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SCHUR CONVEXITY PROPERTIES FOR THE ELLIPTIC NEUMAN MEAN WITH APPLICATIONS

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Abstract. Strictly Schur convexity, Schur multiplicative convexity and Schur harmonic convexity are investigated for the elliptic Neuman mean. As applications, several sharp bounds for the arithmetic, geometric and harmonic means in terms of the elliptic Neuman mean are presented.

1. Introduction

For $x,y\in\mathbb{R}_+^2=(0,+\infty)\times(0,+\infty)$ and $k\in[0,1]$ the elliptic Neuman mean $N_k(x,y)$ [6] is defined by

$$N_k(x,y)=\begin{cases}\frac{\sqrt{y^2-x^2}}{cn^{-1}(x/y,k)}, & x<y,\\ x, & x=y,\\ \frac{\sqrt{x^2-y^2}}{nc^{-1}(x/y,k)}, & y<x,\end{cases}$$

where $cn^{-1}(x,k)=\int\limits_x^1\frac{du}{\sqrt{(1-u^2)(k'^2+k^2u^2)}}$ and $nc^{-1}(x,k)=\int\limits_1^x\frac{du}{\sqrt{(u^2-1)(k^2+k'^2u^2)}}$ are the inverse functions of Jacobian elliptic functions cn and nc [2, 7], respectively, and $k'=\sqrt{1-k^2}$. In particular, $cn^{-1}(0,k)=\mathscr{K}(k)=\int\limits_0^{\pi/2}\frac{dt}{\sqrt{1-k^2\sin^2t}}$ is the well-known complete elliptic integral of the first kind.

In [6] Neuman proved that $N_k(x,y)$ is symmetric and homogeneous of degree 1 on $\mathbb{R}_+^2$, and strictly decreasing with respect to $k\in[0,1]$ for fixed $(x,y)\in\mathbb{R}_+^2$ with $x\neq y$.

Let us recall the notions of Schur convex (concave), Schur multiplicatively convex (concave), and Schur harmonic convex (concave) of a symmetric function on $\mathbb{R}_+^2$.

A real-valued function $f:\mathbb{R}_+^2\to\mathbb{R}$ is said to be strictly Schur convex on $\mathbb{R}_+^2$ if $f(x_1,x_2)<f(y_1,y_2)$ for each pair of 2-tuples $x=(x_1,x_2)$ and $y=(y_1,y_2)$ in $\mathbb{R}_+^2$ with

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$x \prec y$, i.e., $\max\{x_1, x_2\} < \max\{y_1, y_2\}$ and $x_1 + x_2 = y_1 + y_2$. f is said to be strictly Schur concave if $-f$ is strictly Schur convex.

A real-valued function $g : \mathbb{R}_+^2 \to \mathbb{R}$ is said to be strictly Schur multiplicatively convex on $\mathbb{R}_+^2$ if $g(x_1, x_2) < g(y_1, y_2)$ for each pair of 2-tuples $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}_+^2$ with $\log x = (\log x_1, \log x_2) \prec \log y = (\log y_1, \log y_2)$. g is said to be strictly Schur multiplicatively concave if $1/g$ is strictly Schur multiplicatively convex.

A real-valued function $h : \mathbb{R}_+^2 \to \mathbb{R}$ is said to be strictly Schur harmonic convex (or concave, respectively) on $\mathbb{R}_+^2$ if $h(x_1, x_2) < h(y_1, y_2)$ (or $h(x_1, x_2) > h(y_1, y_2)$, respectively) for each pair of 2-tuples $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}_+^2$ with $\frac{1}{x} = (\frac{1}{x_1}, \frac{1}{x_2}) \prec \frac{1}{y} = (\frac{1}{y_1}, \frac{1}{y_2})$.

The main purpose of this paper is to discuss the Schur convexity, Schur multiplicative and Schur harmonic convexity properties of the elliptic Neuman mean $N_k(x,y)$, and present the sharp bounds for the arithmetic, geometric and harmonic means in terms of the elliptic Neuman mean.

