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APPLICATIONS OF REFINED HARDY–TYPE INEQUALITIES

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Abstract. This paper is to provide the broad range of Hardy-type inequalities and their refinements for linear differential operator, Widder's derivative and more generalized fractional integral operator using convex and monotone convex functions. As special cases we give results for Saigo, Riemann-Liouville and Erdélyi-Kober fractional integral operators.

1. Introduction

Let $(\Sigma_1,\Omega_1,\mu_1)$ and $(\Sigma_2,\Omega_2,\mu_2)$ be measure spaces with positive σ-finite measures. Let $U(f,k)$ denote the class of functions $g:\Omega_1\to\mathbb{R}$ with the representation

$$g(x)=\int_{\Omega_2} k(x,t)f(t)d\mu_2(t),$$

and A_k be an integral operator defined by

$$A_kf(x):=\frac{g(x)}{K(x)}=\frac{1}{K(x)}\int_{\Omega_2} k(x,t)f(t)d\mu_2(t), \tag{1}$$

where $k:\Omega_1\times\Omega_2\to\mathbb{R}$ is measurable and non-negative kernel, $f:\Omega_2\to\mathbb{R}$ is measurable function and

$$0<K(x):=\int_{\Omega_2} k(x,t)d\mu_2(t),\quad x\in\Omega_1. \tag{2}$$

Hardy-type inequalities attracted the attention of many mathematicians and they gave a lot of interesting generalizations and improvements of these inequalities and has added the rich literature in this field. Čižmešija, Krulić Himmelreich, Pečarić and Persson ([2], [6], [14], [16]) has studied a lot of Hardy-type inequalities which is an incredible contribution in theory of inequalities. But we give such type of inequalities for linear differential operator, Widder's derivative and more general fractional integral operator using convex and monotone convex functions. For more detail we refer [1], [4], [10], [13], [15] and the references cited therein.

The upcoming theorem is given in [14].

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THEOREM 1. *Let* $(\Omega_1, \Sigma_1, \mu_1)$ *and* $(\Omega_2, \Sigma_2, \mu_2)$ *be measure spaces with positive* σ*-finite measures,* u *be a weight function on* Ω_1*,* k *a non-negative measurable function on* $\Omega_1 \times \Omega_2$*, and* K *be defined on* Ω_1 *by* (2)*. Suppose that* $K(x) > 0$ *for all* $x \in \Omega_1$*, that the function* $x \mapsto u(x)\frac{k(x,t)}{K(x)}$ *is integrable on* Ω_1 *for each fixed* $t \in \Omega_2$*, and that* v *is defined on* Ω_2 *by*

$$v(t) := \int_{\Omega_1} u(x)\frac{k(x,t)}{K(x)}\,d\mu_1(x) < \infty. \tag{3}$$

If Φ *is a convex function on the interval* $I \subseteq \mathbb{R}$*, then the inequality*

$$\int_{\Omega_1} u(x)\Phi(A_k f(x))\,d\mu_1(x) \leqslant \int_{\Omega_2} v(t)\Phi(f(t))d\mu_2(t), \tag{4}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$*, such that* $Im f \subseteq I$*, where* A_k *is defined by* (1)*.*

Substitute $k(x,t)$ by $k(x,t)f_2(t)$ and f by $\frac{f_1}{f_2}$, where $f_i : \Omega_2 \to \mathbb{R}$, $(i = 1,2)$ are measurable functions in Theorem 1, we obtain the following result (see [11]).

THEOREM 2. *Let* $(\Omega_1, \Sigma_1, \mu_1)$ *and* $(\Omega_2, \Sigma_2, \mu_2)$ *be measure spaces with* σ*-finite measures,* u *be a weight function on* Ω_1*,* k *be a non-negative measurable function on* $\Omega_1 \times \Omega_2$*. Assume that the function* $x \mapsto u(x)\frac{k(x,t)}{g_2(x)}$ *is integrable on* Ω_1 *for each fixed* $t \in \Omega_2$*. Define* p *on* Ω_2 *by*

$$p(t) := f_2(t)\int_{\Omega_1} u(x)\frac{k(x,t)}{g_2(x)}d\mu_1(x) < \infty.$$

If $\Phi : I \to \mathbb{R}$ *is a convex function and* $\frac{g_1(x)}{g_2(x)}, \frac{f_1(t)}{f_2(t)} \in I$*, then the inequality*

$$\int_{\Omega_1} u(x)\Phi\left(\frac{g_1(x)}{g_2(x)}\right)d\mu_1(x) \leqslant \int_{\Omega_2} p(t)\Phi\left(\frac{f_1(t)}{f_2(t)}\right)d\mu_2(t), \tag{5}$$

holds for all $g_i \in U(f_i, k)$*,* $(i = 1,2)$ *and for all measurable functions* $f_i : \Omega_2 \to \mathbb{R}$*,* $(i = 1,2)$*.*

REMARK 1. If Φ is strictly convex on I and $\frac{f_1(x)}{f_2(x)}$ is non-constant, then the inequality given in (5) is strict.

New refined general weighted Hardy-type inequality with a non-negative kernel and related to an arbitrary convex function is given in the following theorem (see [5]).

THEOREM 3. *Let the assumptions of Theorem 1 be satisfied. Moreover, if* Φ *is a convex function on an interval* $I \subseteq \mathbb{R}$ *and* $\varphi : I \to \mathbb{R}$ *is any function, such that* $\varphi(x) \in \partial\Phi(x)$ *for all* $x \in \operatorname{Int} I$*, then the inequality*

$$\int_{\Omega_2} v(t)\Phi(f(t))\,d\mu_2(t) - \int_{\Omega_1} u(x)\Phi(A_k f(x))\,d\mu_1(x)$$
$$\geqslant \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} k(x,t)\,\big|\,|\Phi(f(t)) - \Phi(A_k f(x))|$$
$$- |\varphi(A_k f(x))| \cdot |f(t) - A_k f(x)|\,\big|\,d\mu_2(t)\,d\mu_1(x),$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$*, such that* $f(t) \in I$ *for all* $t \in \Omega_2$*.*

If Φ *is a monotone convex function on an interval* $I \subseteq \mathbb{R}$*, then the inequality*

$$\int_{\Omega_2} v(t)\Phi(f(t))\,d\mu_2(t) - \int_{\Omega_1} u(x)\Phi(A_k f(x))\,d\mu_1(x)$$
$$\geqslant \Bigg| \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} sgn(f(t) - A_k f(x)) k(x,t) \Big[\Phi(f(t)) - \Phi(A_k f(x))$$
$$-|\varphi(A_k f(x))| \cdot (f(t) - A_k f(x))\Big]\,d\mu_2(t)\,d\mu_1(x)\Bigg|,$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$*, such that* $f(t) \in I$*, for all fixed* $t \in \Omega_2$ *where* $A_k f$ *is defined by* (1)*.*

In the following theorem, a refinement of a Hardy–type inequality obtained by S. Kaijser et al. in [13].

