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VOLTERRA TYPE OPERATORS ON MORREY TYPE SPACES

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Abstract. In this paper, we investigate the boundedness of Volterra type operators on Morrey type spaces H_K^2.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ its boundary and $H(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$. For $a \in \mathbb{D}$, let φ_a be the automorphism of $\mathbb{D}$ exchanging 0 for a, namely $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$, $z \in \mathbb{D}$. Let μ denote a positive Borel measure on $\mathbb{D}$. For a subarc $I \in \partial\mathbb{D}$, let $S(I)$ be the Carleson box based on I with

$$S(I) = \{z \in \mathbb{D} : 1 - |I| \leqslant |z| < 1 \text{ and } \frac{z}{|z|} \in I\}.$$

If $I = \partial\mathbb{D}$, let $S(I) = \mathbb{D}$. We say that μ is a Carleson measure on $\mathbb{D}$ if

$$\sup_{I \subset \partial\mathbb{D}} \mu(S(I))/|I| < \infty.$$

Here and henceforth $\sup_{I \subset \partial\mathbb{D}}$ indicates the supremum taken over all subarcs I of $\partial\mathbb{D}$.

For $0 < p < \infty$, the Hardy space H^p consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

As usual, H^∞ denote the space of bounded analytic function.

Throughout this paper, we assume that $K : [0,\infty) \to [0,\infty)$ is a right-continuous and nondecreasing function such that

$$\int_0^{1/e} K(\log(1/\rho))\rho d\rho = \int_1^\infty K(t)e^{-2t} dt < \infty.$$

We say that f belongs to the space Q_K if (see, for example [4, 5])

$$\|f\|_{Q_K}^2 = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(1 - |\varphi_a(z)|^2)\, dA(z) < \infty.$$

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Q_K is a Banach space under the norm $|f(0)|+\|f\|_{Q_K}$. For $0<p<\infty$, $K(t)=t^p$ gives the space Q_p, $Q_1=BMOA$, the space of those analytic functions f in the Hardy space H^p whose boundary functions have bounded mean oscillation on $\partial\mathbb{D}$ (see, for example [6, 28, 31]).

We say that a function $f\in H^2$ belongs to Morrey type space H^2_K if

$$\|f\|^2_{H^2_K}=|f(0)|+\sup_{I\subseteq\partial\mathbb{D}}\frac{1}{K(|I|)}\int_I|f(\zeta)-f_I|^2\frac{d\zeta}{2\pi}<\infty,$$

where

$$f_I=\frac{1}{|I|}\int_I f(\zeta)\frac{d\zeta}{2\pi},\ \ I\subseteq\partial\mathbb{D}.$$

This space was recently introduced by Wulan and Zhou in [27]. When $K(t)=t$, it also gives the $BMOA$ space. When $K(t)=t^\lambda$ $(0<\lambda<1)$, the space H^2_K gives the classical Morrey space $\mathscr{L}^{2,\lambda}$, which was first studied by Wu and Xie in [26] in the case of the unit disk. Morrey space was first studied by Morrey for solutions of partial differential equations (PDE) in [16].

Let $g\in H(\mathbb{D})$. The multiplication operator M_g is defined by

$$M_gf(z)=f(z)g(z),\ \ f\in H(\mathbb{D}).$$

An integral operator introduced by Pommerenke in [18] is defined as following:

$$J_gf(z)=\int_0^z f(\xi)g'(\xi)d\xi,\quad f\in H(\mathbb{D}),\ \ z\in\mathbb{D}.$$

We call J_g Volterra type operator (see, e.g. [19]), which can be viewed as a generalization of the Cesàro operator (see, e.g. [3]). Similarly, another integral operator was defined by

$$I_gf(z)=\int_0^z f'(\xi)g(\xi)d\xi.$$

The importance of the operators J_g and I_g comes from the fact that

$$J_gf+I_gf+f(0)g(0)=M_gf.$$

Pommerenke showed that J_g is bounded on H^2 if and only if $g\in$ BMOA in [18]. Recently, the boundedness, compactness, norm and essential norm of J_g and I_g between some spaces of analytic functions, as well as their n-dimensional extensions on the unit ball in $\mathbb{C}^n$, have been investigated considerably (see, e.g. [1, 2, 3, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 19, 20, 21, 22, 23, 24, 25, 29, 32, 33] and also the related references therein).

