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ON A NEW DISCRETE HILBERT–TYPE INEQUALITY AND ITS APPLICATIONS

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Abstract. In this paper, a theorem related to the Hilbert-type inequality is corrected. By introducing parameters, and using Euler-Maclaurin summation formula, we give a discrete form of the Hilbert-type inequality involving a non-homogeneous kernel. Furthermore, we prove that our result is a concise generalization of the corrected theorem and some known results. As applications, some particular new results are presented.

1. Introduction

Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_n,\ b_n\geqslant 0$, $0<\sum_{n=1}^{\infty}a_n^p<\infty$, $0<\sum_{n=1}^{\infty}b_n^q<\infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{m+n}<\frac{\pi}{\sin\frac{\pi}{p}}\Big[\sum_{n=1}^{\infty}a_n^p\Big]^{\frac{1}{p}}\Big[\sum_{n=1}^{\infty}b_n^q\Big]^{\frac{1}{q}};\tag{1.1}$$

The more accurate form of (1.1) is written as follows:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_mb_n}{m+n+1}<\frac{\pi}{\sin\frac{\pi}{p}}\Big[\sum_{n=0}^{\infty}a_n^p\Big]^{\frac{1}{p}}\Big[\sum_{n=0}^{\infty}b_n^q\Big]^{\frac{1}{q}}.\tag{1.2}$$

The constant factor $\frac{\pi}{\sin\frac{\pi}{p}}$ in (1.1) and (1.2) is the best possible. Inequality (1.1) and (1.2) are well known as Hilbert's inequality (see [2]). Both of them are very important in analysis and its applications (see [9]).

In recent years, by introducing parameters and β function, the researchers established quite a lot of extensions of (1.1) and (1.2) (see [3–6, 10–11]). For example, in paper [11], Yang et. al. proved that if $0<\lambda\leqslant\min\{p,q\}$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{(m+n)^{\lambda}}<B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\Big[\sum_{n=1}^{\infty}n^{p-1-\lambda}a_n^p\Big]^{\frac{1}{p}}\Big[\sum_{n=1}^{\infty}n^{q-1-\lambda}b_n^q\Big]^{\frac{1}{q}},\tag{1.3}$$

where $B(u,v)$ is the β function, that is

$$B(u,v):=\int_0^{\infty}\frac{x^{u-1}}{(1+x)^{u+v}}dx=B(v,u)\quad(u,v>0).$$

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Another extension of inequality (1.1) was established by Krnić and Pečarić as follows (see [4]): if $2-\min\{p,q\}<\lambda\leqslant 2+\min\{p,q\}$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}}<B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)[\sum_{n=1}^{\infty}n^{1-\lambda}a_n^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty}n^{1-\lambda}b_n^q]^{\frac{1}{q}}. \quad (1.4)$$

At the same time, some new extensions of inequality (1.2) were also established by the researchers. For instance, recently, Pečarić et. al. gave the following inequality (see [5,10]): if $\mu\geqslant 0$, $2-\min\{p,q\}<\lambda<2$, then the following inequality holds:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n+2\mu)^{\lambda}}$$

$$<B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)[\sum_{n=1}^{\infty}(n+\mu)^{1-\lambda}a_n^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty}(n+\mu)^{1-\lambda}b_n^q]^{\frac{1}{q}}. \quad (1.5)$$

In addition, by introducing two pairs of conjugate parameters, Yang [12] established another extension of inequality (1.2) as follows:

THEOREM A. *Let* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $r>1$, $\frac{1}{r}+\frac{1}{s}=1$, $t\in[0,1]$, *such that*

$$(2-\min\{s,r\})t<\lambda\leqslant(2-\min\{s,r\})t+\min\{s,r\},$$

and

$$0<\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{p(1-t+\frac{2t-\lambda}{r})-1}a_n^p<\infty,\quad 0<\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{q(1-t+\frac{2t-\lambda}{s})-1}b_n^q<\infty,$$

then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}}$$

$$<B\left(\frac{(r-2)t+\lambda}{r},\frac{(s-2)t+\lambda}{s}\right)\left[\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{p(1-t+\frac{2t-\lambda}{r})-1}a_n^p\right]^{\frac{1}{p}}$$

$$\times\left[\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{q(1-t+\frac{2t-\lambda}{s})-1}b_n^q\right]^{\frac{1}{q}}.$$

Inspired by Theorem A, Chen et. al. [1] gave a further generalization of Theorem A:

THEOREM B. *Let* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $r>1$, $\frac{1}{r}+\frac{1}{s}=1$, $t\in[0,1]$, $a,b>0$, $0<c<2$, $2\min\{a,b\}\geqslant c$, *such that*

$$(2-s)t<\lambda\leqslant\frac{s\left(3c+b-\sqrt{b^2+6bc-3c^2}\right)}{2b}-st+2t,$$

$$(2-r)t<\lambda \leqslant \frac{r\left(3c+a-\sqrt{a^2+6ac-3c^2}\right)}{2a}-rt+2t,$$

and

$$0<\sum_{n=0}^{\infty}\left(an+\frac{c}{2}\right)^{p(1-t+\frac{2t-\lambda}{r})-1}a_n^p<\infty, \quad 0<\sum_{n=0}^{\infty}\left(bn+\frac{c}{2}\right)^{q(1-t+\frac{2t-\lambda}{s})-1}b_n^q<\infty,$$

then

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(am+bn+c)^{\lambda}} \\
&< \frac{1}{b^{\frac{1}{p}}a^{\frac{1}{q}}}B\left(\frac{(r-2)t+\lambda}{r},\frac{(s-2)t+\lambda}{s}\right)\left[\sum_{n=0}^{\infty}\left(an+\frac{c}{2}\right)^{p(1-t+\frac{2t-\lambda}{r})-1}a_n^p\right]^{\frac{1}{p}} \\
&\quad \times\left[\sum_{n=0}^{\infty}\left(bn+\frac{c}{2}\right)^{q(1-t+\frac{2t-\lambda}{s})-1}b_n^q\right]^{\frac{1}{q}}. \qquad (1.6)
\end{aligned}$$

