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CONVERSES OF COPSON'S INEQUALITIES ON TIME SCALES

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Abstract. In this paper, we will prove some new dynamic inequalities on a time scale $\mathbb{T}$. These inequalities when $\mathbb{T}=\mathbb{N}$ contain the discrete inequalities due to Bennett and Leindler which are converses of Copson's inequalities. The main results will be proved using the Hölder inequality and Keller's chain rule on time scales.

1. Introduction

The classical Hardy inequality states that for $f \geqslant 0$ and integrable over any finite interval $(0,x)$ and f^p is integrable and convergent over $(0,\infty)$ and $p>1$, then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx. \tag{1.1}$$

The constant $(p/(p-1))^p$ is the best possible. This inequality was proved by Hardy in 1925 and it is the continuous version of a discrete inequality discovered by Hardy in 1920

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}a_i\right)^p \leqslant \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty}a_n^p, \quad p>1. \tag{1.2}$$

For generalizations and applications of these inequalities in the literature we refer the reader to the books [1, 19, 20, 26] and the papers [4, 10, 16, 17, 18, 22, 23, 24, 25, 28, 29] Copson in [8, Theorem 1.1, 2.1] proved that if $p>1$, $\lambda(n)>0\ \forall n$ and $c>1$, then

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{\Lambda^c(n)}\left(\sum_{i=1}^{n}a(i)\lambda(i)\right)^p \leqslant \left(\frac{p}{c-1}\right)\sum_{n=1}^{\infty}\lambda(n)\Lambda^{p-c}(n)a^p(n), \tag{1.3}$$

where $\Lambda_n=\sum_{i=1}^{n}\lambda(i)$, and if $p>1$ and $0\leqslant c<1$, then

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{\Lambda^c(n)}\left(\sum_{i=n}^{\infty}\lambda(i)a(i)\right)^p \leqslant \left(\frac{p}{1-c}\right)^p\sum_{n=1}^{\infty}\lambda(n)\Lambda^{p-c}(n)a^p(n). \tag{1.4}$$

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An interesting variant of the Hardy-Copson inequalities was given by Leindler [21] (see also Bennett [2]). Leindler in [21] proved that if $\sum_{i=n}^{\infty}\lambda(i)<\infty$, $p>1$ and $0\leqslant c<1$, then

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{(\Lambda^{*}(n))^{c}}\left(\sum_{i=1}^{n}\lambda(i)a(i)\right)^{p}\leqslant\left(\frac{p}{1-c}\right)^{p}\sum_{n=1}^{\infty}\lambda(n)(\Lambda^{*}(n))^{p-c}a^{p}(n), \tag{1.5}$$

where $\Lambda_n^{*}=\sum_{i=n}^{\infty}\lambda(i)$, and if $1<c\leqslant p$, then

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{(\Lambda^{*}(n))^{c}}\left(\sum_{i=n}^{\infty}\lambda(i)a(i)\right)^{p}\leqslant\left(\frac{p}{c-1}\right)^{p}\sum_{n=1}^{\infty}\lambda(n)(\Lambda^{*}(n))^{p-c}a^{p}(n). \tag{1.6}$$

Bennett [3] and Leindler [21] proved converses of the inequalities (1.3) and (1.4). In particular they proved that if $c\leqslant 0<p<1$, then

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{(\Lambda_n)^{c}}\left(\sum_{i=n}^{\infty}\lambda(i)g(i)\right)^{p}\geqslant\left(\frac{p}{1-c}\right)^{p}\sum_{n=1}^{\infty}\lambda(n)\left(\sum_{i=1}^{n}\lambda(i)\right)^{p-c}g^{p}(n), \tag{1.7}$$

and if $c>1>p>0$ and if $\Lambda_n\to\infty$, then

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{(\Lambda_n)^{c}}\left(\sum_{i=1}^{n}\lambda(i)g(i)\right)^{p}\geqslant\left(\frac{pL}{c-1}\right)^{p}\sum_{n=1}^{\infty}\lambda(n)\left(\Lambda_n\right)^{p-c}g^{p}(n), \tag{1.8}$$

where $L=\inf\frac{\lambda(n)}{\lambda(n+1)}$.