THEOREM 4. *Let* $u : (0,b) \to \mathbb{R}$ *be a weight function such that the function* $x \mapsto \frac{u(x)}{x} \cdot \frac{k(x,t)}{K(x)}$ *is integrable on* (t,b) *for each fixed* $t \in (0,b)$*, and let the function* $w : (0,b) \to \mathbb{R}$ *be defined by*

$$w(t) := t \int_t^b \frac{k(x,t)}{K(x)} u(x)\,\frac{dx}{x},$$

where $0 < b \leqslant \infty$ *and* $k : (0,b) \times (0,b) \to \mathbb{R}$ *be a non-negative measurable function, such that*

$$K(x) = \int_0^x k(x,t)\,dt > 0,\ x \in (0,b).$$

If Φ *is a convex function on an interval* $I \subseteq \mathbb{R}$ *and* $\varphi : I \to \mathbb{R}$ *is such that* $\varphi(x) \in \partial\Phi(x)$

for all $x \in \operatorname{Int} I$, *then the inequality*

$$\int_0^b w(t)\Phi(f(t))\,\frac{dt}{t} - \int_0^b u(x)\Phi(A_k f(x))\,\frac{dx}{x} \tag{6}$$
$$\geqslant \int_0^b \frac{u(x)}{K(x)} \int_0^x k(x,t)\,\big|\,|\Phi(f(t)) - \Phi(A_k f(x))| - |\varphi(A_k f(x))|\,|f(t) - A_k f(x)|\,\big|\,dt\,\frac{dx}{x},$$

holds for all measurable functions $f:(0,b)\to\mathbb{R}$ *with values in* I *and for* $A_k f$ *defined by*

$$A_k f(x) := \frac{1}{K(x)} \int_0^x k(x,t) f(t)dt, \;\; x \in (0,b).$$

If the function Φ *is concave, the order of integrals on the left-hand side of* (6) *is reversed. If* Φ *is monotone convex on the interval* $I \subseteq \mathbb{R}$, *then the following inequality*

$$\int_0^b w(t)\Phi\left(f(t)\right)\frac{dt}{t} - \int_0^b u(x)\Phi\left(A_k f(x)\right)\frac{dx}{x}$$
$$\geqslant \Big|\int_0^b \frac{u(x)}{K(x)} \int_0^x sgn(f(t) - A_k f(x))k(x,t)\Big[\Phi(f(t) - \Phi(A_k f(x))$$
$$-|\varphi(A_k f(x))|.(f(t) - A_k f(x))\Big]\,dt\,\frac{dx}{x}\Big|,$$

holds for all measurable functions $f:(0,b)\to\mathbb{R}$ *with values in* I.

Next mean value theorem is given in [8].

THEOREM 5. *Let* $(\Omega_1,\Sigma_1,\mu_1)$, $(\Omega_2,\Sigma_2,\mu_2)$ *be measure spaces with* σ*-finite measures and* $u:\Omega_1\to\mathbb{R}$ *be a weight function. Let* I *be compact interval of* $\mathbb{R}$, $\tilde{h}\in C^2(I)$, *and* $f:\Omega_2\to\mathbb{R}$ *a measurable function such that* $Imf \subseteq I$. *Then there exists* $\eta \in I$ *such that*

$$\int_{\Omega_2} v(t)\tilde{h}(f(t))\,d\mu_2(t) - \int_{\Omega_1} u(x)\tilde{h}(A_k f(x))d\mu_1(x)$$
$$= \frac{\tilde{h}''(\eta)}{2}\left[\int_{\Omega_2} v(t)f^2(t)\,d\mu_2(t) - \int_{\Omega_1} u(x)(A_k f(x))^2\,d\mu_1(x)\right],$$

where $A_k f$ *and* v *are defined by* (1) *and* (3) *respectively.*

We continue with the definition of exponentially convex function as originally given in [3] by Bernstein.

DEFINITION 1. A function $\Phi\colon (a,b) \to \mathbb{R}$ is *exponentially convex* if it is continuous and

$$\sum_{i,j=1}^{n} t_i t_j \Phi(x_i + x_j) \geqslant 0,$$

for all $n \in \mathbb{N}$ and all sequences $(t_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ of real numbers, such that $x_i + x_j \in (a,b)$, $1 \leqslant i,j \leqslant n$.

LEMMA 1. *For $s \in \mathbb{R}$, let function $\varphi_s\colon (0,\infty) \to \mathbb{R}$ be defined by*

$$\varphi_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 0,1, \\ -\log x, & s = 0, \\ x\log x, & s = 1. \end{cases} \tag{7}$$

Then $\varphi_s''(x) = x^{s-2}$, that is, φ_s is a convex function.

THEOREM 6. [8] *Let the conditions of Theorem 1 be satisfied and φ_s be defined by (7). Let f be a positive function. Then the function $\xi : \mathbb{R} \to [0,\infty)$ defined by*

$$\xi(s) = \int_{\Omega_2} v(t)\varphi_s(f(t))\,d\mu_2(t) - \int_{\Omega_1} u(x)\varphi_s(A_k f(x))\,d\mu_1(x),$$

is exponentially convex.

THEOREM 7. *Let the conditions of Theorem 5 be satisfied. Moreover, $g, \tilde{h} \in C^2(I)$ such that $\tilde{h}''(x) \neq 0$ for every $x \in I$ and*

$$\int_{\Omega_2} v(t)\,\tilde{h}(f(t))\,d\mu_2(t) - \int_{\Omega_1} u(x)\,\tilde{h}(A_k f(x))\,d\mu_1(x) \neq 0.$$

Then there exists $\eta \in I$ such that it holds

$$\frac{g''(\eta)}{\tilde{h}''(\eta)} = \frac{\int_{\Omega_2} v(t)\,g(f(t))\,d\mu_2(t) - \int_{\Omega_1} u(x)\,g(A_k f(x))\,d\mu_1(x)}{\int_{\Omega_2} v(t)\,\tilde{h}(f(t))\,d\mu_2(t) - \int_{\Omega_1} u(x)\,\tilde{h}(A_k f(x))\,d\mu_1(x)}.$$

Under assumptions of the Theorem 1, we define a linear functional by taking the positive difference of the inequality stated in (4) as:

$$\Delta_1(\Phi) = \int_{\Omega_2} v(t)\Phi(f(t))\,d\mu_2(t) - \int_{\Omega_1} u(x)\Phi(A_k f(x))\,d\mu_1(x). \tag{8}$$

We also define a linear functional by taking the positive difference of left-hand side and right-hand side of the inequality (5) given in Theorem 2 as:

$$\Delta_2(\Phi) = \int_{\Omega_2} p(t)\Phi\left(\frac{f_1(t)}{f_2(t)}\right) d\mu_2(t) - \int_{\Omega_1} u(x)\Phi\left(\frac{g_1(x)}{g_2(x)}\right) d\mu_1(x). \tag{9}$$

First we give some necessary details about the divided differences. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a function. Then for distinct points $z_i \in I$, $i = 0,1,2$, the divided differences of first and second order are defined by:

$$[z_i, z_{i+1}; f] = \frac{f(z_{i+1}) - f(z_i)}{z_{i+1} - z_i}, \quad (i = 0,1),$$

$$[z_0, z_1, z_2; f] = \frac{[z_1, z_2; f] - [z_0, z_1; f]}{z_2 - z_0}. \tag{10}$$

The values of the divided differences are independent of the order of points z_0, z_1, z_2 and may be extended to include the cases when some or all points are equal, that is

$$[z_0, z_0; f] = \lim_{z_1 \to z_0} [z_0, z_1; f] = f'(z_0),$$

provided that f' exists.