REMARK 1.1. The condition $0<c<2$ in Theorem B is unnecessary. In fact, if inequality (1.6) holds under the condition $0<c<2$, then for $c\geqslant 2$, there must exist a constant k $(0<k<1)$ such that $0<kc<2$. Obviously, ka, kb, kc still satisfy the conditions in Theorem B. So for $c\geqslant 2$, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(am+bn+c)^{\lambda}} \\
&= k^{\lambda}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(kam+kbn+kc)^{\lambda}} \\
&< \frac{k^{\lambda}}{(kb)^{\frac{1}{p}}(ka)^{\frac{1}{q}}}B\left(\frac{(r-2)t+\lambda}{r},\frac{(s-2)t+\lambda}{s}\right)\left[\sum_{n=0}^{\infty}\left(kan+\frac{kc}{2}\right)^{p(1-t+\frac{2t-\lambda}{r})-1}a_n^p\right]^{\frac{1}{p}} \\
&\quad \times\left[\sum_{n=0}^{\infty}\left(kbn+\frac{kc}{2}\right)^{q(1-t+\frac{2t-\lambda}{s})-1}b_n^q\right]^{\frac{1}{q}}. \qquad (1.7)
\end{aligned}$$

Simplifying the right side of inequality (1.7), we can get (1.6). So the condition $0<c<2$ is redundant.

REMARK 1.2. There is another mistake in Theorem B because of the mistakes in the proofs of Lemma 2.3 and 2.5 (see [1], p. 392). In fact, the proofs of Lemma 2.3 and 2.5 rely on Lemma 2.1, which requires the function $f(x)$ defined in Lemma 2.3 to Satisfy $(-1)^n f^{(n)}(x)>0$ $(n=0,1,2,3,4)$. Thus, $1-t+\frac{2t-\lambda}{r}\geqslant 0$, and $1-t+\frac{2t-\lambda}{s}\geqslant 0$, that is $\lambda\leqslant r-rt+2t$, and $\lambda\leqslant s-st+2t$. Hence, Theorem B should be corrected as follows:

THEOREM B*. *Let* $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ r>1,\ \frac{1}{r}+\frac{1}{s}=1,\ t\in[0,1],\ a,b>0,$ $2\min\{a,b\}\geqslant c$, *such that*

$$(2-s)t<\lambda\leqslant s\min\left\{1,\frac{3c+b-\sqrt{b^2+6bc-3c^2}}{2b}\right\}-st+2t, \tag{1.8}$$

$$(2-r)t<\lambda\leqslant r\min\left\{1,\frac{3c+a-\sqrt{a^2+6ac-3c^2}}{2a}\right\}-rt+2t, \tag{1.9}$$

and

$$0<\sum_{n=0}^{\infty}\left(an+\frac{c}{2}\right)^{p(1-t+\frac{2t-\lambda}{r})-1}a_n^p<\infty,\quad 0<\sum_{n=0}^{\infty}\left(bn+\frac{c}{2}\right)^{q(1-t+\frac{2t-\lambda}{s})-1}b_n^q<\infty,$$

then

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_mb_n}{(am+bn+c)^{\lambda}}\\ &<\frac{1}{b^{\frac{1}{p}}a^{\frac{1}{q}}}B\left(\frac{(r-2)t+\lambda}{r},\frac{(s-2)t+\lambda}{s}\right)\left[\sum_{n=0}^{\infty}\left(an+\frac{c}{2}\right)^{p(1-t+\frac{2t-\lambda}{r})-1}a_n^p\right]^{\frac{1}{p}}\\ &\quad\times\left[\sum_{n=0}^{\infty}\left(bn+\frac{c}{2}\right)^{q(1-t+\frac{2t-\lambda}{s})-1}b_n^q\right]^{\frac{1}{q}}.\end{aligned} \tag{1.10}$$

It is easy to see there are too many parameters on the right side of Theorem B*, and this makes Theorem B* very complicated. So the further extension of Theorem B* is difficult. In this paper, by introducing only one pair of conjugate parameters, we will establish a new Hilbert-type inequality, which is not only a generalization of Theorem B*, but also a generalization of some known results. As applications, we also consider some particular new results.

2. Some Lemmas

LEMMA 2.1. [8] *Let* $f^{(4)}\in C[0,\infty],\ \int_0^\infty f(x)dx<\infty,\ (-1)^nf^{(n)}(x)>0,\ f^{(n)}(\infty)=0\ (n=0,1,2,3,4)$, *then*

$$\sum_{m=0}^{\infty}f(m)<\int_0^{\infty}f(x)dx+\frac{1}{2}f(0)-\frac{1}{12}f'(0).$$

LEMMA 2.2. *Let* $k_1,\ k_2,\ \lambda_1,\ \lambda_2,\ \lambda>0,\ \lambda_1(1-\beta_2)+\lambda_2(1-\beta_1)=\lambda_1\lambda_2\lambda,$