In [10], the authors studied the boundedness of J_g, I_g and M_g on Morrey space $\mathscr{L}^{2,\lambda}$ $(0<\lambda<1)$. Motivated by [10], in this paper, we investigate the boundedness of J_g, I_g and M_g on Morrey type space H^2_K.

For our aim, we need more constraints on K in the rest of this paper. By [4], we may assume that K is defined on $[0,1]$ and extend its domain to $[0,\infty)$ by setting

$K(t) = K(1)$ for $t > 1$. We also assume that

$$\int_0^1 \frac{\varphi_K(s)}{s} ds < \infty \tag{1.1}$$

$$\int_1^\infty \frac{\varphi_K(s)}{s^2} ds < \infty, \tag{1.2}$$

where

$$\varphi_K(s) = \sup_{0 \leqslant t \leqslant 1} K(st)/K(t), \quad 0 < s < \infty.$$

Finally, we assume that $K(t) \approx K(2t)$. In this paper, the symbol $f \approx g$ means that $f \lesssim g \lesssim f$. We say that $f \lesssim g$ if there exists a constant C such that $f \leqslant Cg$.

2. Main results and proofs

In this section, we give our main results and proofs. For this purpose, we need some auxiliary results. The following lemma can be found in [27, Theorem 3.1].

LEMMA 1. *Let K satisfy the conditions* (1.1) *and* (1.2)*. Then the following statements are equivalent.*

(a) $f \in H_K^2$;

(b)

$$\sup_{I \subseteq \partial\mathbb{D}} \frac{1}{K(|I|)} \int_{S(I)} |f'(z)|^2 (1 - |z|^2) dA(z) < \infty;$$

(c)

$$\sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2) dA(z) < \infty.$$

LEMMA 2. *Let K satisfy the conditions* (1.1) *and* (1.2)*. Then,*

$$|w| \int_0^1 \frac{\sqrt{K(1 - |w|t)}}{(1 - |w|t)^{\frac{3}{2}}} dt \lesssim \frac{\sqrt{K(1 - |w|)}}{\sqrt{1 - |w|}}, \quad w \in \mathbb{D}.$$

Proof. Since K satisfies (1.2), from the proof of Lemma 2.2 of [5], we see that there exists a small enough $c > 0$ such that

$$\varphi_K(t) \lesssim t^{1-c}, \ \ t \geqslant 1.$$

By making change of variables, we have

$$
\begin{aligned}
&|w|\int_0^1 \frac{\sqrt{K(1-|w|t)}}{(1-|w|t)^{\frac{3}{2}}}dt = \int_0^{|w|} \frac{\sqrt{K(1-s)}}{(1-s)^{\frac{3}{2}}}ds\\
=&\sqrt{K(1-|w|)}\int_0^{|w|}\sqrt{\frac{K(1-s)}{K(1-|w|)}}(1-s)^{-\frac{3}{2}}ds\\
\lesssim&\sqrt{K(1-|w|)}\int_0^{|w|}\sqrt{\varphi_K\left(\frac{1-s}{1-|w|}\right)}(1-s)^{-\frac{3}{2}}ds\\
\lesssim&\sqrt{K(1-|w|)}\int_0^{|w|}\sqrt{\left(\frac{1-s}{1-|w|}\right)^{1-c}}(1-s)^{-\frac{3}{2}}ds\\
\lesssim&\sqrt{\frac{K(1-|w|)}{1-|w|}},
\end{aligned}
$$

completing the proof. □

LEMMA 3. *Let K satisfy the conditions* (1.1) *and* (1.2)*. Suppose that $f \in H_K^2$, then*

$$
|f(z)| \lesssim \frac{\|f\|_{H_K^2}\sqrt{K(1-|z|^2)}}{\sqrt{(1-|z|^2)}}, \quad z \in \mathbb{D}.
$$