Dynamic inequalities of Hardy type were established in [27, 30, 31] on a time scale $\mathbb{T}$, which is an arbitrary closed subset of the real numbers $\mathbb{R}$. In this paper, without loss of generality, we assume that $\sup\mathbb{T}=\infty$, and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$. For more details of time scale analysis, we refer the reader to the two books by Bohner and Peterson [5], [6] which summarize and organize much of the time scale calculus.

A natural question now is to ask if it is possible to prove new dynamic inequalities on time scales which contain the inequalities (1.7) and (1.8). The main aim of this paper is to give an affirmative answer to this question. In particular we prove the converse of (1.5) and (1.6) on time scales and we establish the time scale versions of the inequalities (1.7) and (1.8). It is worth remarking that converses of the inequalities (1.5) and (1.6) were not considered in the literature when $\mathbb{T}=\mathbb{N}$. The main results will be proved by using Hölder's inequality and Keller's chain rule on time scales. The technique in our paper is different from the techniques used by Bennett and Leindler to prove their main results.

2. Main results

For completeness, before we prove the main results, we recall the following concepts related to the notion of time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Without loss of generality, we assume that $\sup\mathbb{T} = \infty$, and define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}$. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e, when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by:

$$\sigma(t) := \inf\{s \in \mathbb{T} :\ s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} :\ s < t\},$$

where $\sup\emptyset = \inf\mathbb{T}$. A point $t \in \mathbb{T}$, is said to be left–dense if $\rho(t) = t$ and $t > \inf\mathbb{T}$, is right–dense if $\sigma(t) = t$, is left–scattered if $\rho(t) < t$ and right–scattered if $\sigma(t) > t$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be right–dense continuous (rd–continuous) provided g is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$.

The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. Fix $t \in \mathbb{T}$ and let $x : \mathbb{T} \to \mathbb{R}$. Define $x^{\Delta}(t)$ to be the number (if it exists) with the property that given any $\varepsilon > 0$ there is a neighborhood U of t with

$$|[x(\sigma(t)) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]| \leqslant \varepsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

In this case, we say $x^{\Delta}(t)$ is the (delta) derivative of x at t and that x is (delta) differentiable at t. We will frequently use the following results due to Hilger [15]. Throughout the paper will assume that $g : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}$.

(i) If g is differentiable at t, then g is continuous at t.

(ii) If g is continuous at t and t is right-scattered, then g is differentiable at t with $g^{\Delta}(t) = \frac{g(\sigma(t)) - g(t)}{\mu(t)}$.

(iii) If g is differentiable and t is right-dense, then $g^{\Delta}(t) = \lim_{s \to t} \frac{g(t) - g(s)}{t - s}$.

(iv) If g is differentiable at t, then $g(\sigma(t)) = g(t) + \mu(t)g^{\Delta}(t)$.

Note that if $\mathbb{T} = \mathbb{R}$ then

$$\sigma(t) = t,\ \mu(t) = 0,\ f^{\Delta}(t) = f'(t),\ \int_a^b f(t)\Delta t = \int_a^b f(t)dt$$

if $\mathbb{T} = \mathbb{Z}$, then

$$\sigma(t) = t + 1,\ \mu(t) = 1,\ f^{\Delta}(t) = \Delta f(t),\ \int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t),$$

if $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $\sigma(t) = t + h$, $\mu(t) = h$, and

$$y^{\Delta}(t) = \Delta_h y(t) := \frac{y(t+h) - y(t)}{h},\ \int_a^b f(t)\Delta t = \sum_{k=0}^{\frac{b-a-h}{h}} f(a + kh)h,$$

and if $\mathbb{T} = \{t : t = q^k,\ k \in \mathbb{N}_0,\ q > 1\}$, then $\sigma(t) = qt,\ \mu(t) = (q-1)t$,

$$x^{\Delta}(t) = \Delta_q x(t) = \frac{(x(qt) - x(t))}{(q-1)t},\ \int_{t_0}^{\infty} f(t)\Delta t = \sum_{k=n_0}^{\infty} f(q^k)\mu(q^k),$$

where $t_0 = q^{n_0}$, and if $\mathbb{T} = \mathbb{N}_0^2 := \{n^2 : n \in \mathbb{N}_0\}$, then $\sigma(t) = (\sqrt{t}+1)^2$,

$$\mu(t) = 1 + 2\sqrt{t},\ \Delta_N y(t) = \frac{y((\sqrt{t}+1)^2) - y(t)}{1+2\sqrt{t}}.$$