2. Lemmas

In order to establish our main results we need several lemmas, which we present in this section.

LEMMA 2.1. (See [5]) *Suppose that $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a continuous symmetric function. If f is differentiable in $\mathbb{R}_+^2$, then f is strictly Schur convex on $\mathbb{R}_+^2$ if and only if*

$$\frac{\partial f(x_1, x_2)}{\partial x_2} - \frac{\partial f(x_1, x_2)}{\partial x_1} > 0 \tag{2.1}$$

for all $x = (x_1, x_2) \in \mathbb{R}_+^2$ with $x_2 > x_1$. And f is strictly Schur concave on $\mathbb{R}_+^2$ if and only if inequality (2.1) is reversed. Here, f is a symmetric function in $\mathbb{R}_+^2$ means that $f(xP) = f(x)$ for all $x \in \mathbb{R}_+^2$ and any 2×2 permutation matrix P.

LEMMA 2.2. (see [4, Lemma 2.2] and [3, Lemma 2.10]) *Suppose that $g : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a continuous symmetric function. If g is differentiable in $\mathbb{R}_+^2$, then g is strictly Schur multiplicatively convex on $\mathbb{R}_+^2$ if and only if*

$$x_2 \frac{\partial g(x_1, x_2)}{\partial x_2} - x_1 \frac{\partial g(x_1, x_2)}{\partial x_1} > 0 \tag{2.2}$$

for all $x = (x_1, x_2) \in \mathbb{R}_+^2$ with $x_2 > x_1$. And g is strictly Schur multiplicatively concave on $\mathbb{R}_+^2$ if and only if inequality (2.2) is reversed.

LEMMA 2.3. (see [4, Lemma 2.2] and [3, Lemma 2.11]) *Suppose that $h : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a continuous symmetric function. If h is differentiable in $\mathbb{R}_+^2$, then h is strictly Schur harmonic convex on $\mathbb{R}_+^2$ if and only if*

$${x_2}^2 \frac{\partial h(x_1, x_2)}{\partial x_2} - {x_1}^2 \frac{\partial h(x_1, x_2)}{\partial x_1} > 0 \tag{2.3}$$

for all $x = (x_1, x_2) \in \mathbb{R}^2_+$ *with* $x_2 > x_1$. *And* h *is strictly Schur harmonic concave on* $\mathbb{R}^2_+$ *if and only if inequality* (2.3) *is reversed.*

LEMMA 2.4. (see [1, Lemma 3.21(7)]) *If* $c \geqslant 1/4$, *then the function* $k \to (1-k^2)^c \mathscr{K}(k)$ *is strictly decreasing from* $(0,1)$ *onto* $(0, \pi/2)$.

LEMMA 2.5. *Let* $t \in (0,1)$, $k \in [0,1]$, $k_0 \in (0.897, 0.898)$ *be the unique solution of the equation* $\mathscr{K}(k) - 1/\sqrt{1-k^2} = 0$ *and*

$$f_k(t) = cn^{-1}(t,k) - \frac{\sqrt{1-t^2}}{\sqrt{1-k^2+k^2t^2}}.$$

Then $f_k(t) > 0$ *for all* $t \in (0,1)$ *if and only if* $k \leqslant \sqrt{2}/2$, *and* $f_k(t) < 0$ *for all* $t \in (0,1)$ *if and only if* $k \geqslant k_0$.

Proof. We divide the proof into two cases.

Case 1 $k = 1$. Then we clearly see that

$$\begin{aligned} f_1(t) =& cn^{-1}(t,1) - \frac{\sqrt{1-t^2}}{t} = \cosh^{-1}\left(\frac{1}{t}\right) - \frac{\sqrt{1-t^2}}{t} \\ =& \log(1+\sqrt{1-t^2}) - \log t - \frac{\sqrt{1-t^2}}{t}, \end{aligned}$$

$$f_1(1^-) = 0, \tag{2.4}$$

$$f_1{}'(t) = \frac{1-t}{t^2\sqrt{1-t^2}} > 0 \tag{2.5}$$

for all $t \in (0,1)$.