Now passing through the limit $z_1 \to z_0$ and replacing z_2 by z in (10), we have (see [16, p. 16])

$$[z_0, z_0, z; f] = \lim_{z_1 \to z_0} [z_0, z_1, z; f] = \frac{f(z) - f(z_0) - (z - z_0) f'(z_0)}{(z - z_0)^2}, \quad z \neq z_0,$$

provided that f' exists. Also passing to the limit $z_i \to z$ $(i = 0,1,2)$ in (10), we have

$$[z, z, z; f] = \lim_{z_i \to z} [z_0, z_1, z_2; f] = \frac{f''(z)}{2},$$

provided that f'' exists.

One can observe that if for all $z_0, z_1 \in I$, $[z_0, z_1, f] \geqslant 0$, then f is increasing on I and if for all $z_0, z_1, z_2 \in I$, $[z_0, z_1, z_2; f] \geqslant 0$, then f is convex on I. The following theorem is given in [12].

THEOREM 8. *Let $\Gamma = \{\Phi_p : p \in J\}$ be a family of functions defined on I, such that the function $p \mapsto [z_0, z_1, z_2; \Phi_p]$ is n-exponentially convex in the Jensen sense on J for every three distinct points $z_0, z_1, z_2 \in I$. Let Δ_i $(i = 1,2)$ be linear functionals defined by (8), (9). Then the function $p \mapsto \Delta_i(\Phi_p)$ $(i = 1,2)$ is n-exponentially convex in the Jensen sense on J, if it is continuous on J.*

The rest of the paper is planned in the following way: In Section 2, we prove new Hardy-type inequalities and their refinements involving linear differential operator. Section 3 deals with Hardy-type, refined Hardy-type inequalities for Widder's derivative. Section 4 contain refined Hardy-type inequalities for generalized fractional integral operator. As special case we obtain the results for Saigo, Riemann-Liouville and Erdélyi-Kober fractional integral operators.

2. Hardy-type inequalities for linear differential operator

Let $[a,b]\subset\mathbb{R}$, $a_i(x)$, $i=0,1,\dots,n-1$ $(n\in\mathbb{N})$, $h(x)$ be continuous functions on $[a,b]$. Let

$$L=D^n+a_{n-1}(x)D^{n-1}+\dots+a_0(x),\ \ x\in(a,b),$$

be a fixed linear differential operator on $C^n[a,b]$. Let $y_1(x),\dots,y_n(x)$ be a set of linearly independent solution to $Ly=0$ and the associated Green's function for L is

$$H(x,t):=\frac{\begin{vmatrix} y_1(t) & \cdots & y_n(t)\\ y_1'(t) & \cdots & y_n'(t)\\ \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\\ y_1^{(n-2)}(t) & \cdot & y_n^{(n-2)}(t)\\ y_1(x) & \cdots & y_n(x)\end{vmatrix}}{\begin{vmatrix} y_1(t) & \cdots & y_n(t)\\ y_1'(t) & \cdots & y_n'(t)\\ \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\\ y_1^{(n-2)}(t) & \cdot & y_n^{(n-2)}(t)\\ y_1^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t)\end{vmatrix}},$$

which is continuous function on $[a,b]^2$, then

$$y(x)=\int_a^x H(x,t)h(t)dt,\ \ \text{for all}\ \ x\in[a,b],$$

is the unique solution to the initial value problem

$$Ly=h,\quad y^{(i)}(a)=0,\quad i=0,1,\dots,n-1.$$

In our first upcoming main result we establish the refinement of the Theorem 2.1 of [9].

THEOREM 9. *Let u be a weight function on (a,b), $H(x,t)$ be a non-negative measurable Green function associated to linear differential operator L. Suppose that $\widetilde{H}(x)>0$ for all $x\in(a,b)$, that the function $x\mapsto u(x)\frac{H(x,t)}{\widetilde{H}(x)}$ is integrable on (a,b) for each fixed $t\in(a,b)$ and $\bar{v}$ is defined on (a,b) by*

$$\bar{v}(t):=\int_t^b u(x)\frac{H(x,t)}{\widetilde{H}(x)}dx<\infty. \tag{11}$$

If Φ is a convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$, then the inequality

$$\begin{aligned} &\int_a^b \bar{v}(t)\Phi(h(t))\,dt - \int_a^b u(x)\Phi\left(\frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)dx \\ &\geqslant \int_a^b \frac{u(x)}{\widetilde{H}(x)}\int_a^x H(x,t)\left|\left|\Phi(h(t)) - \Phi\left(\frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)\right|\right. \\ &\quad \left. - \left|\varphi\left(\frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)\right| \cdot \left|h(t) - \frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right|\right| dt dx, \end{aligned} \tag{12}$$

holds for all measurable functions $h : (a,b) \to \mathbb{R}$, such that $h(t) \in I$ for all $t \in (a,b)$.

If Φ is a monotone convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\begin{aligned} &\int_a^b \bar{v}(t)\Phi(h(t))\,dt - \int_a^b u(x)\Phi\left(\frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)dx \\ &\geqslant \left| \int_a^b \frac{u(x)}{\widetilde{H}(x)}\int_a^x sgn\left(h(t) - \frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)\right. \\ &\times H(x,t)\left[\Phi(h(t)) - \Phi\left(\frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)\right. \\ &\left. - \left|\varphi\left(\frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)\right| \cdot \left(h(t) - \frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)\right] dt\,dx \Bigg|, \end{aligned} \tag{13}$$

holds for all measurable functions $h : (a,b) \to \mathbb{R}$, such that $h(t) \in I$, for all fixed $t \in (a,b)$ and $\widetilde{H}(x)$ is defined as

$$0 < \widetilde{H}(x) := \int_a^x H(x,t)dt.$$

Proof. Applying Theorem 3 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$ and $k(x,t) = H(x,t)$, we get inequality (12) and inequality (13). □

REMARK 2. Choose the particular convex function $\Phi(x) = x^\nu$, $\nu \geqslant 1$ and weight function $u(x) = \widetilde{H}(x)$ in Theorem 9 we obtain, $\bar{v}(t) = \int_t^b H(x,t)dx =: K_1(t)$. Since right hand side of inequality (12) and inequality (13) is non negative, we obtain

$$\int_a^b \widetilde{H}^{1-\nu}(x)\left(\int_a^x H(x,t)h(t)dt\right)^\nu dx \leqslant \int_a^b K_1(t)h^\nu(t)dt. \tag{14}$$

Inequality (14) gives

$$\widetilde{H}^{1-\nu}(b)\int_a^b y^\nu(x)dx \leqslant K_1(a)\int_a^b h^\nu(t)dt.$$

This implies that

$$\|y\|_\nu(a,b) \leqslant \left(\frac{K_1(a)}{\widetilde{H}^{1-\nu}(b)}\right)^{\frac{1}{\nu}} \|h\|_\nu(a,b).$$

One-dimensional setting give refined Hardy and Pólya-Knopp-type inequalities. In the following theorem, a refinement of a Hardy-type inequality obtained by S. Kaijser et al. in [13] is given for linear differential operator.