$(1-\lambda_2^2)k_2^2+12k_2-12 \geqslant 0$, $\max\left\{0, \frac{k_2-6+\sqrt{k_2^2+12k_2-12}}{2k_2}\right\} \leqslant \beta_2 < 1$, *then*

$$R(m) := \frac{1}{k_2\lambda_2}(k_1m+1)^{\beta_1-1}\int_0^{\frac{1}{(k_1m+1)^{\lambda_1}}} \frac{u^{\frac{1-\beta_2}{\lambda_2}-1}}{(1+u)^{\lambda}}du - \frac{1}{2[(k_1m+1)^{\lambda_1}+1]^{\lambda}} - \frac{k_2\beta_2}{12[(k_1m+1)^{\lambda_1}+1]^{\lambda}} - \frac{k_2\lambda_2\lambda}{12[(k_1m+1)^{\lambda_1}+1]^{\lambda+1}} > 0.$$

Proof. Since $\lambda_1(1-\beta_2)+\lambda_2(1-\beta_1) = \lambda_1\lambda_2\lambda$, we obtain $\lambda\lambda_1 - \frac{\lambda_1(1-\beta_2)}{\lambda_2} = 1-\beta_1$. Integration by parts yields

$$\begin{aligned}
&\int_0^{\frac{1}{(k_1m+1)^{\lambda_1}}} \frac{u^{\frac{1-\beta_2}{\lambda_2}-1}}{(1+u)^{\lambda}}du \\
&= \frac{\lambda_2}{1-\beta_2}\int_0^{\frac{1}{(k_1m+1)^{\lambda_1}}} \frac{1}{(1+u)^{\lambda}}du^{\frac{1-\beta_2}{\lambda_2}} \\
&= \frac{\lambda_2}{1-\beta_2}\left\{\frac{(k_1m+1)^{\lambda\lambda_1-\frac{\lambda_1(1-\beta_2)}{\lambda_2}}}{[(k_1m+1)^{\lambda_1}+1]^{\lambda}} + \frac{\lambda\lambda_2}{1+\lambda_2-\beta_2}\int_0^{\frac{1}{(k_1m+1)^{\lambda_1}}} \frac{1}{(1+u)^{\lambda+1}}du^{\frac{1-\beta_2}{\lambda_2}+1}\right\} \\
&> \frac{\lambda_2}{1-\beta_2}\left\{\frac{(k_1m+1)^{1-\beta_1}}{[(k_1m+1)^{\lambda_1}+1]^{\lambda}} + \frac{\lambda\lambda_2}{1+\lambda_2-\beta_2}\frac{(k_1m+1)^{1-\beta_1}}{[(k_1m+1)^{\lambda_1}+1]^{\lambda+1}}\right\}.
\end{aligned}$$

So we have

$$\begin{aligned}
R(m) &> \frac{1}{[(k_1m+1)^{\lambda_1}+1]^{\lambda}}\left[\frac{1}{k_2(1-\beta_2)} - \frac{1}{2} - \frac{k_2\beta_2}{12}\right] \\
&\quad + \frac{\lambda\lambda_2}{[(k_1m+1)^{\lambda_1}+1]^{\lambda+1}}\left[\frac{1}{k_2(1-\beta_2)(1+\lambda_2-\beta_2)} - \frac{k_2}{12}\right] \\
&= \frac{k_2^2\beta_2^2+(6k_2-k_2^2)\beta_2+12-6k_2}{12k_2[(k_1m+1)^{\lambda_1}+1]^{\lambda}(1-\beta_2)} \\
&\quad + \frac{\lambda\lambda_2[12-k_2^2(1-\beta_2)(1+\lambda_2-\beta_2)]}{12k_2[(k_1m+1)^{\lambda_1}+1]^{\lambda+1}(1-\beta_2)(1+\lambda_2-\beta_2)}. \qquad (2.1)
\end{aligned}$$

In view of $(1-\lambda_2^2)k_2^2+12k_2-12 \geqslant 0$, we have $k_2^2+12k_2-12 \geqslant 0$. So for $\frac{k_2-6+\sqrt{k_2^2+12k_2-12}}{2k_2} \leqslant \beta_2 < 1$, we obtain

$$\begin{aligned}
&k_2^2\beta_2^2+(6k_2-k_2^2)\beta_2+12-6k_2 \\
&= k_2^2\left[\beta_2 - \frac{k_2-6-\sqrt{k_2^2+12k_2-12}}{2k_2}\right]\left[\beta_2 - \frac{k_2-6+\sqrt{k_2^2+12k_2-12}}{2k_2}\right] \geqslant 0. \qquad (2.2)
\end{aligned}$$

Since $(1-\lambda_2^2)k_2^2+12k_2-12 \geqslant 0$, then $\lambda_2 k_2 - \sqrt{k_2^2+12k_2-12} \leqslant 0$. Therefore

$$\begin{aligned}\frac{48}{k_2+6-\sqrt{k_2^2+12k_2-12}} &= k_2+6+\sqrt{k_2^2+12k_2-12}\\ &\geqslant k_2+6+\sqrt{k_2^2+12k_2-12}+2\left(\lambda_2 k_2-\sqrt{k_2^2+12k_2-12}\right)\\ &= (2\lambda_2+1)k_2+6-\sqrt{k_2^2+12k_2-12}.\end{aligned}$$

Hence

$$48-\left(k_2+6-\sqrt{k_2^2+12k_2-12}\right)\left[(2\lambda_2+1)k_2+6-\sqrt{k_2^2+12k_2-12}\right] \geqslant 0.$$