Proof. By Lemma 1, for any $w \in \mathbb{D}$, we have

$$
\begin{aligned}
&\frac{1-|w|^2}{K(1-|w|^2)}\int_{\mathbb{D}} |f'(z)|^2(1-|\varphi_w(z)|^2)dA(z)\\
\lesssim&\sup_{a\in\mathbb{D}}\frac{1-|a|^2}{K(1-|a|^2)}\int_{\mathbb{D}} |f'(z)|^2(1-|\varphi_a(z)|^2)dA(z)\\
\lesssim&\|f\|_{H_K^2}^2.
\end{aligned}
$$

Let $E(w,r) = \{z \in \mathbb{D} : |\varphi_w(z)| < r\}$. Using the sub-mean value property of $|f'|^2$ and notice the fact that

$$
|1-\bar{z}w| \approx 1-|z|^2 \approx 1-|w|^2, \quad z \in E(w,r),
$$

we get

$$
\begin{aligned}
|f'(w)|^2 \lesssim&\frac{1}{(1-|w|^2)^2}\int_{E(w,r)} |f'(z)|^2 dA(z)\\
\lesssim&\frac{1}{(1-|w|^2)^2}\int_{E(w,r)} |f'(z)|^2(1-|\varphi_w(z)|^2)dA(z)\\
\leqslant&\frac{1}{(1-|w|^2)^2}\int_{\mathbb{D}} |f'(z)|^2(1-|\varphi_w(z)|^2)dA(z).
\end{aligned}
$$

Hence, we have

$$|f'(w)| \lesssim \frac{\|f\|_{H_K^2}\sqrt{K(1-|w|^2)}}{(1-|w|^2)^{\frac{3}{2}}}.$$

Since

$$|f(w)-f(0)| = |w\int_0^1 f'(wt)dt| \leqslant |w|\int_0^1 |f'(wt)|dt,$$

combine with Lemma 2, we easy to get that

$$|f(w)-f(0)| \lesssim \|f\|_{H_K^2}|w|\int_0^1 \frac{\sqrt{K(1-|wt|^2)}}{(1-|wt|^2)^{\frac{3}{2}}}dt \lesssim \|f\|_{H_K^2}\frac{\sqrt{K(1-|w|^2)}}{\sqrt{1-|w|^2}}.$$

By [5, Lemma 2.3], we know that there exists a K_3 such that $K_3 \approx K$ and $\frac{K_3(t)}{t}$ is nonincreasing on $(0,\infty)$. Thus,

$$\frac{K(t)}{t} \approx \frac{K_3(t)}{t} \gtrsim 1, \quad t\in(0,1).$$

Since the point w is arbitrary, it follows that

$$|f(z)| \lesssim \|f\|_{H_K^2}\frac{\sqrt{K(1-|z|^2)}}{\sqrt{1-|z|^2}}, \ z\in\mathbb{D}.$$

The proof is completed. □

LEMMA 4. ([30, Lemma 1]) *Suppose that $\alpha > -1$ and $s, t > 0$. If $t < \alpha+2 < s$, then we have*

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^{\alpha}}{|1-\overline{a}z|^s|1-\overline{b}z|^t}dA(z) \lesssim \frac{1}{(1-|a|^2)^{s-\alpha-2}|1-\overline{b}a|^t}.$$

LEMMA 5. [6] *Suppose that μ is a non-negative measure on $\mathbb{D}$. Then μ is a Carleson measure if and only if the following inequality*

$$\int_{\mathbb{D}} |f(z)|^2 d\mu \lesssim \|f\|_{H^2}^2$$

holds for all $f\in H^2$. Moreover,

$$\sup_{\|f\|_{H^2}=1}\int_{\mathbb{D}} |f(z)|^2 d\mu \approx \sup_{I\subseteq\partial\mathbb{D}} \frac{\mu(S(I))}{|I|}.$$

LEMMA 6. [6] *Suppose that $f\in H(\mathbb{D})$, then $f\in BMOA$ if and only if the measure $\mu_f = |f'(z)|^2(1-|z|^2)dA(z)$ is a Carleson measure.*