THEOREM 10. *Let* $0 < b \leqslant \infty$ *and* $H : (0,b)\times(0,b) \to \mathbb{R}$ *be a non-negative measurable function, such that*

$$\bar{H}(x) = \int_0^x H(x,t)\,dt > 0,\; x \in (0,b).$$

Let a weight $u : (0,b) \to \mathbb{R}$ *be such that the function* $x \mapsto \frac{u(x)}{x}\cdot\frac{H(x,t)}{\bar{H}(x)}$ *is integrable on* (t,b) *for each fixed* $t \in (0,b)$, *and let the function* $\bar{w} : (0,b) \to \mathbb{R}$ *be defined by*

$$\bar{w}(t) := t\int_t^b \frac{H(x,t)}{\bar{H}(x)}u(x)\,\frac{dx}{x}.$$

If Φ *is a convex function on an interval* $I \subseteq \mathbb{R}$ *and* $\varphi : I \to \mathbb{R}$ *is such that* $\varphi(x) \in \partial\Phi(x)$ *for all* $x \in \mathrm{Int}\,I$, *then the inequality*

$$\int_0^b \bar{w}(t)\Phi(h(t))\,\frac{dt}{t} - \int_0^b u(x)\Phi\left(\frac{1}{\bar{H}(x)}\int_0^x H(x,t)h(t)dt\right)\frac{dx}{x} \tag{15}$$
$$\geqslant \int_0^b \frac{u(x)}{\bar{H}(x)}\int_0^x H(x,t)\left|\left|\Phi(h(t)) - \Phi\left(\frac{1}{\bar{H}(x)}\int_0^x H(x,t)h(t)dt\right)\right|\right.$$
$$\left. - \left|\varphi\left(\frac{1}{\bar{H}(x)}\int_0^x H(x,t)h(t)dt\right)\right|\cdot\left|h(t) - \frac{1}{\bar{H}(x)}\int_0^x H(x,t)h(t)dt\right|\right|dt\frac{dx}{x},$$

holds for all measurable functions $h : (0,b) \to \mathbb{R}$ *with values in* I. *If the function* Φ *is concave, the order of integrals on the left-hand side of* (15) *is reversed. If* Φ *is monotone convex on the interval* $I \subseteq \mathbb{R}$ *and* $\varphi : I \to \mathbb{R}$ *is such that* $\varphi(x) \in \partial\Phi(x)$ *for* *all $x \in IntI$, then the following inequality*

$$\int_0^b \bar{w}(t)\Phi(h(t))\frac{dt}{t} - \int_0^b u(x)\Phi\left(\frac{1}{\bar{H}(x)}\int_0^x H(x,t)h(t)dt\right)\frac{dx}{x} \tag{16}$$
$$\geqslant \left| \int_0^b \frac{u(x)}{\bar{H}(x)} \int_0^x sgn\left(h(t) - \frac{1}{\bar{H}(x)}\int_0^x H(x,t)h(t)dt\right) \right.$$
$$\times H(x,t)\left[\Phi(h(t)) - \Phi\left(\frac{1}{\bar{H}(x)}\int_0^x H(x,t)h(t)dt\right)\right.$$
$$\left.\left. - \left|\varphi\left(\frac{1}{\bar{H}(x)}\int_0^x H(x,t)h(t)dt\right)\right| \cdot \left(h(t) - \frac{1}{\bar{H}(x)}\int_0^x H(x,t)h(t)dt\right)\right] dt\frac{dx}{x}\right|,$$

holds for all measurable functions $h:(0,b)\to\mathbb{R}$ with values in I.

Proof. Applying Theorem 4 with $k(x,t) = H(x,t)$, we get inequality (15) and inequality (16). □

REMARK 3. Choose the particular convex function $\Phi(x) = x^{\nu}$, $\nu \geqslant 1$ and weight function $u(x) = x\bar{H}(x)$ in Theorem 10 we obtain, $\bar{w}(t) = t\int_t^b H(x,t)dx = tK_1(t)$. Since the right hand side of inequality (15) and inequality (16) is non negative, we obtain

$$\|y\|_{\nu}(0,b) \leqslant \left(\frac{K_1(0)}{\bar{H}^{1-\nu}(b)}\right)^{\frac{1}{\nu}} \|h\|_{\nu}(0,b).$$

Next we give the mean value theorem's for linear differential operators.

THEOREM 11. *Let $u:(a,b)\to\mathbb{R}$ be a weight function. Let I be a compact interval of $\mathbb{R}$, $\tilde{h}\in C^2(I)$, and $h:(a,b)\to\mathbb{R}$ a measurable function such that $Imh \subseteq I$. Then there exists $\eta \in I$ such that*

$$\int_a^b \bar{v}(t)\tilde{h}(h(t))\,dt - \int_a^b u(x)\tilde{h}\left(\frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)dx \tag{17}$$
$$= \frac{\tilde{h}''(\eta)}{2}\left[\int_a^b \bar{v}(t)h^2(t)\,dt - \int_a^b u(x)\left(\frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)^2 dx\right],$$

where $\bar{v}$ is defined by (11).

Proof. Applying Theorem 5 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$ and $k(x,t) = H(x,t)$, we get (17). □

THEOREM 12. *Let the conditions of Theorem 9 be satisfied and φ_s be defined by (7). Let h be a positive function. Then the function $\tilde{\xi}:\mathbb{R}\to[0,\infty)$ for linear differential operator defined by*

$$\tilde{\xi}(s)=\int_a^b \bar{v}(t)\varphi_s(h(t))\,dt-\int_a^b u(x)\varphi_s\left(\frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)dx, \tag{18}$$

is exponentially convex.

Proof. Applying Theorem 6 with $\Omega_1=\Omega_2=(a,b),\ d\mu_1(x)=dx,\ d\mu_2(t)=dt$ and $k(x,t)=H(x,t)$, we get (18). □

THEOREM 13. *Let $u:(a,b)\to\mathbb{R}$ be a weight function. Let I be a compact interval in $\mathbb{R}$ and $g,\tilde{h}\in C^2(I)$ such that $\tilde{h}''(x)\neq 0$ for every $x\in I$. Let $h:(a,b)\to\mathbb{R}$ be a measurable function such that $Imh\subseteq I$ and*

$$\int_a^b \bar{v}(t)\tilde{h}(h(t))\,dt-\int_a^b u(x)\tilde{h}\left(\frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)dx\neq 0.$$

Then there exists $\eta\in I$ such that it holds

$$\frac{g''(\eta)}{\tilde{h}''(\eta)}=\frac{\int_a^b \bar{v}(t)g(h(t))\,dt-\int_a^b u(x)g\left(\frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)dx}{\int_a^b \bar{v}(t)\tilde{h}(h(t))\,dt-\int_a^b u(x)\tilde{h}\left(\frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)dx}. \tag{19}$$

Proof. Applying Theorem 7 with $\Omega_1=\Omega_2=(a,b),\ d\mu_1(x)=dx,\ d\mu_2(t)=dt$ and $k(x,t)=H(x,t)$, we get (19). □

Under assumptions of the Theorem 9, we define a linear functional by taking the positive difference of the inequality stated in (12) as:

$$\Upsilon_1(\Phi)=\int_a^b \bar{v}(t)\Phi(h(t))\,dt-\int_a^b u(x)\Phi\left(\frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)dx. \tag{20}$$

Also we take a linear functional by taking the positive difference of left-hand side and right-hand side of the inequality (5) given in Theorem 2 for linear differential operators as:

$$\Upsilon_2(\Phi)=\int_a^b p(t)\Phi\left(\frac{h_1(t)}{h_2(t)}\right)dt-\int_a^b u(x)\Phi\left(\frac{y_1(x)}{y_2(x)}\right)dx. \tag{21}$$