Since $\frac{k_2-6+\sqrt{k_2^2+12k_2-12}}{2k_2} \leqslant \beta_2 < 1$, we have

$$\begin{aligned}&12-k_2^2(1-\beta_2)(1+\lambda_2-\beta_2)\\ &\geqslant 12-k_2^2\left(1-\frac{k_2-6+\sqrt{k_2^2+12k_2-12}}{2k_2}\right)\left(1+\lambda_2-\frac{k_2-6+\sqrt{k_2^2+12k_2-12}}{2k_2}\right)\\ &=\frac{1}{4}\left\{48-\left(k_2+6-\sqrt{k_2^2+12k_2-12}\right)\left[(2\lambda_2+1)k_2+6-\sqrt{k_2^2+12k_2-12}\right]\right\}\\ &\geqslant 0. \qquad (2.3)\end{aligned}$$

Combining (2.1), (2.2) and (2.3), we have $R(m) > 0$. The lemma is proved. □

LEMMA 2.3. *Under the assumption of Lemma 2.2, and $\beta_1 < 1$, the following inequality holds:*

$$\begin{aligned}F(k_1 m+1) &:= \sum_{n=0}^{\infty}\frac{(k_2 n+1)^{-\beta_2}}{[(k_1 m+1)^{\lambda_1}+(k_2 n+1)^{\lambda_2}]^{\lambda}}\\ &< \frac{1}{k_2\lambda_2}(k_1 m+1)^{\beta_1-1}B\left(\frac{1-\beta_1}{\lambda_1},\frac{1-\beta_2}{\lambda_2}\right).\end{aligned}$$

Proof. Since $\beta_2 \geqslant 0$, then

$$f(x) := \frac{(k_2 x+1)^{-\beta_2}}{[(k_1 m+1)^{\lambda_1}+(k_2 x+1)^{\lambda_2}]^{\lambda}}$$

satisfies the condition of Lemma 2.1. Thus, by Lemma 2.1, we have

$$\begin{aligned}F(k_1 m+1) &= \sum_{n=0}^{\infty} f(n) < \int_0^{\infty} f(x)dx+\frac{1}{2}f(0)-\frac{1}{12}f'(0)\\ &= \int_0^{\infty} f(x)dx+\frac{1}{2[(k_1 m+1)^{\lambda_1}+1]^{\lambda}}+\frac{k_2\beta_2}{12[(k_1 m+1)^{\lambda_1}+1]^{\lambda}}\\ &\quad+\frac{k_2\lambda_2\lambda}{12[(k_1 m+1)^{\lambda_1}+1]^{\lambda+1}}. \qquad (2.4)\end{aligned}$$

In view of $\lambda_1(1-\beta_2)+\lambda_2(1-\beta_1)=\lambda_1\lambda_2\lambda$, and $\beta_1,\ \beta_2<1$, setting $(k_2x+1)^{\lambda_2}=u(k_1m+1)^{\lambda_1}$, we obtain

$$\begin{aligned}
&\int_0^\infty f(x)dx=\int_0^\infty \frac{(k_2x+1)^{-\beta_2}}{[(k_1m+1)^{\lambda_1}+(k_2x+1)^{\lambda_2}]^\lambda}dx\\
&=\frac{1}{k_2\lambda_2}(k_1m+1)^{\frac{\lambda_1(1-\beta_2)}{\lambda_2}-\lambda\lambda_1}\int_{\frac{1}{(k_1m+1)^{\lambda_1}}}^\infty \frac{u^{\frac{1-\beta_2}{\lambda_2}-1}}{(1+u)^\lambda}du\\
&=\frac{1}{k_2\lambda_2}(k_1m+1)^{\beta_1-1}\left[\int_0^\infty \frac{u^{\frac{1-\beta_2}{\lambda_2}-1}}{(1+u)^\lambda}du-\int_0^{\frac{1}{(k_1m+1)^{\lambda_1}}} \frac{u^{\frac{1-\beta_2}{\lambda_2}-1}}{(1+u)^\lambda}du\right]\\
&=\frac{1}{k_2\lambda_2}(k_1m+1)^{\beta_1-1}\left[B\left(\frac{1-\beta_1}{\lambda_1},\frac{1-\beta_2}{\lambda_2}\right)-\int_0^{\frac{1}{(k_1m+1)^{\lambda_1}}} \frac{u^{\frac{1-\beta_2}{\lambda_2}-1}}{(1+u)^\lambda}du\right]. \qquad (2.5)
\end{aligned}$$

Applying (2.5) to (2.4), and combining Lemma 2.2, we have

$$\begin{aligned}
F(k_1m+1)&<\frac{1}{k_2\lambda_2}(k_1m+1)^{\beta_1-1}B\left(\frac{1-\beta_1}{\lambda_1},\frac{1-\beta_2}{\lambda_2}\right)-R(m)\\
&<\frac{1}{k_2\lambda_2}(k_1m+1)^{\beta_1-1}B\left(\frac{1-\beta_1}{\lambda_1},\frac{1-\beta_2}{\lambda_2}\right).
\end{aligned}$$

Lemma 2.3 is proved. Similarly, it can be proved that the lemma below also holds. □

LEMMA 2.4. *Let* $k_1,k_2,\lambda_1,\lambda_2,\lambda>0,\ \lambda_1(1-\beta_2)+\lambda_2(1-\beta_1)=\lambda_1\lambda_2\lambda,\ (1-\lambda_1^2)k_1^2+12k_1-12\geqslant 0,\ \beta_2<1,\ \max\left\{0,\frac{k_1-6+\sqrt{k_1^2+12k_1-12}}{2k_1}\right\}\leqslant\beta_1<1$, *then*

$$\begin{aligned}
G(k_2n+1)&:=\sum_{m=0}^\infty \frac{(k_1m+1)^{-\beta_1}}{[(k_1m+1)^{\lambda_1}+(k_2n+1)^{\lambda_2}]^\lambda}\\
&<\frac{1}{k_1\lambda_1}(k_2n+1)^{\beta_2-1}B\left(\frac{1-\beta_1}{\lambda_1},\frac{1-\beta_2}{\lambda_2}\right).
\end{aligned}$$

