\Omega_i(id \cdot \psi_0)}{\Omega_i(\psi_0)}\right), & t = q = 0, \end{cases} \quad i = 1, 2$$

where "id" is the identity function. By Corollary 2 $\mu_{t,q}(\Omega_i, \Theta_1)$ is a monotone function in parameters t and q.

Since

$$\left(\frac{\frac{d^n f_t}{dx^n}}{\frac{d^n f_q}{dx^n}}\right)^{\frac{1}{t-q}} (\log x) = x,$$

using Theorem 13 it follows that:

$$M_{t,q}(\Omega_i, \Theta_1) = \log \mu_{t,q}(\Omega_i, \Theta_1), \quad i = 1, 2$$

satisfies

$$\alpha \leqslant M_{t,q}(\Omega_i, \Theta_1) \leqslant \beta, \quad i = 1, 2.$$

Hence $M_{t,q}(\Omega_i, \Theta_1)$ $(i = 1, 2)$ are monotonic means.

EXAMPLE 2. Let us consider a family of functions

$$\Theta_2 = \{g_t : (0,\infty) \to \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$g_t(x) = \begin{cases} \frac{x^t}{t(t-1)\cdots(t-n+1)}, & t \notin \{0,1,\ldots,n-1\}, \\ \frac{x^j \log x}{(-1)^{n-1-j} j!(n-1-j)!}, & t = j \in \{0,1,\ldots,n-1\}. \end{cases}$$

Since $\frac{d^n g_t}{dx^n}(x) = x^{t-n} > 0$, the function g_t is n-convex for $x > 0$ and $t \mapsto \frac{d^n g_t}{dx^n}(x)$ is exponentially convex by definition. Arguing as in Example 1 we get that the mappings $t \mapsto \Omega_i(g_t)$ is exponentially convex for each $i = 1,2$. Hence, for this family of functions $\mu_{p,q}(\Omega_i, \Theta_2)$ $(i = 1,2)$, from (42), are equal to

$$\mu_{t,q}(\Omega_i, \Theta_2) = \begin{cases} \left(\frac{\Omega_i(g_t)}{\Omega_i(g_q)}\right)^{\frac{1}{t-q}}, & t \neq q, \\ \exp\left((-1)^{n-1}(n-1)!\frac{\Omega_i(g_0 g_t)}{\Omega_i(g_t)} + \sum_{k=0}^{n-1} \frac{1}{k-t}\right), & t = q \notin \{0,1,\ldots,n-1\}, \\ \exp\left((-1)^{n-1}(n-1)!\frac{\Omega_i(g_0 g_t)}{2\Omega_i(g_t)} + \sum_{\substack{k=0 \\ k\neq t}}^{n-1} \frac{1}{k-t}\right), & t = q \in \{0,1,\ldots,n-1\}. \end{cases}$$

Again, using Theorem 13 we conclude that

$$\alpha \leqslant \left(\frac{\Omega_i(g_t)}{\Omega_i(g_q)}\right)^{\frac{1}{t-q}} \leqslant \beta, \quad i = 1,2.$$

Hence $\mu_{t,q}(\Omega_i, \Theta_2)$ $(i = 1,2)$ are means and their monotonicity is followed by (41).

EXAMPLE 3. Let

$$\Theta_3 = \{\zeta_t : (0,\infty) \to \mathbb{R} : t \in (0,\infty)\}$$

be a family of functions defined by

$$\zeta_t(x) = \begin{cases} \frac{t^{-x}}{(-\log t)^n}, & t \neq 1; \\ \frac{x^n}{(n)!}, & t = 1. \end{cases}$$

Since $\frac{d^n \zeta_t}{dx^n}(x) = t^{-x}$ is the Laplace transform of a non-negative function (see [16]) it is exponentially convex. Obviously ζ_t are n-convex functions for every $t > 0$.

For this family of functions, $\mu_{t,q}(\Omega_i, \Theta_3)$, in this case for $[\alpha,\beta] \subset \mathbb{R}^+$, from (42) become

$$\mu_{t,q}(\Omega_i, \Theta_3) = \begin{cases} \left(\frac{\Omega_i(\zeta_t)}{\Omega_i(\zeta_q)}\right)^{\frac{1}{t-q}}, & t \neq q; \\ \exp\left(-\frac{\Omega_i(id.\zeta_t)}{t\Omega_i(\zeta_t)} - \frac{n}{t \log t}\right), & t = q \neq 1; \quad i = 1,2 \\ \exp\left(-\frac{1}{n+1}\frac{\Omega_i(id.\zeta_1)}{\Omega_i(\zeta_1)}\right), & t = q = 1, \end{cases}$$

where "id" is the identity function. By Corollary 2 $\mu_{p,q}(\Omega_i,\Theta_3)$ $(i=1,2)$ are monotone functions in parameters t and q.

Using Theorem 13 it follows that

$$M_{t,q}(\Omega_i,\Theta_3) = -L(t,q)log\mu_{t,q}(\Omega_i,\Theta_3), \quad i=1,2$$

satisfy

$$\alpha \leqslant M_{t,q}(\Omega_i,\Theta_3) \leqslant \beta, \quad i=1,2.$$

This shows that $M_{t,q}(\Omega_i,\Theta_3)$ is a mean for each $i=1,2$. Because of the inequality (41), these means are monotonic. Furthermore, $L(t,q)$ is logarithmic mean defined by

$$L(t,q) = \begin{cases} \frac{t-q}{\log t-\log q}, & t \neq q; \\ t, & t=q. \end{cases}$$

EXAMPLE 4. Let

$$\Theta_4 = \{\gamma_t : (0,\infty) \to \mathbb{R} : t \in (0,\infty)\}$$

be a family of functions defined by

$$\gamma_t(x) = \frac{e^{-x\sqrt{t}}}{(-\sqrt{t})^n}.$$

Since $\frac{d^n\gamma_t}{dx^n}(x) = e^{-x\sqrt{t}}$ is the Laplace transform of a non-negative function (see [16]) it is exponentially convex. Obviously γ_t are n-convex function for every $t>0$.

For this family of functions, $\mu_{t,q}(\Omega_i,\Theta_4)$ $(i=1,2)$, in this case for $[\alpha,\beta] \subset \mathbb{R}^+$, from (42) become

$$\mu_{t,q}(\Omega_i,\Theta_4) = \begin{cases} \left(\frac{\Omega_i(\gamma_t)}{\Omega_i(\gamma_q)}\right)^{\frac{1}{t-q}}, & t \neq q; \\ \exp\left(-\frac{\Omega_i(id.\gamma_t)}{2\sqrt{t}\Omega_i(\gamma_t)} - \frac{n}{2t}\right), & t=q; \end{cases} \qquad i=1,2.$$

By Corollary 2, these are monotone functions in parameters t and q.

Using Theorem 13 it follows that

$$M_{t,q}(\Omega_i,\Theta_4) = -\left(\sqrt{t}+\sqrt{q}\right) ln\mu_{t,q}(\Omega_i,\Theta_4), \ i=1,2$$

satisfy

$$\alpha \leqslant M_{t,q}(\Omega_i,\Theta_4) \leqslant \beta, \ i=1,2.$$

This shows that $M_{t,q}(\Omega_i,\Theta_4)$ $(i=1,2)$ are means. Because of the above inequality (41), these means are monotonic in nature.

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