It follows from (2.4) and (2.5) that $f_1(t) < 0$ for all $t \in (0,1)$.

Case 2 $0 \leqslant k < 1$. Then simple computations lead to

$$f_k(0^+) = \mathscr{K}(k) - \frac{1}{\sqrt{1-k^2}}, \tag{2.6}$$

$$f_k(1^-) = 0, \tag{2.7}$$

$$\begin{aligned} f_k{}'(t) =& -\frac{1}{\sqrt{(1-t^2)(1-k^2+k^2t^2)}} + \frac{t(1-k^2+k^2t^2) + k^2t(1-t^2)}{\sqrt{1-t^2}(1-k^2+k^2t^2)^{3/2}} \\ =& \frac{\sqrt{1-t^2}}{(1-k^2+k^2t^2)^{3/2}}\left(k^2 - \frac{1}{1+t}\right). \end{aligned} \tag{2.8}$$

We divide the proof into three subcases.

Subcase 2.1 $0 \leqslant k \leqslant \sqrt{2}/2$. Then (2.8) implies that

$$f_k{}'(t) < 0 \tag{2.9}$$

for all $t \in (0,1)$.

Therefore, $f_k(t) > 0$ for all $t \in (0,1)$ follows from (2.7) and (2.9).

Subcase 2.2 $k_0 \leqslant k < 1$. Then from Lemma 2.4 and (2.6) we get

$$f_k(0^+) = \frac{\sqrt{1-k^2}\mathscr{K}(k) - 1}{\sqrt{1-k^2}} \leqslant 0. \tag{2.10}$$

Let $t_0 = t_0(k) = 1/k^2 - 1$, then from (2.8) and $k \geqslant k_0 > \sqrt{2}/2$ we clearly see that $t_0 \in (0,1)$, and $f_k(t)$ is strictly decreasing in $(0, t_0]$ and strictly increasing in $[t_0, 1)$.

Therefore, $f_k(t) < 0$ follows from (2.7) and (2.10) together with the piecewise monotonicity of $f_k(t)$.

Subcase 2.3 $\sqrt{2}/2 < k < k_0$. Then from Lemma 2.4 and (2.6) together with (2.8) we know that

$$f_k(0^+) = \frac{\sqrt{1-k^2}\mathscr{K}(k) - 1}{\sqrt{1-k^2}} > 0, \tag{2.11}$$

and $f_k(t)$ is strictly decreasing in $(0, t_0]$ and strictly increasing in $[t_0, 1)$ with $t_0 = 1/k^2 - 1 \in (0,1)$.

From (2.7) and (2.11) together with the piecewise monotonicity of $f_k(t)$ we clearly see that there exists unique $\lambda \in (0, t_0)$ such that $f_k(t) > 0$ for $t \in (0, \lambda)$ and $f_k(t) < 0$ for $t \in (\lambda, 1)$. □

LEMMA 2.6. *Let $t \in (0,1)$, $k \in [0,1]$, and*

$$g_k(t) = cn^{-1}(t,k) - \frac{2t\sqrt{1-t^2}}{(1+t^2)\sqrt{1-k^2+k^2t^2}}.$$

Then $g_k(t) > 0$ for all $t \in (0,1)$ if and only if $k \leqslant \sqrt{2}/2$, and there exists $\mu = \mu(k) \in (0,1)$ such that $g_k(t) > 0$ for $t \in (0, \mu)$ and $g_k(t) < 0$ for $t \in (\mu, 1)$ if $\sqrt{2}/2 < k \leqslant 1$.

Proof. We divide the proof into two cases.