In this paper, we will refer to the (delta) integral which we can define as follows. If $G^{\Delta}(t) = g(t)$, then the Cauchy (delta) integral of g is defined by $\int_a^t g(s)\Delta s := G(t) - G(a)$. It can be shown (see [5]) that if $g \in C_{rd}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $G^{\Delta}(t) = g(t)$, $t \in \mathbb{T}$. An infinite integral is defined as $\int_a^{\infty} f(t)\Delta t = \lim_{b\to\infty} \int_a^b f(t)\Delta t$. We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$, here $g^{\sigma} = g \circ \sigma$) of two differentiable function f and g

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma}, \text{ and } \left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}. \tag{2.1}$$

We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided $1 + \mu(t)p(t) \neq 0,\ t \in \mathbb{T}$. The chain rule formula that we will use in this paper is

$$(x^{\gamma}(t))^{\Delta} = \gamma \int_0^1 [hx^{\sigma} + (1-h)x]^{\gamma-1}\, dh x^{\Delta}(t), \tag{2.2}$$

which is a simple consequence of Keller's chain rule [5, Theorem 1.90]. The integration by parts formula is given by

$$\int_a^b u(t)v^{\Delta}(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^{\Delta}(t)v^{\sigma}(t)\Delta t. \tag{2.3}$$

To prove the main results, we will use the following Hölder inequality [5, Theorem 6.13]. Let $a,\ b \in \mathbb{T}$. For $u, v \in C_{rd}(\mathbb{T}, \mathbb{R})$, we have

$$\int_a^b |u(t)v(t)|\Delta t \leqslant \left[\int_a^b |u(t)|^q \Delta t\right]^{\frac{1}{q}} \left[\int_a^b |v(t)|^p \Delta t\right]^{\frac{1}{p}}, \tag{2.4}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Throughout the paper, we will assume that the functions in the statements of the theorems are nonnegative and rd-continuous functions and the integrals considered are assumed to exist.

In the following theorem we will prove a converse of the inequality (1.5) due to Leindler on time scales.

THEOREM 2.1. *Let $\mathbb{T}$ be a time scale with $a \in (0,\infty)_{\mathbb{T}}$ and $c \leqslant 0 < p < 1$. Let $\Omega(t) = \int_t^\infty \lambda(s)\Delta s$, and $\Psi(t) = \int_a^t \lambda(s)g(s)\Delta s$. Then*

$$\int_a^\infty \frac{\lambda(t)}{\Omega^c(t)}(\Psi^\sigma(t))^p\Delta t \geqslant \left(\frac{p}{1-c}\right)^p \left[\int_a^\infty \lambda(t)\,(\Omega(t))^{p-c}\,g^p(t)\Delta t\right]. \tag{2.5}$$

Proof. Integrating the left hand side of (2.5) by the parts formula (2.3) with $u^\Delta(t) = \lambda(t)/\Omega^c(t)$ and $v^\sigma(t) = (\Psi^\sigma(t))^p$, we obtain

$$\int_a^\infty \frac{\lambda(t)}{\Omega^c(t)}(\Psi^\sigma(t))^p\Delta t = u(t)\Psi^p(t)|_a^\infty + \int_a^\infty (-u(t))(\Psi^p(t))^\Delta\Delta t, \tag{2.6}$$

where $u(t) = -\int_t^\infty \frac{\lambda(s)}{\Omega^c(s)}\Delta s$. Using $\Psi(a) = 0$ and $u(\infty) = 0$ in (2.6), we have that

$$\int_a^\infty \frac{\lambda(t)}{\Omega^c(t)}(\Psi^\sigma(t))^p\Delta t = -\int_a^\infty u(t)(\Psi^p(t))^\Delta\Delta t. \tag{2.7}$$

Applying the chain rule ([5, Theorem 1.87]) $f^\Delta(\delta(t)) = f'(\delta(d))\delta^\Delta(t)$, where $d \in [t,\sigma(t)]$, we see that there exists $d \in [t,\sigma(t)]$ such that

$$(\Psi^p(t))^\Delta = \frac{p}{\Psi^{1-p}(d)}(\Psi^\Delta(t)) = \frac{pg(t)\lambda(t)}{\Psi^{1-p}(d)}. \tag{2.8}$$