LEMMA 2.5. *Let* $k_1,k_2,\lambda_1,\lambda_2,\lambda,\varepsilon>0,\ \beta_1,\beta_2<1$, *such that* $\lambda_1(1-\beta_2)+\lambda_2(1-\beta_1)=\lambda_1\lambda_2\lambda$, *and* $\frac{\varepsilon}{q}<\frac{1-\beta_2}{\lambda_2}$, *then*

$$\begin{aligned}
I&:=\int_0^\infty\int_0^\infty \frac{(k_1x+1)^{\frac{-p\beta_1-\lambda_1\varepsilon}{p}}(k_2y+1)^{\frac{-q\beta_2-\lambda_2\varepsilon}{q}}}{[(k_1x+1)^{\lambda_1}+(k_2y+1)^{\lambda_2}]^\lambda}dxdy\\
&>\frac{1}{k_1k_2\lambda_1\lambda_2\varepsilon}\left[B\left(\frac{1-\beta_1}{\lambda_1}+\frac{\varepsilon}{q},\frac{1-\beta_2}{\lambda_2}-\frac{\varepsilon}{q}\right)-o(1)\right],\ \ (\varepsilon\to 0^+).
\end{aligned}$$

Proof. Since $\lambda_1(1-\beta_2)+\lambda_2(1-\beta_1)=\lambda_1\lambda_2\lambda$, then $\frac{\lambda_1(1-\beta_2)}{\lambda_2}+(1-\beta_1)-\lambda\lambda_1=0$. Let $(k_2y+1)^{\lambda_2}=t(k_1x+1)^{\lambda_1}$, we obtain

$$I=\frac{1}{k_2\lambda_2}\int_0^\infty (k_1x+1)^{\frac{\lambda_1(1-\beta_2)}{\lambda_2}+(1-\beta_1)-\lambda\lambda_1-\lambda_1\varepsilon-1}dx\int_{\frac{1}{(k_1x+1)^{\lambda_1}}}^\infty \frac{t^{\frac{1-\beta_2}{\lambda_2}-\frac{\varepsilon}{q}-1}}{(1+t)^\lambda}dt$$
$$=\frac{1}{k_2\lambda_2}\int_0^\infty (k_1x+1)^{-\lambda_1\varepsilon-1}dx\left[\int_0^\infty \frac{t^{\frac{1-\beta_2}{\lambda_2}-\frac{\varepsilon}{q}-1}}{(1+t)^\lambda}dt-\int_0^{\frac{1}{(k_1x+1)^{\lambda_1}}} \frac{t^{\frac{1-\beta_2}{\lambda_2}-\frac{\varepsilon}{q}-1}}{(1+t)^\lambda}dt\right]. \quad (2.6)$$

It is easy to show that

$$\int_0^\infty (k_1x+1)^{-\lambda_1\varepsilon-1}dx\int_0^\infty \frac{t^{\frac{1-\beta_2}{\lambda_2}-\frac{\varepsilon}{q}-1}}{(1+t)^\lambda}dt$$
$$=\frac{1}{k_1\lambda_1\varepsilon}B\left(\frac{1-\beta_1}{\lambda_1}+\frac{\varepsilon}{q},\frac{1-\beta_2}{\lambda_2}-\frac{\varepsilon}{q}\right); \quad (2.7)$$

and

$$\int_0^\infty (k_1x+1)^{-\lambda_1\varepsilon-1}dx\int_0^{\frac{1}{(k_1x+1)^{\lambda_1}}} \frac{t^{\frac{1-\beta_2}{\lambda_2}-\frac{\varepsilon}{q}-1}}{(1+t)^\lambda}dt$$
$$<\int_0^\infty (k_1x+1)^{-1}dx\int_0^{\frac{1}{(k_1x+1)^{\lambda_1}}} t^{\frac{1-\beta_2}{\lambda_2}-\frac{\varepsilon}{q}-1}dt=\frac{1}{k_1\lambda_1}\left(\frac{1-\beta_2}{\lambda_2}-\frac{\varepsilon}{q}\right)^{-2}. \quad (2.8)$$

Combining (2.6), (2.7) and (2.8), Lemma 2.5 is obtained. □

3. Main result and applications

THEOREM 3.1. *Let* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, a_n, $b_n\geqslant 0$, $i=1,2$, k_i, λ_i, $\lambda>0$, *such that* $\lambda_1(1-\beta_2)+\lambda_2(1-\beta_1)=\lambda_1\lambda_2\lambda, (1-\lambda_i^2)k_i^2+12k_i-12\geqslant 0$,

$$\max\left\{0,\frac{k_i-6+\sqrt{k_i^2+12k_i-12}}{2k_i}\right\}\leqslant\beta_i<1,$$

and $0<\sum_{n=0}^\infty(k_1n+1)^{p\beta_1-1}a_n^p<\infty$, $0<\sum_{n=0}^\infty(k_2n+1)^{q\beta_2-1}b_n^q<\infty$, *then*

$$\sum_{n=0}^\infty\sum_{m=0}^\infty\frac{a_mb_n}{[(k_1m+1)^{\lambda_1}+(k_2n+1)^{\lambda_2}]^\lambda}$$
$$<\frac{1}{(k_1\lambda_1)^{\frac{1}{q}}(k_2\lambda_2)^{\frac{1}{p}}}B\left(\frac{1-\beta_1}{\lambda_1},\frac{1-\beta_2}{\lambda_2}\right)[\sum_{n=0}^\infty(k_1n+1)^{p\beta_1-1}a_m^p]^{\frac{1}{p}}[\sum_{n=0}^\infty(k_2n+1)^{q\beta_2-1}b_n^q]^{\frac{1}{q}}, \quad (3.1)$$

where the constant factor $\frac{1}{(k_1\lambda_1)^{\frac{1}{q}}(k_2\lambda_2)^{\frac{1}{p}}}B(\frac{1-\beta_1}{\lambda_1},\frac{1-\beta_2}{\lambda_2})$ *is the best possible.*