Case 1 $k = 1$. Then we clearly see that

$$g_1(t) = cn^{-1}(t,1) - \frac{2\sqrt{1-t^2}}{1+t^2} = \log(1+\sqrt{1-t^2}) - \log t - \frac{2\sqrt{1-t^2}}{1+t^2},$$

$$g_1(0^+) = +\infty, \tag{2.12}$$

$$g_1(1^-) = 0, \tag{2.13}$$

$$g_1{}'(t) = \frac{(3t^2-1)\sqrt{1-t^2}}{t(1+t^2)^2}. \tag{2.14}$$

Equation (2.14) implies that $g_1(t)$ is strictly decreasing in $(0, \sqrt{3}/3]$ and strictly increasing in $[\sqrt{3}/3, 1)$. It follows from (2.12) and (2.13) together with the piecewise monotonicity of $g_1(t)$ that there exists unique $\mu \in (0, \sqrt{3}/3)$ such that $g_1(t) > 0$ for $t \in (0, \mu)$ and $g_1(t) < 0$ for $t \in (\mu, 1)$.

Case 2 $0 \leqslant k < 1$. Then simple computations lead to

$$g_k(0^+) = \mathscr{K}(k) > 0, \tag{2.15}$$

$$g_k(1^-) = 0, \tag{2.16}$$

$$\begin{aligned} g_k'(t) =& -\frac{1}{\sqrt{(1-t^2)(1-k^2+k^2t^2)}} - \frac{2k^2(t^6-3t^4+3t^2-1)-6t^2+2}{\sqrt{1-t^2}(1+t^2)^2(1-k^2+k^2t^2)^{3/2}} \\ =& \frac{(3t^4-2t^2+3)\sqrt{1-t^2}}{(1+t^2)^2(1-k^2+k^2t^2)^{3/2}}\left[k^2 - J(t)\right], \end{aligned} \tag{2.17}$$

where

$$J(t) = \frac{3-t^2}{3t^4-2t^2+3} = \left[3\left(3-t^2+\frac{8}{3-t^2}\right)-16\right]^{-1}. \tag{2.18}$$

Equation (2.18) leads to

$$J(0) = 1, \tag{2.19}$$

$$J(1) = 1/2, \tag{2.20}$$

and $J(t)$ is strictly increasing in $(0, \sqrt{2}-1]$ and strictly decreasing in $[\sqrt{2}-1, 1)$.

We divide the proof into two subcases.

Subcase 2.1 $0 \leqslant k \leqslant \sqrt{2}/2$. Then from (2.17), (2.19), (2.20) and the piecewise monotonicity of $J(t)$ we clearly see that $g_k(t)$ is strictly decreasing in $(0,1)$

Therefore, $g_k(t) > 0$ follows from (2.16) and the monotonicity of $g_k(t)$.

Subcase 2.2 $\sqrt{2}/2 < k < 1$. Then from (2.17), (2.19), (2.20) and the piecewise monotonicity of $J(t)$ we know that there exists unique $t_1 \in (0,1)$ such that $g_k(t)$ is strictly decreasing in $(0, t_1]$ and strictly increasing in $[t_1, 1)$.

Equations (2.15) and (2.16) together with the piecewise monotonicity of $g_k(t)$ lead to the conclusion that there exists unique $\mu = \mu(k) \in (0, t_1)$ such that $g_k(t) > 0$ for $t \in (0, \mu)$ and $g_k(t) < 0$ for $t \in (\mu, 1)$. □

LEMMA 2.7. *Let $t \in (0,1)$, $k \in [0,1]$, and*

$$h_k(t) = cn^{-1}(t,k) - \frac{t\sqrt{1-t^2}}{(t^2-t+1)\sqrt{1-k^2+k^2t^2}}.$$

Then $h_k(t) > 0$ for all $t \in (0,1)$ if and only if $k \leqslant \sqrt{2}/2$, and there exists $\eta = \eta(k) \in (0,1)$ such that $h_k(t) > 0$ for $t \in (0,\eta)$ and $h_k(t) < 0$ for $t \in (\eta, 1)$ if $\sqrt{2}/2 < k \leqslant 1$.