Since $\Psi^\Delta(t) = \lambda(t)g(t) \geqslant 0$, and $\sigma(t) \geqslant d$, we see that $\Psi^\sigma(t) \geqslant \Psi(d)$, and then

$$\frac{p}{\Psi^{1-p}(d)} \geqslant \frac{p}{(\Psi^\sigma(t))^{1-p}} \quad (\text{note } p < 1). \tag{2.9}$$

Combining (2.8) and (2.9), we have that

$$(\Psi^p(t))^\Delta \geqslant \frac{pg(t)\lambda(t)}{(\Psi^\sigma(t))^{1-p}}. \tag{2.10}$$

Next note $\Omega^\Delta(t) = -\lambda(t) \leqslant 0$. By the chain rule (2.2), we see that (note $c \leqslant 0$)

$$\begin{aligned}
\left(\Omega^{1-c}(t)\right)^\Delta &= (1-c)\int_0^1 \frac{\Omega^\Delta(t)}{[h\Omega^\sigma(t) + (1-h)\Omega(t)]^c}dh \\
&= -(1-c)\int_0^1 \frac{\lambda(t)}{[h\Omega^\sigma(t) + (1-h)\Omega(t)]^c}dh \\
&\geqslant -(1-c)\int_0^1 \frac{\lambda(t)}{[h\Omega(t) + (1-h)\Omega(t)]^c}dh \\
&= -(1-c)\frac{\lambda(t)}{(\Omega(t))^c}.
\end{aligned}$$

This implies that

$$(\Omega(t))^{-c}\lambda(t) \geqslant \frac{-1}{1-c}\left(\Omega^{1-c}(t)\right)^\Delta,$$

and then, we have

$$-u(t)=\int_t^\infty \frac{\lambda(s)}{(\Omega(s))^c}\Delta s \geqslant \frac{-1}{1-c}\int_t^\infty \left(\Omega^{1-c}(s)\right)^\Delta \Delta s=\frac{1}{(1-c)\Omega^{c-1}(t)}. \tag{2.11}$$

Substituting (2.11), (2.10) into (2.7) yields

$$\left(\int_a^\infty \frac{\lambda(t)}{\Omega^c(t)}(\Psi^\sigma(t))^p\Delta t\right)^p \geqslant \left(\frac{p}{1-c}\right)^p\left[\int_a^\infty\left(\frac{g^p(t)\lambda^p(t)}{(\Omega(t))^{p(c-1)}(\Psi^\sigma(t))^{p(1-p)}}\right)^{1/p}\Delta t\right]^p. \tag{2.12}$$

Applying the Hölder inequality

$$\int_a^b F(t)G(t)\Delta t \leqslant \left[\int_a^b F^q(t)\Delta t\right]^{\frac{1}{q}}\left[\int_a^b G^h(t)\Delta t\right]^{\frac{1}{h}},$$

on the term

$$\left[\int_a^\infty\left(\frac{g^p(t)\lambda^p(t)}{(\Omega(t))^{p(c-1)}(\Psi^\sigma(t))^{p(1-p)}}\right)^{1/p}\Delta t\right]^p,$$

with indices $q=1/p>1$, $h=1/(1-p)$ (note that $\frac{1}{q}+\frac{1}{h}=1$, where $q>1$) and

$$F(t)=\frac{g^p(t)\lambda^p(t)}{(\Omega(t))^{p(c-1)}(\Psi^\sigma(t))^{p(1-p)}}, \text{ and } G(t)=\left(\frac{\lambda(t)}{\Omega^c(t)}\right)^{1-p}(\Psi^\sigma(t))^{p(1-p)},$$