Proof. We start the proof with the following inequality (see [7]) which provides a unified treatment of Hilbert's type inequalities, that is

$$\int_{\Omega\times\Omega}K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)$$
$$\leqslant\left[\int_{\Omega}\varphi^p(x)F(x)f^p(x)d\mu_1(x)\right]^{\frac{1}{p}}\left[\int_{\Omega}\psi^q(y)G(y)g^q(y)d\mu_2(y)\right]^{\frac{1}{q}},\qquad(3.2)$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, μ_1, μ_2 are positive σ-finite measures, $K:\Omega\times\Omega\to\mathbb{R}$, $f,g,\varphi,\psi:\Omega\to\mathbb{R}$ are measurable, non-negative functions and

$$F(x)=\int_{\Omega}\frac{K(x,y)}{\psi^p(y)}d\mu_2(y),\qquad G(y)=\int_{\Omega}\frac{K(x,y)}{\varphi^q(x)}d\mu_1(x).$$

Let $K(k_1m+1,k_2n+1)=[(k_1m+1)^{\lambda_1}+(k_2n+1)^{\lambda_2}]^{-\lambda}$, $\varphi(k_1m+1)=(k_1m+1)^{\frac{\beta_1}{q}}$, $\psi(k_2n+1)=(k_2n+1)^{\frac{\beta_2}{p}}$, $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ are non-negative real sequences. Using inequality (3.2) with counting measure, then, we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_mb_n}{[(k_1m+1)^{\lambda_1}+(k_2n+1)^{\lambda_2}]^{\lambda}}$$
$$=\left\{\sum_{m=0}^{\infty}a_m^p(k_1m+1)^{\frac{p\beta_1}{q}}F(k_1m+1)\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}b_n^q(k_2n+1)^{\frac{q\beta_2}{p}}G(k_2n+1)\right\}^{\frac{1}{q}},$$

where $F(k_1m+1)$ and $G(k_2n+1)$ are defined by Lemma 2.3 and Lemma 2.4 respectively. Thus, by Lemma 2.3 and Lemma 2.4, (3.1) is obtained. Next we will prove that the constant factor in Theorem 3.1 is the best possible.

Let $a_n(\varepsilon)=(k_1n+1)^{\frac{-p\beta_1-\lambda_1\varepsilon}{p}}$, $b_n(\varepsilon)=(k_2n+1)^{\frac{-q\beta_2-\lambda_2\varepsilon}{q}}$, ε is defined by Lemma 2.5, then

$$[\sum_{n=0}^{\infty}(k_1n+1)^{p\beta_1-1}a_n^p(\varepsilon)]^{\frac{1}{p}}[\sum_{n=0}^{\infty}(k_2n+1)^{q\beta_2-1}b_n^q(\varepsilon)]^{\frac{1}{q}}$$
$$=[\sum_{n=0}^{\infty}(k_1n+1)^{-\lambda_1\varepsilon-1}]^{\frac{1}{p}}[\sum_{n=0}^{\infty}(k_2n+1)^{-\lambda_2\varepsilon-1}]^{\frac{1}{q}}$$
$$=[1+\sum_{n=1}^{\infty}(k_1n+1)^{-\lambda_1\varepsilon-1}]^{\frac{1}{p}}[1+\sum_{n=1}^{\infty}(k_2n+1)^{-\lambda_2\varepsilon-1}]^{\frac{1}{q}}$$
$$<[1+\int_0^{\infty}(k_1x+1)^{-\lambda_1\varepsilon-1}dx]^{\frac{1}{p}}[1+\int_0^{\infty}(k_2x+1)^{-\lambda_2\varepsilon-1}dx]^{\frac{1}{q}}$$
$$=\left(1+\frac{1}{\lambda_1k_1\varepsilon}\right)^{\frac{1}{p}}\left(1+\frac{1}{\lambda_2k_2\varepsilon}\right)^{\frac{1}{q}}.\qquad(3.3)$$

If the constant factor in (3.1) is not the best possible, then there must exist a positive number k,

$$k < \frac{1}{(k_1\lambda_1)^{\frac{1}{q}}(k_2\lambda_2)^{\frac{1}{p}}} B\left(\frac{1-\beta_1}{\lambda_1}, \frac{1-\beta_2}{\lambda_2}\right)$$

such that (3.1) is still valid if we replace the constant factor by k. In particular, by Lemma 2.5 and inequality (3.3), we have

$$\begin{aligned}
&\frac{1}{k_1k_2\lambda_1\lambda_2}\left[B\left(\frac{1-\beta_1}{\lambda_1}+\frac{\varepsilon}{q}, \frac{1-\beta_2}{\lambda_2}-\frac{\varepsilon}{q}\right)-o(1)\right] \\
&< \varepsilon I \\
&< \varepsilon \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m(\varepsilon)b_n(\varepsilon)}{[(k_1m+1)^{\lambda_1}+(k_2n+1)^{\lambda_2}]^{\lambda}} \\
&< \varepsilon k[\sum_{n=0}^{\infty}(k_1n+1)^{p\beta_1-1}a_n^p(\varepsilon)]^{\frac{1}{p}}[\sum_{n=0}^{\infty}(k_2n+1)^{q\beta_2-1}b_n^q(\varepsilon)]^{\frac{1}{q}} \\
&< \varepsilon k\left(1+\frac{1}{\lambda_1k_1\varepsilon}\right)^{\frac{1}{p}}\left(1+\frac{1}{\lambda_2k_2\varepsilon}\right)^{\frac{1}{q}} = k\left(\varepsilon+\frac{1}{\lambda_1k_1}\right)^{\frac{1}{p}}\left(\varepsilon+\frac{1}{\lambda_2k_2}\right)^{\frac{1}{q}}.
\end{aligned}$$