Proof. We divide the proof into two cases.

Case 1 $k = 1$. Then we clearly see that

$$h_1(t) = cn^{-1}(t,1) - \frac{\sqrt{1-t^2}}{(t^2-t+1)} = \log(1+\sqrt{1-t^2}) - \log t - \frac{\sqrt{1-t^2}}{(t^2-t+1)},$$

$$h_1(0^+) = +\infty, \tag{2.21}$$

$$h_1(1^-) = 0, \tag{2.22}$$

$$h_1{}'(t) = \frac{(1-t)(2t^3-1)}{t\sqrt{1-t^2}(t^2-t+1)^2}. \tag{2.23}$$

Equation (2.23) implies that $h_1(t)$ is strictly decreasing in $(0, \sqrt[3]{4}/2)$ and strictly increasing in $(\sqrt[3]{4}/2, 1)$. From (2.21) and (2.22) together with the piecewise monotonicity of $h_1(t)$ we clearly see that there exists unique $\eta \in (0, \sqrt[3]{4}/2)$ such that $h_1(t) > 0$ for $t \in (0, \eta)$ and $h_1(t) < 0$ for $t \in (\eta, 1)$.

Case 2 $0 \leqslant k < 1$. Then simple computations lead to

$$h_k(0^+) = \mathscr{K}(k) > 0, \tag{2.24}$$

$$h_k(1^-) = 0, \tag{2.25}$$

$$\begin{aligned} h_k{}'(t) &= -\frac{1}{\sqrt{(1-t^2)(1-k^2+k^2t^2)}} - \frac{(t^6-3t^4+3t^2-1)k^2+t^3-3t^2+1}{\sqrt{1-t^2}(1-t+t^2)^2(1-k^2+k^2t^2)^{3/2}} \\ &= \frac{(1-t)(2t^5-t^3-t^2+2)}{\sqrt{1-t^2}(1-t+t^2)^2(1-k^2+k^2t^2)^{3/2}} \left[k^2 - I(t)\right], \end{aligned} \tag{2.26}$$

where

$$I(t) = \frac{2-t^3}{2t^5-t^3-t^2+2},$$

$$I(0) = 1, \tag{2.27}$$

$$I(1) = \frac{1}{2}, \tag{2.28}$$

$$I'(t) = \frac{t(4t^6-19t^3+4)}{(2t^5-t^3-t^2+2)^2}. \tag{2.29}$$

From (2.29) we know that $I(t)$ is strictly increasing in $(0, \sqrt[3]{(19-3\sqrt{33})/8})$ and strictly decreasing in $(\sqrt[3]{(19-3\sqrt{33})/8}, 1)$.

We divide the proof into two subcases.

Subase 2.1 $0 \leqslant k \leqslant \sqrt{2}/2$. Then from (2.26)–(2.28) and the piecewise monotonicity of $I(t)$ we clearly see that $h_k(t)$ is strictly decreasing in $(0,1)$.

Therefore, $h_k(t) > 0$ for all $t \in (0,1)$ follows from (2.25) and the monotonicity of $h_k(t)$.

Subcase 2.2 $\sqrt{2}/2 < k < 1$. Then equations (2.26)–(2.28) and the piecewise monotonicity of $I(t)$ lead to the conclusion that there exists unique $t_2 \in (\sqrt[3]{(19-3\sqrt{33})/8}, 1)$ such that $h_k(t)$ is strictly decreasing in $(0, t_2]$ and strictly increasing in $[t_2, 1)$.