we see that

$$\begin{aligned}
&\left(\int_a^\infty F^{1/p}(t)\Delta t\right)^p=\left[\int_a^\infty\left(\frac{g^p(t)\lambda^p(t)}{(\Omega(t))^{p(c-1)}(\Psi^\sigma(t))^{p(1-p)}}\right)^{1/p}\Delta t\right]^p\\
&\geqslant \frac{\int_a^\infty F(t)G(t)\Delta t}{\left[\int_a^\infty (G(t))^{\frac{1}{1-p}}\Delta t\right]^{1-p}}=\left[\int_a^\infty \frac{g^p(t)\left(\lambda(t)\Omega^{-c}(t)\right)^{1-p}\lambda^p(t)(\Psi^\sigma(t))^{p(1-p)}\Delta t}{(\Omega(t))^{p(c-1)}(\Psi^\sigma(t))^{p(1-p)}}\right]\\
&\quad\times\left[\int_a^\infty\left(\left(\frac{\lambda(t)}{\Omega^c(t)}\right)^{1-p}(\Psi^\sigma(t))^{p(1-p)}\right)^{\frac{1}{1-p}}\Delta t\right]^{p-1}\\
&=\left[\int_a^\infty \frac{\lambda(t)g^p(t)}{(\Omega(t))^{p(c-1)}(\Omega^c(t))^{1-p}}\Delta t\right]\left[\int_a^\infty \frac{\lambda(t)}{\Omega^c(t)}(\Psi^\sigma(t))^p\Delta t\right]^{p-1}\\
&=\left[\int_a^\infty \frac{g^p(t)\lambda(t)}{(\Omega(t))^{c-p}}\Delta t\right]\frac{1}{\left[\int_a^\infty \frac{\lambda(t)}{\Omega^c(t)}(\Psi^\sigma(t))^p\Delta t\right]^{1-p}}.
\end{aligned}$$

This implies that

$$\left[\int_a^\infty\left(\frac{g^p(t)\lambda^p(t)}{(\Omega(t))^{p(c-1)}(\Psi^\sigma(t))^{p(1-p)}}\right)^{1/p}\Delta t\right]^p \geqslant \frac{\int_a^\infty g^p(t)\Omega^{p-c}(t)\lambda(t)\Delta t}{\left[\int_a^\infty \frac{\lambda(t)}{\Omega^c(t)}(\Psi^\sigma(t))^p\Delta t\right]^{1-p}}. \tag{2.13}$$

Substituting (2.13) into (2.12) yields

$$\left(\int_a^\infty \frac{\lambda(t)}{\Omega^c(t)}(\Psi^\sigma(t))^p\Delta t\right)^p \geqslant \left(\frac{p}{1-c}\right)^p \frac{\int_a^\infty \lambda(t)(\Omega(t))^{p-c}g^p(t)\Delta t}{\left[\int_a^\infty \frac{\lambda(t)}{\Omega^c(t)}(\Psi^\sigma(t))^p\Delta t\right]^{1-p}}.$$

This implies that

$$\int_a^\infty \frac{\lambda(t)}{\Omega^c(t)}(\Psi^\sigma(t))^p\Delta t \geqslant \left(\frac{p}{1-c}\right)^p \left[\int_a^\infty \lambda(t)(\Omega(t))^{p-c}g^p(t)\Delta t\right],$$

which is the desired inequality (2.5). The proof is complete. □

REMARK 1. As a special case of (2.5), when $\mathbb{T}=\mathbb{R}$ and $c\leqslant 0<p<1$ and $a=1$, we have the following inequality of Leindler type

$$\int_1^\infty \frac{\lambda(t)}{\Omega^c(t)}\left(\int_1^t \lambda(s)g(s)ds\right)^p dt \geqslant \left(\frac{p}{1-c}\right)^p \int_1^\infty \lambda(t)\Omega^{p-c}(t)g^p(t)dt,$$

where $\Omega(t)=\int_t^\infty \lambda(s)ds$.

REMARK 2. As a special case of (2.5), when $\mathbb{T}=\mathbb{N}$ and $c\leqslant 0<p<1$ and $a=1$, we have the following discrete inequality of Leindler type

$$\sum_{n=1}^\infty \frac{\lambda(n)}{\Omega^c(n)}\left(\sum_{k=1}^n \lambda(k)g(k)\right)^p \geqslant \left(\frac{p}{1-c}\right)^p \sum_{n=1}^\infty \lambda(n)\left(\sum_{k=n}^\infty \lambda(k)\right)^{p-c} g^p(n),$$

where $\Omega(n)=\sum_{k=n}^\infty \lambda(k)$.

In the following, we will prove the time scale version of a converse of the inequality (1.6) due to Leindler on time scales.