THEOREM 14. *Let $\Gamma = \{\Phi_p : p \in J\}$ be a family of functions defined on I, such that the function $p \mapsto [z_0, z_1, z_2; \Phi_p]$ is n-exponentially convex in the Jensen sense on J for every three distinct points $z_0, z_1, z_2 \in I$. Let Υ_i $(i = 1,2)$ be linear functionals defined by (20), (21). Then the function $p \mapsto \Upsilon_i(\Phi_p)$ $(i = 1,2)$ is n-exponentially convex in the Jensen sense on J. If the function $p \mapsto \Upsilon_i(\Phi_p)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. Applying Theorem 8 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$ and $k(x,t) = H(x,t)$ to complete the proof. □

3. Refined Hardy-type inequalities for Widder's derivative

First it is necessary to give some important details about Widder's derivative (see [17]). Let $f, u_0, u_1, \ldots, u_n \in C^{n+1}[a,b]$, $n \geqslant 0$, and the Wronskians

$$W_i(x) := W[u_0(x), u_1(x), \ldots, u_i(x)] = \begin{vmatrix} u_0(x) & \cdots & u_i(x) \\ u_0'(x) & \cdots & u_i'(x) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ u_0^{(i)}(x) & \cdots & u_i^{(i)}(x) \end{vmatrix},$$

$i = 0,1,\ldots,n$. Here $W_0(x) = u_0(x)$. Assume $W_i(x) > 0$ over $[a,b]$. For $i \geqslant 0$, the differential operator of order i (Widder's derivative):

$$L_i f(x) := \frac{W[u_0(x), u_1(x), \ldots, u_{i-1}(x), f(x)]}{W_{i-1}(x)},$$

$i = 1,\ldots,n+1$; $L_0 f(x) = f(x)$ for all $x \in [a,b]$. Consider also

$$g_i(x,t) := \frac{1}{W_i(t)} \begin{vmatrix} u_0(t) & \cdots & u_i(t) \\ u_0'(t) & \cdots & u_i'(t) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ u_0^{(i-1)}(t) & \cdots & u_i^{(i-1)}(t) \\ u_0(x) & \cdots & u_i(x) \end{vmatrix},$$

$i = 1,2,\ldots,n$; $g_0(x,t) := \frac{u_0(x)}{u_0(t)}$ for all $x,t \in [a,b]$.

EXAMPLE 1. [17] Sets of the form $\{u_0, u_1, u_2, \ldots, u_n\}$ are $\{1, x, x^2, \ldots, x^n\}$, $\{1, \sin x, \cos x, -\sin 2x, \cos 2x, \ldots, (-1)^{n-1}\sin nx, (-1)^{n-1}\cos nx\}$, etc. fulfill the above theory.

We also mention the generalized Widder-Talylor's formula, see [17].

THEOREM 15. *Let the functions* $f, u_0, u_1, \ldots, u_n \in C^{n+1}[a,b]$, *and the Wronkians* $W_0(x), W_1(x), \ldots, W_n(x) > 0$ *on* $[a,b]$, $x \in [a,b]$. *Then for* $t \in [a,b]$ *we have*

$$f(x) = f(t)\frac{u_0(x)}{u_0(t)} + L_1 f(t) g_1(x,t) + \ldots + L_n f(t) g_n(x,t) + R_n(x),$$

where

$$R_n(x) := \int_s^x g_n(x,s) L_{n+1} f(s) ds.$$

For example (see [17]) one could take $u_0(x) = c > 0$. If $u_i(x) = x^i$, $i = 0, 1, \ldots, n$, defined on [a,b], then

$$L_i f(t) = f^{(i)}(t) \quad \text{and} \quad g_i(x,t) = \frac{(x-t)^i}{i!}, \quad t \in [a,b].$$

COROLLARY 1. *By additionally assuming for fixed* a *that* $L_i f(a) = 0$, $i = 0, 1, \ldots, n$, *we get that*

$$f(x) := \int_a^x g_n(x,t) L_{n+1} f(t) dt \quad for\ all \quad x \in [a,b].$$

In the next result we give refinement of Theorem 3.4 of [9].

THEOREM 16. *Let* u *be a weight function on* (a,b) *and* $g_n(x,t)$ *be a non-negative measurable kernel. Suppose that* $\widetilde{g}_n(x) > 0$ *for all* $x \in (a,b)$, *that the function* $x \mapsto u(x)\frac{g_n(x,t)}{\widetilde{g}_n(x)}$ *is integrable on* (a,b) *for each fixed* $t \in (a,b)$, *and that* $\tilde{v}$ *is defined on* (a,b) *by*

$$\tilde{v}(t) := \int_t^b u(x) \frac{g_n(x,t)}{\widetilde{g}_n(x)} dx < \infty. \tag{22}$$

If Φ *is a convex function on an interval* $I \subseteq \mathbb{R}$ *and* $\varphi : I \to \mathbb{R}$ *is any function, such that* $\varphi(x) \in \partial\Phi(x)$ *for all* $x \in \operatorname{Int} I$, *then the inequality*

$$\begin{aligned}
&\int_a^b \tilde{v}(t)\Phi(L_{n+1}f(t))\,dt - \int_a^b u(x)\Phi\left(\frac{1}{\widetilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)dx \\
&\geqslant \int_a^b \frac{u(x)}{\widetilde{g}_n(x)} \int_a^x g_n(x,t) \left| \left| \Phi(L_{n+1}f(t)) - \Phi\left(\frac{1}{\widetilde{g}_n(x)}\int_a^x g_n(x,t)\, L_{n+1}f(t)dt\right)\right| \right. \\
&\quad - \left|\varphi\left(\frac{1}{\widetilde{g}_n(x)}\int_a^x g_n(x,t)\, L_{n+1}f(t)dt\right)\right| \\
&\quad \times \left. \left| L_{n+1}f(t) - \frac{1}{\widetilde{g}_n(x)}\int_a^x g_n(x,t)\, L_{n+1}f(t)dt \right| \right| dt dx,
\end{aligned} \tag{23}$$

holds for all measurable functions $L_{n+1}f:(a,b)\to\mathbb{R}$*, such that* $L_{n+1}f(t)\in I$ *for all* $t\in(a,b)$*. If* Φ *is a monotone convex function on an interval* $I\subseteq\mathbb{R}$*, then the inequality*

$$\begin{aligned}
&\int_a^b \tilde{v}(t)\Phi(L_{n+1}f(t))\,dt-\int_a^b u(x)\Phi\left(\frac{1}{\widetilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)\,dt\right)dx\\
&\geqslant\left|\int_a^b\frac{u(x)}{\widetilde{g}_n(x)}\int_a^x sgn\left(L_{n+1}f(t)-\frac{1}{\widetilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)\right.\\
&\quad\times g_n(x,t)\left[\Phi(L_{n+1}f(t))-\Phi\left(\frac{1}{\widetilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)\,dt\right)\right.\\
&\quad-\left|\varphi\left(\frac{1}{\widetilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)\,dt\right)\right|\\
&\quad\times\left.\left.\left(L_{n+1}f(t)-\frac{1}{\widetilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)\right]dtdx\right|,
\end{aligned}\tag{24}$$

holds for all measurable functions $L_{n+1}f:(a,b)\to\mathbb{R}$*, such that* $L_{n+1}f(t)\in I$*, for all fixed* $t\in(a,b)$ *and where* $\widetilde{g}_n$ *is defined as*

$$\widetilde{g}_n(x):=\int_a^x g_n(x,t)dt.$$

Proof. Applying Theorem 3 with $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(x)=dx$, $d\mu_2(t)=dt$ and $k(x,t)=g_n(x,t)$, we get inequality (23) and inequality (24). □