THEOREM 1. *Suppose that $g \in H(\mathbb{D})$ and K satisfy the conditions* (1.1) *and* (1.2)*. Then I_g is bounded on H_K^2 if and only if $g \in H^\infty$. Moreover,*

$$\|I_g\| \approx \sup_{z\in\mathbb{D}} |g(z)|.$$

Proof. If $g \in H^\infty$, then by Lemma 1, it is easy to see that I_g is bounded on H_K^2 and

$$\|I_g\| \leqslant \sup_{z\in\mathbb{D}} |g(z)|.$$

Now we assume that I_g is bounded on H_K^2. For any $w \in \mathbb{D}$, we define

$$f_w(z) = \sqrt{\frac{K(1-|w|^2)}{1-|w|^2}}\,(\varphi_w(z)-w)\,, \ \ z \in \mathbb{D}.$$

Since K satisfies (2), by [5, Lemma 2.2], there exists a small enough $c > 0$ such that $\varphi_K(t) \lesssim t^{1-c}$, $t \geqslant 1$. By Lemma 4, we have

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \int_{\mathbb{D}} |f_w'(z)|^2 \left(1-|\varphi_a(z)|^2\right) dA(z)\\
=&\sup_{a\in\mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \int_{\mathbb{D}} \frac{K(1-|w|^2)(1-|w|^2)}{|1-\overline{w}z|^4} \left(1-|\varphi_a(z)|^2\right) dA(z)\\
=&\sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^2 K(1-|w|^2)(1-|w|^2)}{K(1-|a|^2)} \int_{\mathbb{D}} \left(\frac{(1-|z|^2)}{|1-\overline{w}z|^4|1-\overline{a}z|^2}\right) dA(z)\\
\lesssim&\sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^2 K(1-|w|^2)}{K(1-|a|^2)|1-\overline{a}w|^2}\\
\lesssim&\sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^2 K(|1-\overline{a}w|)}{K(1-|a|)|1-\overline{a}w|^2}\\
\lesssim&\sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^2}{|1-\overline{a}w|^2}\varphi_K\left(\frac{|1-\overline{a}w|}{1-|a|}\right)\\
\lesssim&\sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^2}{|1-\overline{a}w|^2}\left(\frac{|1-\overline{a}w|}{1-|a|}\right)^{1-c}\\
=&\sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^{1+c}}{|1-\overline{a}w|^{1+c}} \lesssim 1.
\end{aligned}$$

By Lemma 1, we see that $f_w \in H_K^2$.

Since $g \in H(\mathbb{D})$, then $g \circ \varphi_w \in H(\mathbb{D})$. By sub-mean value property of $|g|^2$, the asymptotic relations

$$|1-\overline{z}w| \approx 1-|z|^2 \approx 1-|w|^2, \ \ z \in E(w,r),$$

and Lemma 1, we obtain

$$\begin{aligned}
\infty >&\|I_g f_w\|_{H_K^2}^2\\
\gtrsim&\sup_{a\in\mathbb{D}}\frac{1-|a|^2}{K(1-|a|^2)}\int_{\mathbb{D}}|f_w'(z)|^2|g(z)|^2\left(1-|\varphi_a(z)|^2\right)dA(z)\\
\gtrsim&\frac{1-|w|^2}{K(1-|w|^2)}\int_{\mathbb{D}}|f_w'(z)|^2|g(z)|^2\left(1-|\varphi_w(z)|^2\right)dA(z)\\
\gtrsim&\int_{\mathbb{D}}|g(z)|^2|\varphi_w'(z)|^2\left(1-|\varphi_w(z)|^2\right)dA(z)\\
\geqslant&\int_{E(w,r)}|g(z)|^2|\varphi_w'(z)|^2\left(1-|\varphi_w(z)|^2\right)dA(z)\\
\gtrsim&|g(w)|^2.
\end{aligned}$$

Since $w\in\mathbb{D}$ is arbitrary, we have $\infty>\|I_g f_w\|_{H_K^2}\gtrsim\|g\|_{H^\infty}$. Moreover, from the proof we see that $\|I_g\|\approx\sup_{z\in\mathbb{D}}|g(z)|$, completing the proof. □