THEOREM 2.2. *Let $\mathbb{T}$ be a time scale with $a\in(0,\infty)_{\mathbb{T}}$ and $0<p<1<c$. Let $\Omega(t)$ be defined as in Theorem 2.1 such that*

$$K=\inf_{t\in\mathbb{T}}\frac{\Omega^\sigma(t)}{\Omega(t)}>0, \tag{2.14}$$

and define $\overline{\Psi}(t):=\int_t^\infty \lambda(s)g(s)\Delta s$. Then

$$\int_a^\infty \frac{\lambda(t)}{\Omega^c(t)}(\overline{\Psi}(t))^p\Delta t \geqslant \left(\frac{pK^c}{c-1}\right)^p \left[\int_a^\infty (\Omega(t))^{p-c}g^p(t)\lambda(t)\Delta t\right]. \tag{2.15}$$

Proof. Integrating the left hand side of (2.15) by the parts formula (2.3) with $v^\Delta(t)=\lambda(t)/\Omega^c(t)$ and $u(t)=\left(\overline{\Psi}(t)\right)^p$, we obtain

$$\int_a^\infty \frac{\lambda(t)}{\Omega^c(t)}(\overline{\Psi}(t))^p\Delta t = v(t)\overline{\Psi}^p(t)\Big|_a^\infty + \int_a^\infty (v^\sigma(t))(-\overline{\Psi}^p(t))^\Delta\Delta t,$$

where $v(t)=\int_a^t \frac{\lambda(s)}{\Omega^c(s)}\Delta s$. This with $\overline{\Psi}(\infty)=0$ and $v(a)=0$ imply that

$$\int_a^\infty \frac{\lambda(t)}{\Omega^c(t)}(\overline{\Psi}(t))^p\Delta t=\int_a^\infty v^\sigma(t)(-\overline{\Psi}^p(t))^\Delta\Delta t. \tag{2.16}$$

Applying the chain rule ([5, Theorem 1.87]) $f^\Delta(\delta(t))=f^{'}(\delta(d))\delta^\Delta(t)$, where $d\in[t,\sigma(t)]$, we see that there exists $d\in[t,\sigma(t)]$ such that

$$(-\overline{\Psi}^p(t))^\Delta=\frac{p}{\overline{\Psi}^{1-p}(d)}(-\overline{\Psi}^\Delta(t))=\frac{pg(t)\lambda(t)}{\overline{\Psi}^{1-p}(d)}. \tag{2.17}$$

Since $\overline{\Psi}^\Delta(t)=-\lambda(t)g(t)\leqslant 0$, and $d\geqslant t$, we see that $\overline{\Psi}(t)\geqslant\overline{\Psi}(d)$, and then

$$\frac{p}{\overline{\Psi}^{1-p}(d)}\geqslant\frac{p}{(\overline{\Psi}(t))^{1-p}}\quad(\text{note } 0<p<1). \tag{2.18}$$

Combining (2.17) and (2.18), we have that

$$(-\overline{\Psi}^p(t))^\Delta\geqslant\frac{pg(t)\lambda(t)}{(\overline{\Psi}(t))^{1-p}}. \tag{2.19}$$

Substituting (2.19) into (2.16) and using the fact that $v^\Delta(t)\geqslant 0$, we have that

$$\int_a^\infty \frac{\lambda(t)}{\Omega^c(t)}(\overline{\Psi}(t))^p\Delta t\geqslant p\int_a^\infty v(t)\frac{g(t)\lambda(t)}{(\overline{\Psi}(t))^{1-p}}\Delta t. \tag{2.20}$$

By (2.14) and the chain rule (2.2), since $(\Omega(t))^\Delta=-\lambda(t)\leqslant 0$ and $c>1$, we see that

$$\begin{aligned}\left((\Omega(t))^{1-c}\right)^\Delta&=(1-c)\int_0^1\frac{-\lambda(t)}{[h\Omega^\sigma(t)+(1-h)\Omega(t)]^c}dh\\&=(c-1)\int_0^1\frac{\lambda(t)}{[h\Omega^\sigma(t)+(1-h)\Omega(t)]^c}dh\\&\leqslant(c-1)\frac{\lambda(t)}{[\Omega^\sigma(t)]^c}\\&=(c-1)\frac{\lambda(t)}{[\Omega(t)]^c}\frac{[\Omega(t)]^c}{[\Omega^\sigma(t)]^c}\\&\leqslant\frac{(c-1)}{K^c}\frac{\lambda(t)}{[\Omega(t)]^c}.\end{aligned}$$