REMARK 4. Choose the particular convex function $\Phi(x)=x^{\nu}$, $\nu\geqslant1$ and weight function $u(x)=\widetilde{g}_n(x)$ in Theorem 16 we obtain, $\tilde{v}(t)=\int_t^b g_n(x,t)dx=:K_2(t)$. Since right hand side of inequality (23) and inequality (24) is non negative, therefore we get that

$$\int_a^b\widetilde{g}_n^{1-\nu}(x)\left(\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)^{\nu}dx\leqslant\int_a^b K_2(t)L_{n+1}f^{\nu}(t)dt.\tag{25}$$

Inequality (25) gives

$$\widetilde{g}_n^{1-\nu}(b)\int_a^b f^{\nu}(x)dx\leqslant K_2(a)\int_a^b L_{n+1}f^{\nu}(t)dt.$$

This implies that

$$\|f\|_{\nu}(a,b)\leqslant\left(\frac{K_2(a)}{\widetilde{g}_n^{1-\nu}(b)}\right)^{\frac{1}{\nu}}\|L_{n+1}f(t)\|_{\nu}(a,b).$$

In the following theorem, a refinement of a Hardy-type inequality is given for Widder's derivative.

THEOREM 17. *Let* $0 < b \leqslant \infty$ *and* $g_n : (0,b) \times (0,b) \to \mathbb{R}$ *be a non-negative measurable function, such that*

$$\overline{g}_n(x) := \int_0^x g_n(x,t)\, dt > 0,\ x \in (0,b).$$

Let a weight $u : (0,b) \to \mathbb{R}$ *be such that the function* $x \mapsto \frac{u(x)}{x} \cdot \frac{g_n(x,t)}{\overline{g}_n(x)}$ *is integrable on* (t,b) *for each fixed* $t \in (0,b)$*, and let the function* $\tilde{w} : (0,b) \to \mathbb{R}$ *be defined by*

$$\tilde{w}(t) := t \int_t^b \frac{g_n(x,t)}{\overline{g}_n(x)} u(x)\, \frac{dx}{x}.$$

If Φ *is a convex function on an interval* $I \subseteq \mathbb{R}$ *and* $\varphi : I \to \mathbb{R}$ *is such that* $\varphi(x) \in \partial\Phi(x)$ *for all* $x \in \operatorname{Int} I$*, then the inequality*

$$\begin{aligned}
&\int_0^b \tilde{w}(t)\Phi(L_{n+1}f(t))\, \frac{dt}{t} - \int_0^b u(x)\Phi\left(\frac{1}{\overline{g}_n(x)}\int_0^x g_n(x,t)L_{n+1}f(t)\, dt\right)\frac{dx}{x} \\
&\geqslant \int_0^b \frac{u(x)}{\overline{g}_n(x)} \int_0^x g_n(x,t)\left|\left|\Phi(L_{n+1}f(t)) - \Phi\left(\frac{1}{\overline{g}_n(x)}\int_0^x g_n(x,t)L_{n+1}f(t)\, dt\right)\right|\right. \\
&\quad - \left|\varphi\left(\frac{1}{\overline{g}_n(x)}\int_0^x g_n(x,t)L_{n+1}f(t)\, dt\right)\right| \\
&\quad \times \left.\left|L_{n+1}f(t) - \frac{1}{\overline{g}_n(x)}\int_0^x g_n(x,t)L_{n+1}f(t)\, dt\right|\right| dt\frac{dx}{x},
\end{aligned} \tag{26}$$

holds for all measurable functions $L_{n+1}f : (0,b) \to \mathbb{R}$ *with values in* I*. If the function* Φ *is concave, the order of integrals on the left-hand side of* (26) *is reversed. If* Φ *is monotone convex on the interval* $I \subseteq \mathbb{R}$ *then the following inequality*

$$\int_0^b \tilde{w}(t)\Phi\left(L_{n+1}f(t)\right)\frac{dt}{t} - \int_0^b u(x)\Phi\left(\frac{1}{\overline{g}_n(x)}\int_0^x g_n(x,t)L_{n+1}f(t)dt\right)\frac{dx}{x} \tag{27}$$

$$\begin{aligned}
&\geqslant \left|\int_0^b \frac{u(x)}{\overline{g}_n(x)}\int_0^x sgn\left(L_{n+1}f(t) - \frac{1}{\overline{g}_n(x)}\int_0^x g_n(x,t)L_{n+1}f(t)\, dt\right)\right. \\
&\quad \times g_n(x,t)\left[\Phi(L_{n+1}f(t) - \Phi\left(\frac{1}{\overline{g}_n(x)}\int_0^x g_n(x,t)L_{n+1}f(t)\, dt\right)\right.
\end{aligned}$$

$$- \left| \varphi \left(\frac{1}{\bar{g}_n(x)} \int_0^x g_n(x,t) L_{n+1} f(t)\,dt \right) \right|$$
$$\times \left(L_{n+1} f(t) - \frac{1}{\bar{g}_n(x)} \int_0^x g_n(x,t) L_{n+1} f(t)\,dt \right) \Bigg] dt \frac{dx}{x} \Bigg|,$$

holds for all measurable functions $L_{n+1}f : (0,b) \to \mathbb{R}$ with values in I.

Proof. Applying Theorem 4 with $k(x,t) = g_n(x,t)$, we get inequality (26) and inequality (27). □

REMARK 5. Choose the particular convex function $\Phi(x) = x^{\nu}$, $\nu \geqslant 1$ and weight function $u(x) = x\bar{g}_n(x)$ in Theorem 17 we obtain, $\tilde{w}(t) = t \int_t^b g_n(x,t)dx =: tK_2(t)$. Since right hand side of inequality (26) and (27) is non negative, so we obtain

$$\|f\|_{\nu}(0,b) \leqslant \left(\frac{K_2(0)}{\bar{g}_n(b)^{1-\nu}} \right)^{\frac{1}{\nu}} \|L_{n+1} f(t)\|_{\nu}(0,b).$$

The upcoming results are mean value theorems for Widder's derivative.

THEOREM 18. *Let $u : (a,b) \to \mathbb{R}$ be a weight function. Let I be a compact interval of $\mathbb{R}$, $\tilde{h} \in C^2(I)$, and $L_{n+1}f : (a,b) \to \mathbb{R}$ a measurable function such that $ImL_{n+1}f \subseteq I$. Then there exists $\eta \in I$ such that*

$$\int_a^b \tilde{v}(t)\tilde{h}(L_{n+1}f(t))\,dt - \int_a^b u(x)\tilde{h}\left(\frac{1}{\tilde{g}_n(x)} \int_a^x g_n(x,t)L_{n+1}f(t)\,dt \right) dx \tag{28}$$
$$= \frac{\tilde{h}''(\eta)}{2} \left[\int_a^b \tilde{v}(t)(L_{n+1}f(t))^2\,dt - \int_a^b u(x) \left(\frac{1}{\tilde{g}_n(x)} \int_a^x g_n(x,t)L_{n+1}f(t)\,dt \right)^2 dx \right],$$

where $\tilde{v}(t)$ is defined by (22).

Proof. Applying Theorem 5 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$ and $k(x,t) = g_n(x,t)$, we get equality (28). □

THEOREM 19. *Let the conditions of Theorem 16 be satisfied and φ_s be defined by (7). Let f be a positive function. Then the function $\bar{\xi} : \mathbb{R} \to [0,\infty)$ defined by*

$$\bar{\xi}(s) = \int_a^b \tilde{v}(t)\varphi_s(L_{n+1}f(t))\,dt - \int_a^b u(x)\varphi_s\left(\frac{1}{\tilde{g}_n(x)} \int_a^x g_n(x,t)L_{n+1}f(t)\,dt \right) dx, \tag{29}$$

is exponentially convex.