It follows from (2.24) and (2.25) together with the piecewise monotonicity of $h_k(t)$ that there exists $\eta = \eta(k) \in (0, t_2)$ such that $h_k(t) > 0$ for $t \in (0, \eta)$ and $h_k(t) < 0$ for $t \in (\eta, 1)$. $\square$

3. Main Results

THEOREM 3.1. *The elliptic Neuman mean $N_k(x,y)$ is strictly Schur convex on $\mathbb{R}^2_+$ if and only if $k \leqslant \sqrt{2}/2$, and strictly Schur concave on $\mathbb{R}^2_+$ if and only if $k \geqslant k_0 \in (0.897, 0.898)$. Here, k_0 is the unique solution of the equation $\mathscr{K}(k) - 1/\sqrt{1-k^2} = 0$.*

Proof. Since $N_k(x,y)$ is symmetric and homogeneous of degree 1, without loss of generality, we assume that $x < y$. Let $t = x/y \in (0,1)$, then

$$N_k(x,y) = yN_k(t,1), \qquad \frac{\partial t}{\partial y} = -\frac{x}{y^2}, \qquad \frac{\partial t}{\partial x} = \frac{1}{y}, \tag{3.1}$$

$$\frac{\partial N_k(x,y)}{\partial y} = N_k(t,1) - t\frac{dN_k(t,1)}{dt}, \qquad \frac{\partial N_k(x,y)}{\partial x} = \frac{dN_k(t,1)}{dt}. \tag{3.2}$$

Note that

$$\frac{dN_k(t,1)}{dt} = -\frac{t}{\sqrt{1-t^2}cn^{-1}(t,k)} + \frac{1}{(cn^{-1}(t,k))^2\sqrt{1-k^2+k^2t^2}}. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\begin{aligned}
&\frac{\partial N_k(x,y)}{\partial y} - \frac{\partial N_k(x,y)}{\partial x} \\
=&N_k(t,1) - (t+1)\frac{dN_k(t,1)}{dt} \\
=&\frac{1+t}{\sqrt{1-t^2}\left[cn^{-1}(t,k)\right]^2}\left(cn^{-1}(t,k) - \frac{\sqrt{1-t^2}}{\sqrt{1-k^2+k^2t^2}}\right).
\end{aligned} \tag{3.4}$$

Therefore, Theorem 3.1 follows from Lemmas 2.1 and 2.5 together with equation (3.4). □

THEOREM 3.2. *The elliptic Neuman mean $N_k(x,y)$ is strictly Schur multiplicatively convex on $\mathbb{R}^2_+$ if and only if $k \leqslant \sqrt{2}/2$, and $N_k(x,y)$ is not Schur multiplicatively concave on $\mathbb{R}^2_+$ for any $\sqrt{2}/2 < k \leqslant 1$.*

Proof. We follow the lines of proof in Theorem 3.1. Let $t = x/y \in (0,1)$, then from (3.2) and (3.3) we have

$$\begin{aligned}
&y\frac{\partial N_k(x,y)}{\partial y} - x\frac{\partial N_k(x,y)}{\partial x} \\
=&y\left(N_k(t,1) - 2t\frac{dN_k(t,1)}{dt}\right) \\
=&\frac{y(1+t^2)}{\sqrt{1-t^2}\left[cn^{-1}(t,k)\right]^2}\left(cn^{-1}(t,k) - \frac{2t\sqrt{1-t^2}}{(1+t^2)\sqrt{1-k^2+k^2t^2}}\right).
\end{aligned} \tag{3.5}$$

Therefore, Theorem 3.2 follows directly from Lemmas 2.2 and 2.6 together with equation (3.5). □

THEOREM 3.3. *The elliptic Neuman mean $N_k(x,y)$ is strictly Schur harmonic convex on $\mathbb{R}^2_+$ if and only if $k \leqslant \sqrt{2}/2$, and $N_k(x,y)$ is not Schur harmonic concave on $\mathbb{R}^2_+$ for any $\sqrt{2}/2 < k \leqslant 1$.*