This implies that

$$v(t)=\int_a^t\frac{\lambda(s)}{(\Omega(s))^c}\Delta s\geqslant\left(\frac{K^c}{c-1}\right)\int_a^t\left(\Omega^{1-c}(s)\right)^\Delta\Delta s=\left(\frac{K^c}{c-1}\right)(\Omega(t))^{1-c}. \tag{2.21}$$

Substituting (2.21) into (2.20) yields

$$\left(\int_a^\infty\frac{\lambda(t)}{\Omega^c(t)}\left(\overline{\Psi}(t)\right)^p\Delta t\right)^p\geqslant\left(\frac{pK^c}{c-1}\right)^p\left[\int_a^\infty\left(\frac{g^p(t)\lambda^p(t)}{(\Omega(t))^{p(c-1)}(\overline{\Psi}(t))^{p(1-p)}}\right)^{1/p}\Delta t\right]^p.$$

The rest of the proof is similar to the proof of Theorem 2.1 and hence is omitted. The proof is complete. □

REMARK 3. As a special case of (2.15), when $\mathbb{T}=\mathbb{R}$ and $0<p<1<c$ and $a=1$, we have the following inequality of Leindler type (note that in $\mathbb{R}$ we have $\Omega^{\sigma}(t)=\Omega(t)$ and so $K=1$)

$$\int_1^{\infty}\frac{\lambda(t)}{\Omega^c(t)}\left(\int_t^{\infty}\lambda(s)g(s)ds\right)^p dt \geqslant \left(\frac{p}{c-1}\right)^p\int_1^{\infty}\lambda(t)\Omega^{p-c}(t)g^p(t)dt,$$

where $\Omega(t)=\int_t^{\infty}\lambda(s)ds$.

REMARK 4. Assume that $\mathbb{T}=\mathbb{N}$ in Theorem 2.2 and $0<p\leqslant 1<c$ and $a=1$. In this case inequality (2.15) becomes the following Leindler type discrete inequality

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{\Omega^c(n)}\left(\sum_{k=n}^{\infty}\lambda(k)g(k)\right)^p \geqslant \left(\frac{pK^c}{c-1}\right)^p\sum_{n=1}^{\infty}\lambda(n)\left(\sum_{k=n}^{\infty}\lambda(k)\right)^{p-c}g^p(n),$$

where $\Omega(n)=\sum\limits_{k=n}^{\infty}\lambda(k)$ and $K=\inf_{n\in\mathbb{N}}\Omega(n+1)/\Omega(n)$.

In the following theorem, we will prove a time scale version of the Bennett-Leindler inequality (1.7) on time scales.

THEOREM 2.3. *Let $\mathbb{T}$ be a time scale with $a\in(0,\infty)_{\mathbb{T}}$ and $c\leqslant 0<p<1$. Let $\Lambda(t)=\int_a^t\lambda(s)\Delta s$, and*

$$\overline{\Psi}(t)=\int_t^{\infty}\lambda(s)g(s)\Delta s. \tag{2.22}$$

Then

$$\int_a^{\infty}\frac{\lambda(t)}{(\Lambda^{\sigma}(t))^c}(\overline{\Psi}(t))^p\Delta t \geqslant \left(\frac{p}{1-c}\right)^p\int_a^{\infty}\lambda(t)(\Lambda^{\sigma}(t))^{p-c}g^p(t)\Delta t. \tag{2.23}$$