Proof. Applying Theorem 6 with $\Omega_1 = \Omega_2 = (a,b),\ d\mu_1(x) = dx,\ d\mu_2(t) = dt$ and $k(x,t) = g_n(x,t)$, we get equality (29). □

THEOREM 20. *Assume that all conditions of Theorem 18 are satisfied. Let I be a compact interval in $\mathbb{R}$ and $g,\tilde{h} \in C^2(I)$ such that $\tilde{h}''(x) \neq 0$ for every $x \in I$. Let $L_{n+1}f : (a,b) \to \mathbb{R}$ be a measurable function such that $ImL_{n+1}f \subseteq I$ and*

$$\int_a^b \tilde{v}(t)\tilde{h}(L_{n+1}f(t))\,dt - \int_a^b u(x)\tilde{h}\left(\frac{1}{\widetilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)\,dt\right)dx \neq 0.$$

Then there exists $\eta \in I$ such that it holds

$$\frac{g''(\eta)}{\tilde{h}''(\eta)} = \frac{\int_a^b \tilde{v}(t)g(L_{n+1}f(t))\,dt - \int_a^b u(x)g\left(\frac{1}{\widetilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)\,dt\right)dx}{\int_a^b \tilde{v}(t)\tilde{h}(L_{n+1}f(t))\,dt - \int_a^b u(x)\tilde{h}\left(\frac{1}{\widetilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)\,dt\right)dx}. \tag{30}$$

Proof. Applying Theorem 7 with $\Omega_1 = \Omega_2 = (a,b),\ d\mu_1(x) = dx,\ d\mu_2(t) = dt$ and $k(x,t) = g_n(x,t)$, we get equality (30). □

Under assumptions of the Theorem 16, we define a linear functional by taking the positive difference of the inequality stated in (23) as:

$$\Lambda_1(\Phi) = \int_a^b \tilde{v}(t)\Phi(L_{n+1}f(t))\,dt - \int_a^b u(x)\Phi\left(\frac{1}{\widetilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)dx. \tag{31}$$

We also define a linear functional by taking the positive difference of left-hand side and right-hand side of the inequality (5) given in Theorem 2 for Widder's derivative as:

$$\Lambda_2(\Phi) = \int_a^b p(t)\Phi\left(\frac{L_{n+1}f_1(t)}{L_{n+1}f_2(t)}\right)dt - \int_a^b u(x)\Phi\left(\frac{f_1(x)}{f_2(x)}\right)dx. \tag{32}$$

THEOREM 21. *Let $\Gamma = \{\Phi_p : p \in J\}$ be a family of functions defined on I, such that the function $p \mapsto [z_0,z_1,z_2;\Phi_p]$ is n-exponentially convex in the Jensen sense on J for every three distinct points $z_0, z_1, z_2 \in I$. Let Λ_i $(i = 1,2)$ be linear functionals defined by (31), (32). Then the function $p \mapsto \Lambda_i(\Phi_p)$ $(i = 1,2)$ is n-exponentially convex in the Jensen sense on J. If the function $p \mapsto \Lambda_i(\Phi_p)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. Applying Theorem 8 with $\Omega_1 = \Omega_2 = (a,b),\ d\mu_1(x) = dx,\ d\mu_2(t) = dt$ and $k(x,t) = g_n(x,t)$ to complete the proof. □

4. Refined Hardy-type inequalities for generalized fractional integral operator

In the following we give the definition of generalized fractional integral operator involving hypergeometric function in its kernel (see, [7]).

DEFINITION 2. Let $\alpha > 0$, $\mu > -1$, $\beta, \eta \in \mathbb{R}$. Then the generalized fractional integral $I_{a,x}^{\alpha,\beta,\eta,\mu}$ of order α, for a real-valued continuous function f is defined by:

$$I_{a,x}^{\alpha,\beta,\eta,\mu} f(x) \tag{33}$$

$$:= \frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)} \int_a^x t^{\mu}(x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta+\mu, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)dt, \ \ x \in [a,b],$$

where, the function ${}_2F_1(.,.,;.)$ appearing in kernel for operator (33) is the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;t) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} t^n,$$

and $(a)_n$ is the Pochhammer symbol: $(a)_n = a(a+1)\dots(a+n-1), (a)_0 = 1$.

The operator (33) includes Saigo, Riemann-Liouville and Erdélyi-Kober fractional integral operators i.e.

$$I_{a,x}^{\alpha,\beta,\eta} f(x) = I_{a,x}^{\alpha,\beta,\eta,0} f(x)$$
$$= \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)dt, \ \ x \in [a,b],$$

$$R^{\alpha} f(x) = I_{a,x}^{\alpha,-\alpha,\eta} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \ \ x \in [a,b],$$

and

$$I^{\alpha,\eta} f(x) = I_{a,x}^{\alpha,0,\eta} f(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} t^{\eta} f(t)dt, \ \ x \in [a,b].$$

First we give our general result for generalized fractional integral of order α, then as special cases we establish the inequalities for Saigo, Riemann-Liouville and Erdélyi-Kober fractional integral operators.

THEOREM 22. *Let $\alpha > 0$, $\mu > -1$, $\beta, \eta \in \mathbb{R}$, $I_{a,x}^{\alpha,\beta,\eta,\mu}$ denotes the generalized fractional integral of order α and u be a weight function defined on (a,b). Moreover for each fixed $t \in (a,b)$ define $\hat{v}$ by*

$$\hat{v}(t) := \frac{1}{\Gamma(\alpha)} \int_t^b u(x) \frac{x^{-\alpha-\beta-2\mu} {}_2F_1\left(\alpha+\beta+\mu, -\eta, \alpha; 1-\frac{t}{x}\right) t^{\mu}(x-t)^{\alpha-1}}{\hat{K}(x)} dx < \infty. \tag{34}$$

If Φ *is a convex function on the interval* $I \subseteq \mathbb{R}$, *then the inequality*

$$\int_a^b u(x)\Phi\left(\frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f(x)}{\hat{K}(x)}\right) dx \leqslant \int_a^b \hat{v}(t)\Phi(f(t))\, dt, \tag{35}$$

where

$$\hat{K}(x) := \int_a^x \frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)} {}_2F_1\left(\alpha+\beta+\mu, -\eta, \alpha; 1-\frac{t}{x}\right) t^{\mu}(x-t)^{\alpha-1} dt. \tag{36}$$

Proof. Applying Theorem 1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$,

$$k(x,t) = \begin{cases} \frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)} {}_2F_1(\alpha+\beta+\mu, -\eta, \alpha; 1-\frac{t}{x}) t^{\mu}(x-t)^{\alpha-1}, & a \leqslant t \leqslant x; \\ 0, & x < t \leqslant b, \end{cases}$$

and $g(x) = I_{a,x}^{\alpha,\beta,\eta,\mu} f(x)$, so inequality (35) follows. □

Now we obtain the fractional inequality for generalized fractional integral.