Proof. Let $t = x/y \in (0,1)$, then from (3.2) and (3.3) we have

$$\begin{aligned} & y^2 \frac{\partial N_k(x,y)}{\partial y} - x^2 \frac{\partial N_k(x,y)}{\partial x} \\ =& y^2 \left(N_k(t,1) - t(t+1) \frac{dN_k(t,1)}{dt} \right) \\ =& \frac{y^2(1+t^3)}{\sqrt{1-t^2}\,[cn^{-1}(t,k)]^2} \left(cn^{-1}(t,k) - \frac{t\sqrt{1-t^2}}{(t^2-t+1)\sqrt{1-k^2+k^2t^2}} \right). \end{aligned} \tag{3.6}$$

Therefore, Theorem 3.3 follows from Lemmas 2.3 and 2.7 together with equation (3.6). □

4. Applications

In this section, we give several sharp bounds for the arithmetic, harmonic and geometric means in terms of the elliptic Neuman mean. But, we first establish the following Lemma 4.1, which will be used in the proof of Theorem 4.1.

LEMMA 4.1. *(1) $N_k(x,y)$ is strictly Schur convex (or concave, respectively) on $\mathbb{R}^2_+$ if and only if the function $N_k(t,1)/A(t,1)$ is strictly decreasing (or increasing, respectively) in $(0,1)$;*

(2) $N_k(x,y)$ is strictly Schur multiplicatively convex (or concave, respectively) on $\mathbb{R}^2_+$ if and only if the function $N_k(t,1)/G(t,1)$ is strictly decreasing (or increasing, respectively) in $(0,1)$;

(3) $N_k(x,y)$ is strictly Schur harmonic convex (or concave, respectively) on $\mathbb{R}^2_+$ if and only if the function $N_k(t,1)/H(t,1)$ is strictly decreasing (or increasing, respectively) in $(0,1)$.

Here, $A(x,y) = (x+y)/2$, $G(x,y) = \sqrt{xy}$, and $H(x,y) = 2xy/(x+y)$ are the classical arithmetic, geometric, and harmonic means of x and y, respectively.

Proof. Without loss of generality, we assume that $x < y$. Let $t = x/y \in (0,1)$, then from (3.1) and (3.2) one has

$$\frac{d(N_k(t,1)/A(t,1))}{dt} = -\frac{2}{(t+1)^2}\left(N_k(t,1) - (t+1)\frac{dN_k(t,1)}{dt} \right), \tag{4.1}$$

$$\frac{d(N_k(t,1)/G(t,1))}{dt} = -\frac{1}{2t^{3/2}}\left(N_k(t,1) - 2t\frac{dN_k(t,1)}{dt} \right), \tag{4.2}$$

$$\frac{d(N_k(t,1)/H(t,1))}{dt} = -\frac{1}{2t^2}\left(N_k(t,1) - t(t+1)\frac{dN_k(t,1)}{dt} \right). \tag{4.3}$$

Therefore, part (1) follows from (3.4) and (4.1) together with Lemma 2.1, part (2) follows from (3.5) and (4.2) together with Lemma 2.2, and part (3) follows from (3.6) and (4.3) together with Lemma 2.3. □

REMARK 4.1. The results for the elliptic Neuman mans $N_k(x,y)$ in Lemma 4.1 can be generalized to more general symmetric, homogeneous and differentiable bivariate mean $M(x,y)$ of degree 1. The proof is similar to the proof in Lemma 4.1, we omit the details

THEOREM 4.1. *Let* $k_0 \in (0.897, 0.898)$ *be the unique solution of the equation* $\mathscr{K}(k) - 1/\sqrt{1-k^2} = 0$*, and* $k_1 \in (0.802, 0.803)$ *be the unique solution of the equation* $\mathscr{K}(k) = 2$*. Then the inequalities*

$$\frac{\mathscr{K}(\sqrt{2}/2)}{2} N_{\sqrt{2}/2}(x,y) < A(x,y) < N_{\sqrt{2}/2}(x,y), \tag{4.4}$$

$$N_{k_0}(x,y) < A(x,y) < \frac{\mathscr{K}(k_0)}{2} N_{k_0}(x,y) \tag{4.5}$$

and

$$N_{k_1}(x,y) < A(x,y) \tag{4.6}$$

hold for all $x, y > 0$ *with* $x \neq y$*,* $N_{k_1}(x,y)$ *and* $N_{\sqrt{2}/2}(x,y)$ *are the best possible lower and upper elliptic Neuman mean bounds for the arithmetic mean* $A(x,y)$*, respectively.*