THEOREM 2. *Suppose that $g\in H(\mathbb{D})$ and K satisfy the conditions (1.1) and (1.2). Then J_g is bounded on H_K^2 if and only if $g\in BMOA$. Moreover,*

$$\|J_g\|\approx\|g\|_{BMOA}.$$

Proof. First, we assume that J_g is bounded on H_K^2. For any $I\subseteq\partial\mathbb{D}$, let $w=(1-|I|)\zeta\in\mathbb{D}$, where ζ is the center of I. Then

$$1-|w|\approx|1-\overline{w}z|\approx|I|,\ \ z\in S(I).$$

Thus, by double condition and nondecreasing of weighted function K, we have

$$K(1-|w|)\approx K(|I|),\ \ z\in S(I).$$

Take

$$h_w(z)=\frac{(1-|w|^2)\sqrt{K(1-|w|^2)}}{(1-\overline{w}z)^{\frac{3}{2}}},\quad z\in\mathbb{D}.$$

Similarly to the proof of Theorem 1, we get

$$\begin{aligned}
&\sup_{a\in\mathbb{D}}\frac{1-|a|^2}{K(1-|a|^2)}\int_{\mathbb{D}}|h_w'(z)|^2\left(1-|\varphi_a(z)|^2\right)dA(z)\\
=&\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^2K(1-|w|^2)(1-|w|^2)^2}{K(1-|a|^2)}\int_{\mathbb{D}}\left(\frac{1-|z|^2}{|1-\overline{w}z|^5|1-\overline{a}z|^2}\right)dA(z)\\
\lesssim&\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^2K(1-|w|^2)}{K(1-|a|^2)|1-\overline{a}w|^2}\\
\lesssim&1.
\end{aligned}$$

Thus, $h_w \in H_K^2$. By Lemma 1 and the boundedness of J_g, we have

$$\begin{aligned}
&\frac{1}{|I|}\int_{S(I)} |g'(z)|^2(1-|z|^2)dA(z)\\
\lesssim&\frac{1}{K(|I|)}\int_{S(I)} |h_w(z)|^2|g'(z)|^2(1-|z|^2)dA(z)\\
\lesssim&\frac{1}{K(|I|)}\int_{S(I)} |(J_gh_w)'(z)|^2(1-|z|^2)dA(z)\\
\lesssim&\|J_gh_w\|_{H_K^2}^2<\infty.
\end{aligned}$$

Thus, $g \in BMOA$.

Conversely, suppose that $g \in BMOA$. For any $I \subseteq \partial\mathbb{D}$ and $f \in H_K^2$, let ζ be the center of I and $w=(1-|I|)\zeta \in \mathbb{D}$. We have

$$\begin{aligned}
&\frac{1}{K(|I|)}\int_{S(I)} |(J_gf)'(z)|^2(1-|z|^2)dA(z)\\
=&\frac{1}{K(|I|)}\int_{S(I)} |f(z)|^2|g'(z)|^2(1-|z|^2)dA(z)\lesssim A+B,
\end{aligned}$$

where

$$A:=\frac{1}{K(|I|)}\int_{S(I)} |f(w)|^2|g'(z)|^2(1-|z|^2)dA(z)$$

and

$$B:=\frac{1}{K(|I|)}\int_{S(I)} |f(z)-f(w)|^2|g'(z)|^2(1-|z|^2)dA(z).$$

By Lemma 3, we get

$$|f(w)|\lesssim\frac{\|f\|_{H_K^2}\sqrt{K(1-|w|^2)}}{\sqrt{(1-|w|^2)}}\lesssim\frac{\|f\|_{H_K^2}\sqrt{K(|I|)}}{\sqrt{|I|}},\ w\in S(I).$$