THEOREM 23. *Let* $\alpha > 0$, $\mu > -1$, $\beta, \eta \in \mathbb{R}$ *and* $I_{a,x}^{\alpha,\beta,\eta,\mu}$ *denotes the generalized fractional integral of order* α. *Define* $\hat{p}$ *on* (a,b) *by*

$$\hat{p}(t) := \frac{f_2(t)}{\Gamma(\alpha)} \int_t^b u(x) \frac{x^{-\alpha-\beta-2\mu} {}_2F_1(\alpha+\beta+\mu, -\eta, \alpha; 1-\frac{t}{x}) t^{\mu}(x-t)^{\alpha-1}}{I_{a,x}^{\alpha,\beta,\eta,\mu} f_2(x)} dx < \infty.$$

If $\Phi : I \to \mathbb{R}$ *is a convex function and* $\frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f_1(x)}{I_{a,x}^{\alpha,\beta,\eta,\mu} f_2(x)}, \frac{f_1(t)}{f_2(t)} \in I$, *then the inequality*

$$\int_a^b u(x)\Phi\left(\frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f_1(x)}{I_{a,x}^{\alpha,\beta,\eta,\mu} f_2(x)}\right) dx \leqslant \int_a^b \hat{p}(t)\Phi\left(\frac{f_1(t)}{f_2(t)}\right) dt, \tag{37}$$

Proof. Applying Theorem 2 with the same technique used in Theorem 22. □

Refinement of Theorem 22 is given in next theorem.

THEOREM 24. *Let the assumption of Theorem* 22 *be satisfied. Moreover, if* Φ *is a convex function on an interval* $I \subseteq \mathbb{R}$ *and* $\varphi : I \to \mathbb{R}$ *is any function, such that* $\varphi(x) \in \partial\Phi(x)$ *for all* $x \in \operatorname{Int} I$, *then the inequality*

$$\int_a^b \hat{v}(t)\Phi(f(t))dt - \int_a^b u(x)\Phi\left(\frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f(x)}{\hat{K}(x)}\right) dx \tag{38}$$

$$\geqslant \frac{1}{\Gamma(\alpha)} \int_a^b \frac{u(x)}{\hat{K}(x)} \int_a^x x^{-\alpha-\beta-2\mu} {}_2F_1\left(\alpha+\beta+\mu, -\eta, \alpha; 1-\frac{t}{x}\right) t^{\mu}(x-t)^{\alpha-1}$$

$$\times \left| \left| \Phi(f(t)) - \Phi\left(\frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f(x)}{\hat{K}(x)}\right) \right| - \left| \varphi\left(\frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f(x)}{\hat{K}(x)}\right) \right| \cdot \left| f(t) - \frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f(x)}{\hat{K}(x)} \right| \right| dt dx,$$

holds for all measurable functions $f : (a,b) \to \mathbb{R}$, *such that* $f(t) \in I$ *for all* $t \in (a,b)$.

If Φ *is a monotone convex function on an interval* $I \subseteq \mathbb{R}$, *then the following inequality holds:*

$$\int_a^b \hat{v}(t)\Phi(f(t))\,dt - \int_a^b u(x)\Phi\left(\frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f(x)}{\hat{K}(x)}\right) dx \tag{39}$$
$$\geqslant \frac{1}{\Gamma(\alpha)} \left| \int_a^b \frac{u(x)}{\hat{K}(x)} \int_a^x sgn\left(f(t) - \frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f(x)}{\hat{K}(x)}\right) \right.$$
$$\times \int_a^x x^{-\alpha-\beta-2\mu} {}_2F_1\left(\alpha+\beta+\mu, -\eta, \alpha; 1-\frac{t}{x}\right) t^{\mu}(x-t)^{\alpha-1}$$
$$\times \left[\Phi(f(t)) - \Phi\left(\frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f(x)}{\hat{K}(x)}\right) - \left| \varphi\left(\frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f(x)}{\hat{K}(x)}\right) \right| \right.$$
$$\left. \left. \times \left(f(t) - \frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f(x)}{\hat{K}(x)}\right) \right] dt\, dx \right|,$$

where $\hat{v}$ *and* $\hat{K}$ *are defined by* (34) *and* (36).

Proof. Applying Theorem 3 with the same technique used in Theorem 22. □

Similar results can be given for one dimensional setting but we omit the details. Here we give the mean value theorems for generalized fractional integral.

THEOREM 25. *Let the assumption of Theorem 22 be satisfied. Moreover, suppose* I *be a compact interval of* $\mathbb{R}$, $\tilde{h} \in C^2(I)$. *Then there exists* $\eta \in I$ *such that*

$$\int_a^b \hat{v}(t)\tilde{h}(f(t))\,dt - \int_a^b u(x)\tilde{h}\left(\frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f(x)}{\hat{K}(x)}\right) dx \tag{40}$$
$$= \frac{\tilde{h}''(\eta)}{2}\left[\int_a^b \hat{v}(t) f^2(t)\,dt - \int_a^b u(x)\left(\frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f(x)}{\hat{K}(x)}\right)^2 dx\right],$$

$\hat{v}$ *and* $\hat{K}$ *are defined by* (34) *and* (36) *respectively.*

Proof. Applying Theorem 5 with the same technique used in Theorem 22. □

THEOREM 26. *Let the assumption of Theorem 22 be satisfied and φ_s be defined by (7). Let f be a positive function. Then the function $\hat{\xi}:\mathbb{R}\to[0,\infty)$ defined by*

$$\hat{\xi}(s)=\int_a^b \hat{v}(t)\varphi_s(f(t))\,dt-\int_a^b u(x)\varphi_s\left(\frac{I_{a,x}^{\alpha,\beta,\eta,\mu}f(x)}{\hat{K}(x)}\right)dx, \tag{41}$$

is exponentially convex.

Proof. Applying Theorem 6 with the same technique used in Theorem 22. □

THEOREM 27. *Let the assumption of Theorem 25 be satisfied. Moreover, $g,\tilde{h}\in C^2(I)$ such that $\tilde{h}''(x)\neq 0$ for every $x\in I$ and*

$$\int_a^b \hat{v}(t)\tilde{h}(f(t))\,dt-\int_a^b u(x)\tilde{h}\left(\frac{I_{a,x}^{\alpha,\beta,\eta,\mu}f(x)}{\hat{K}(x)}\right)dx\neq 0.$$

Then there exists $\eta\in I$ such that it holds

$$\frac{g''(\eta)}{\tilde{h}''(\eta)}=\frac{\int_a^b \hat{v}(t)g(f(t))\,dt-\int_a^b u(x)g\left(\frac{I_{a,x}^{\alpha,\beta,\eta,\mu}f(x)}{\hat{K}(x)}\right)dx}{\int_a^b \hat{v}(t)\tilde{h}(f(t))\,dt-\int_a^b u(x)\tilde{h}\left(\frac{I_{a,x}^{\alpha,\beta,\eta,\mu}f(x)}{\hat{K}(x)}\right)dx}. \tag{42}$$

Proof. Applying Theorem 7 with the same technique used in Theorem 22. □

COROLLARY 2. *If we take $\mu=0$ in inequalities (35), (37), (38), (39), (40), (41), and (42) we get the inequalities for Saigo fractional integral.*

COROLLARY 3. *If along $\mu=0$ we take $\beta=-\alpha$ in inequalities (35), (37), (38), (39), (40), (41), and (42) we get the inequalities for Riemann-Liouvill's fractional integral.*

COROLLARY 4. *If we take $\beta=0$ and $\mu=0$ in inequalities (35), (37), (38), (39), (40), (41), and (42) we get the inequalities for Erdélyi-Kober fractional integral operator.*

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