Let $\varepsilon \to 0^+$, then

$$\frac{1}{(k_1\lambda_1)^{\frac{1}{q}}(k_2\lambda_2)^{\frac{1}{p}}} B\left(\frac{1-\beta_1}{\lambda_1}, \frac{1-\beta_2}{\lambda_2}\right) \leqslant k.$$

This contradicts the fact

$$k < \frac{1}{(k_1\lambda_1)^{\frac{1}{q}}(k_2\lambda_2)^{\frac{1}{p}}} B\left(\frac{1-\beta_1}{\lambda_1}, \frac{1-\beta_2}{\lambda_2}\right).$$

Hence the constant factor in (3.1) is the best possible. The proof of Theorem 3.1 is completed. □

Setting $\lambda_1 = \lambda_2 = 1$ in Theorem 3.1, we can obtain the following corollary:

COROLLARY 3.2. *Let* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, a_n, $b_n \geqslant 0$, $\lambda > 0$, $i = 1,2$, $k_i \geqslant 1$, *such that* $(1-\beta_1)+(1-\beta_2) = \lambda$, $\max\left\{0, \frac{k_i-6+\sqrt{k_i^2+12k_i-12}}{2k_i}\right\} \leqslant \beta_i < 1$, *and* $0 < \sum_{n=0}^{\infty}(k_1n+1)^{p\beta_1-1}a_n^p < \infty$, $0 < \sum_{n=0}^{\infty}(k_2n+1)^{q\beta_2-1}b_n^q < \infty$, *then*

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_mb_n}{(k_1m+k_2n+2)^{\lambda}} &< \frac{1}{k_1^{\frac{1}{q}}k_2^{\frac{1}{p}}} B(1-\beta_1, 1-\beta_2)[\sum_{n=0}^{\infty}(k_1n+1)^{p\beta_1-1}a_n^p]^{\frac{1}{p}} \\
&\quad \times [\sum_{n=0}^{\infty}(k_2n+1)^{q\beta_2-1}b_n^q]^{\frac{1}{q}}. \qquad (3.4)
\end{aligned}$$

REMARK 3.3. Corollary 3.2 and Theorem B* are equivalent. In fact, in Theorem B*, we set $r=\frac{2t-\lambda}{\beta_1+t-1}$, $s=\frac{2t-\lambda}{\beta_2+t-1}$, $c=1$, $a=\frac{k_1}{2}$, $b=\frac{k_2}{2}$. Since $\frac{1}{r}+\frac{1}{s}=1$, then

$$\frac{\beta_1+t-1}{2t-\lambda}+\frac{\beta_2+t-1}{2t-\lambda}=1,$$

that is

$$(1-\beta_1)+(1-\beta_2)=\lambda.$$

Since

$$r=\frac{2t-\lambda}{\beta_1+t-1},\qquad s=\frac{2t-\lambda}{\beta_2+t-1},$$

then

$$\lambda=2t-rt-\beta_1 r+r=2t-st-\beta_2 s+s.$$

Thus, (1.8) and (1.9) can be transformed into the following inequalities:

$$(2-s)t<2t-st-\beta_2 s+s\leqslant s\min\left\{1,\frac{6+k_2-\sqrt{k_2^2+12k_2-12}}{2k_2}\right\}-st+2t;$$

and

$$(2-r)t<2t-rt-\beta_1 r+r\leqslant r\min\left\{1,\frac{6+k_1-\sqrt{k_1^2+12k_1-12}}{2k_1}\right\}-rt+2t.$$

That is

$$\max\left\{0,\frac{k_i-6+\sqrt{k_i^2+12k_i-12}}{2k_i}\right\}\leqslant\beta_i<1,\qquad(i=1,2).$$

By simple calculation, it is also shown that (1.10) can be transformed into (3.4). So Theorem B* implies Corollary 3.2.

On the other hand, in Corollary 3.2, we set $k_1=\frac{2a}{c}$, $k_2=\frac{2b}{c}$, $\beta_1=1-t+\frac{2t-\lambda}{r}$, $\beta_2=1-t+\frac{2t-\lambda}{s}$, where $\frac{1}{r}+\frac{1}{s}=1$, $t\in[0,1]$. By easy and careful computation, we can get Theorem B*. Thus, Corollary 3.2 and Theorem B* are equivalent, and Theorem 3.1 is therefore an extension of Theorem B*.

REMARK 3.4. 1) In Corollary 3.2, let $k_1=k_2=1$, $\beta_1=1-\frac{\lambda}{p}$, $\beta_2=1-\frac{\lambda}{q}$, then $0<\lambda\leqslant\min\{p,q\}$, and (3.4) reduces to (1.3) by easy substitution. So Theorem 3.1 is an extension of (1.3).

2) In Corollary 3.2, let $k_1=k_2=1$, $\beta_1=\frac{2-\lambda}{p}$, $\beta_2=\frac{2-\lambda}{q}$, then $2-\min\{p,q\}<\lambda\leqslant 2$, and (3.4) reduces to (1.4). So Theorem 3.1 is an extension of (1.4) for $2-\min\{p,q\}<\lambda\leqslant 2$.