Combine with Lemma 6, it easy to get

$$A\lesssim\|f\|_{H_K^2}^2\|g\|_{BMOA}^2.$$

Since

$$\frac{1-|z|^2}{|I|}\lesssim 1-|\varphi_w(z)|^2,\ \ z\in S(I),$$

we obtain

$$\begin{aligned}
B\lesssim&\frac{|I|}{K(|I|)}\int_{S(I)} |f(z)-f(w)|^2|g'(z)|^2(1-|\varphi_w(z)|^2)dA(z)\\
\lesssim&\frac{|I|}{K(|I|)}\int_{S(I)} |f\circ\varphi_w(\eta)-f(w)|^2|(g\circ\varphi_w)'(\eta)|^2(1-|\eta|^2)dA(\eta)\\
\lesssim&\frac{1-|w|^2}{K(1-|w|^2)}\int_{\mathbb{D}} |f\circ\varphi_w(\eta)-f(w)|^2|(g\circ\varphi_w)'(\eta)|^2(1-|\eta|^2)dA(\eta).
\end{aligned}$$

Since $g \in BMOA$, then $g \circ \varphi_w \in BMOA$ and $|(g \circ \varphi_w)'(\eta)|^2(1-|\eta|^2)dA(\eta)$ is a Carleson measure by Lemma 6. From Lemma 1, if $a = 0$, it is not hard to deduce that $f \in H^2_K \subseteq H^2$. Then $(f \circ \varphi_w)(\eta) - f(w) \in H^2$. Combining this with Lemma 5 gives

$$\begin{aligned} B &\lesssim \frac{1-|w|^2}{K(1-|w|^2)} \|g \circ \varphi_w\|^2_{BMOA} \int_0^{2\pi} |f \circ \varphi_w(e^{i\theta}) - f(w)|^2 d\theta \\ &\lesssim \frac{1-|w|^2}{K(1-|w|^2)} \|g \circ \varphi_w\|^2_{BMOA} \int_{\mathbb{D}} |f'(z)|^2(1-|\varphi_w(z)|^2)dA(z) \\ &\lesssim \|g\|^2_{BMOA}\|f\|^2_{H^2_K}, \end{aligned}$$

where we used the Littlewood-Paley identity in the second inequality (see [6, page 236]). Therefore,

$$\|J_g f\|^2_{H^2_K} \lesssim A + B \lesssim \|g\|^2_{BMOA}\|f\|^2_{H^2_K}.$$

The proof is complete. □

THEOREM 3. *Suppose that* $g \in H(\mathbb{D})$ *and* K *satisfy the conditions* (1.1) *and* (1.2). *Then* M_g *is bounded on* H^2_K *if and only if* $g \in H^\infty$.

Proof. Suppose M_g is bounded on H^2_K. For any $w \in \mathbb{D}$, consider the function h_w defined in Theorem 2. Using Lemma 3, it gives

$$\begin{aligned} |M_g h_w| &= \left| \frac{(1-|w|^2)\sqrt{K(1-|w|^2)}}{(1-\overline{w}z)^{\frac{3}{2}}} g(z) \right| \\ &\lesssim \frac{\|M_g h_w\|_{H^2_K}\sqrt{K(1-|z|^2)}}{\sqrt{(1-|z|^2)}} \\ &\lesssim \frac{\|M_g\|\sqrt{K(1-|z|^2)}}{\sqrt{(1-|z|^2)}}. \end{aligned}$$

Taking $z = w$, we get $|g(w)| \lesssim \|M_g\|$. Since $w \in \mathbb{D}$ is arbitrary, we deduce that $g \in H^\infty$. The other side is obvious. The proof is complete. □

REMARK 1. If $K(t) = t^\lambda$ $(0 < \lambda < 1)$, then K satisfies our conditions and H^2_K is just Morrey space. Hence our results generalize the results in [10]. If $K(t) = t$, H^2_K is just BMOA space. However K does not satisfy the condition (1.2). Hence, our results do not include the case of BMOA space. In [19], Siskakis and Zhao proved that $J_g : BMOA \to BMOA$ is bounded if only if

$$\sup_{I \subset \partial\mathbb{D}} \frac{(\log\frac{2}{|I|})^2}{|I|} \int_{S(I)} |g'(z)|^2(1-|z|^2)dA(z) < \infty.$$

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