Proof. Without loss of generality, we assume that $y > x > 0$. Let $t = x/y \in (0,1)$, and $L_k(t) = N_k(t,1)/A(t,1)$. Then

$$\frac{N_k(x,y)}{A(x,y)} = L_k(t), \tag{4.7}$$

$$L_k(0) = \frac{2}{\mathscr{K}(k)}, \tag{4.8}$$

$$L_k(1) = 1. \tag{4.9}$$

Lemma 4.1(1) and Theorem 3.1 lead to the conclusion that $L_{\sqrt{2}/2}(t)$ is strictly decreasing and $L_{k_0}(t)$ is strictly increasing in $(0,1)$. Therefore, inequalities (4.4) and (4.5) follow from (4.7)–(4.9) and the monotonicities of $L_{\sqrt{2}/2}(t)$ and $L_{k_0}(t)$.

If $\sqrt{2}/2 < k < k_0$, then from (3.4) and (4.1) together with Subcase 2.3 in Lemma 2.5 we conclude that there exists $\lambda \in (0,1)$ such that $L_k(t)$ is strictly decreasing in $(0,\lambda]$ and strictly increasing in $[\lambda,1)$. In particular, if $k_1 \leqslant k < k_0$, then (4.8) implies

$$L_k(0) \leqslant \frac{2}{\mathscr{K}(k_1)} = 1. \tag{4.10}$$

Therefore, inequality (4.6) and the optimality follow from (4.9) and (4.10) together with the piecewise monotonicity of $L_k(t)$. □

Similarly, from Theorems 3.2 and 3.3, and Lemma 4.1(2) and (3) we have

THEOREM 4.2. *The inequalities*

$$G(x,y) < N_{\sqrt{2}/2}(x,y), \quad H(x,y) < N_{\sqrt{2}/2}(x,y)$$

hold for all $x,y > 0$ *with* $x \neq y$, $N_{\sqrt{2}/2}(x,y)$ *is the best possible upper elliptic Neuman mean bound for the geometric and harmonic means, there does not exist lower elliptic Neuman mean bound for the geometric and harmonic means, and there does not exist constant* $\lambda > 0$ *such that the inequality* $G(x,y) > \lambda N_{\sqrt{2}/2}(x,y)$ *or* $H(x,y) > \lambda N_{\sqrt{2}/2}(x,y)$ *holds for all* $x,y > 0$ *with* $x \neq y$.

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REFERENCES

[1] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. K. VUORINEN, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, John Wiley & Sons, New York, 1997.
[2] B. C. CARLSON, *Special Functions of Applied Mathematics*, Academic Press, New York, 1977.
[3] Y.-M. CHU, G.-D. WANG AND X.-H. ZHANG, *The Schur multiplicative and harmonic convexities of the complete symmetric function*, Math. Nachr. **284** (2011), 653–663.
[4] Y.-M. CHU, W.-F. XIA AND X.-H. ZHANG, *The Schur concavity, Schur multiplicative and harmonic convexities of the second dual form of the Hamy symmetric function with applications*, J. Multivariate Anal. **105** (2012), 412–421.
[5] A. W. MARSHALL, I. OLKIN AND B. C. ARNOLD, *Inequalities: Theory of Majorization and Its Applications*, Springer, New York, 2011.
[6] E. NEUMAN, *On one-parameter family of bivariate means*, Aequationes Math. **83** (2012), 191–197.
[7] F. W. J. OLVER, D. W. LOZIER, R. F. BOISVERT AND C. W. CLARK (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.

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