3) In Corollary 3.2, let $k_1 = k_2 = \frac{1}{\mu+1}$, $\beta_1 = \frac{2-\lambda}{p}$, $\beta_2 = \frac{2-\lambda}{q}$, then (3.4) reduces to (1.5) under the conditions $-1 < \mu \leqslant 0$ and

$$2 - \min\{p,q\} < \lambda \leqslant 2 - \max\{p,q\} \max\left\{0, \frac{-6\mu - 5 + \sqrt{-12\mu^2 - 12\mu + 1}}{2}\right\}.$$

Obviously, the range of λ and μ above in our result are different from Pečarić's in [5, 10]. so our result is a complement to Pečarić's.

Setting $k_1 = k_2 = 2$ in Corollary 3.2, we can obtain the following corollary which is equivalent to Theorem A.

COROLLARY 3.5. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, a_n, $b_n \geqslant 0$, $\lambda > 0$, $0 \leqslant \beta_1, \beta_2 < 1$, *such that* $(1-\beta_1) + (1-\beta_2) = \lambda$, *and* $0 < \sum_{n=0}^{\infty}(n+\frac{1}{2})^{p\beta_1 - 1} a_n^p < \infty$, $0 < \sum_{n=0}^{\infty}(n+\frac{1}{2})^{q\beta_2 - 1} b_n^q < \infty$, *then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^{\lambda}}$$
$$< B(1-\beta_1, 1-\beta_2) \left[\sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^{p\beta_1 - 1} a_n^p\right]^{\frac{1}{p}} \left[\sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^{q\beta_2 - 1} b_n^q\right]^{\frac{1}{q}}.$$

By variation substitution, we can also show that Corollary 3.5 is equivalent to Theorem A. Obviously, Corollary 3.5 has more concise form than Theorem A. In Corollary 3.5, let $\lambda = 1$, then we have Corollary 3.6.

COROLLARY 3.6. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, a_n, $b_n \geqslant 0$, β_1, $\beta_2 \geqslant 0$, $\beta_1 + \beta_2 = 1$ *such that* $0 < \sum_{n=0}^{\infty}(n+\frac{1}{2})^{p\beta_1 - 1} a_n^p < \infty$, *and* $0 < \sum_{n=0}^{\infty}(n+\frac{1}{2})^{q\beta_2 - 1} b_n^q < \infty$, *then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin \beta_1 \pi} \left[\sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^{p\beta_1 - 1} a_n^p\right]^{\frac{1}{p}} \left[\sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^{q\beta_2 - 1} b_n^q\right]^{\frac{1}{q}}. \tag{3.5}$$

Moreover, setting $\lambda_1 = \lambda_2 = \lambda = 1$, $k_1 = k_2 = 1$ in Theorem 3.1, and by easy variation substitution, we have:

COROLLARY 3.7. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n, b_n \geqslant 0$, $\beta_1, \beta_2 \geqslant 0$, $\beta_1 + \beta_2 = 1$ *such that* $0 < \sum_{n=1}^{\infty} n^{p\beta_1 - 1} a_n^p < \infty$, *and* $0 < \sum_{n=1}^{\infty} n^{q\beta_2 - 1} b_n^q < \infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin \beta_1 \pi} [\sum_{n=1}^{\infty} n^{p\beta_1 - 1} a_n^p]^{\frac{1}{p}} [\sum_{n=1}^{\infty} n^{q\beta_2 - 1} b_n^q]^{\frac{1}{q}}. \tag{3.6}$$

REMARK 3.8. (3.6) and (3.5) are two new and meaningful inequalities. They are new extensions of (1.1) and (1.2) respectively. In fact, let $\beta_1 = \frac{1}{p}$, $\beta_2 = \frac{1}{q}$, then (3.6) and (3.5) reduce to (1.1) and (1.2).

Similarly, let $\lambda_1 = \lambda_2 = \frac{1}{2}$, $k_1 = k_2 = 2$, $\lambda = 1$, by Theorem 3.1, we obtain:

COROLLARY 3.9. *Let* $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$, a_n, $b_n \geqslant 0$, $0 \leqslant \beta_1, \beta_2 < 1$, $\beta_1+\beta_2=\frac{3}{2}$ *such that* $0<\sum_{n=0}^{\infty}(2n+1)^{p\beta_1-1}a_n^p<\infty$, *and* $0<\sum_{n=0}^{\infty}(2n+1)^{q\beta_2-1}b_n^q<\infty$, *then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{\sqrt{2m+1}+\sqrt{2n+1}} < \frac{-\pi}{\sin 2\beta_1\pi}\Big[\sum_{n=0}^{\infty}(2n+1)^{p\beta_1-1}a_n^p\Big]^{\frac{1}{p}}\Big[\sum_{n=0}^{\infty}(2n+1)^{q\beta_2-1}b_n^q\Big]^{\frac{1}{q}}.$$

Finally, let $\lambda_1=\lambda_2=2$, $k_1=k_2=2$, $\beta_1=\beta_2=0$, $\lambda=1$, by Theorem 3.1, we have:

COROLLARY 3.10. *Let* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, a_n, $b_n \geqslant 0$, $0<\sum_{n=0}^{\infty}\frac{a_n^p}{2n+1}<\infty$, *and* $0<\sum_{n=0}^{\infty}\frac{b_n^q}{2n+1}<\infty$, *then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(2m+1)^2+(2n+1)^2} < \frac{\pi}{4}\Big[\sum_{n=0}^{\infty}\frac{a_n^p}{2n+1}\Big]^{\frac{1}{p}}\Big[\sum_{n=0}^{\infty}\frac{b_n^q}{2n+1}\Big]^{\frac{1}{q}}.$$

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