Proof. Integrating the left hand side of (2.23) by the parts formula (2.3) with $v^{\Delta}(t)=\lambda(t)/(\Lambda^{\sigma}(t))^c$, and $u(t)=\left(\overline{\Psi}(t)\right)^p$, we obtain

$$\int_a^{\infty}\frac{\lambda(t)}{(\Lambda^{\sigma}(t))^c}(\overline{\Psi}(t))^p\Delta t = v(t)\overline{\Psi}^p(t)\Big|_a^{\infty}+\int_a^{\infty}(v^{\sigma}(t))(-\overline{\Psi}^p(t))^{\Delta}\Delta t, \tag{2.24}$$

where $v(t)=\int_a^t\frac{\lambda(s)}{(\Lambda^{\sigma}(s))^c}\Delta s$. From the inequality (2.24) and $\overline{\Psi}(\infty)=v(a)=0$, we have

$$\int_a^{\infty}\frac{\lambda(t)}{(\Lambda^{\sigma}(t))^c}(\overline{\Psi}(t))^p\Delta t = \int_a^{\infty}v^{\sigma}(t)(-\overline{\Psi}^p(t))^{\Delta}\Delta t. \tag{2.25}$$

Applying the chain rule $f^{\Delta}(\delta(t))=f'(\delta(d))\delta^{\Delta}(t)$, where $d\in[t,\sigma(t)]$, we see that there exists $d\in[t,\sigma(t)]$ such that

$$-(\overline{\Psi}^p(t))^{\Delta}=\frac{-p}{\overline{\Psi}^{1-p}(d)}(\overline{\Psi}^{\Delta}(t))=\frac{p\lambda(t)g(t)}{\overline{\Psi}^{1-p}(d)}. \tag{2.26}$$

Since $\overline{\Psi}^{\Delta}(t) = -\lambda(t)g(t) \leqslant 0$, and $d \geqslant t$, we see that $\overline{\Psi}(t) \geqslant \overline{\Psi}(d)$, and then

$$\frac{p\lambda(t)g(t)}{\overline{\Psi}^{1-p}(d)} \geqslant \frac{p\lambda(t)g(t)}{(\overline{\Psi}(t))^{1-p}} \quad \text{(note } 0 < p < 1\text{)}.$$

This and (2.26) imply that

$$(-\overline{\Psi}^{p}(t))^{\Delta} \geqslant \frac{pg(t)\lambda(t)}{(\overline{\Psi}(t))^{1-p}}. \tag{2.27}$$

By the chain rule (2.2) and the fact that $(\Lambda(t))^{\Delta} = \lambda(t) \geqslant 0$ and $c \leqslant 0$, we see that

$$\begin{aligned}\left((\Lambda(t))^{1-c}\right)^{\Delta} &= (1-c)\int_0^1 \frac{\lambda(t)}{[h\Lambda^{\sigma}(t)+(1-h)\Lambda(t)]^c}dh \\ &\leqslant (1-c)\int_0^1 \frac{\lambda(t)}{[h\Lambda^{\sigma}(t)+(1-h)\Lambda^{\sigma}(t)]^c}dh \\ &= (1-c)\frac{\lambda(t)}{[\Lambda^{\sigma}(t)]^c}.\end{aligned}$$

This implies that

$$\begin{aligned} v^{\sigma}(t) &= \int_a^{\sigma(t)} \frac{\lambda(s)}{(\Lambda^{\sigma}(s))^c}\Delta s \\ &\geqslant \left(\frac{1}{1-c}\right)\int_a^{\sigma(t)} \left(\Lambda^{1-c}(s)\right)^{\Delta}\Delta s \\ &= \left(\frac{1}{1-c}\right)(\Lambda^{\sigma}(t))^{1-c}. \end{aligned} \tag{2.28}$$

Substituting (2.27) and (2.28) into (2.25) yields

$$\left(\int_a^{\infty} \frac{\lambda(t)}{(\Lambda^{\sigma}(t))^c}\left(\overline{\Psi}(t)\right)^p \Delta t\right)^p \geqslant \left(\frac{p}{1-c}\right)^p \left[\int_a^{\infty}\left(\frac{g^p(t)\lambda^p(t)}{(\overline{\Psi}(t))^{p(1-p)}(\Lambda^{\sigma}(t))^{p(c-1)}}\right)^{1/p}\Delta t\right]^p.$$

The rest of the proof is similar to the proof of Theorem 2.1 and hence is omitted. The proof is complete. □

REMARK 5. Assume that $\mathbb{T} = \mathbb{R}$ in Theorem 2.3, $c \leqslant 0 < p < 1$ and $a = 1$. In this case, we have the following integral inequality of Bennett-Leindler type (note that when $\mathbb{T} = \mathbb{R}$, we have $\Lambda^{\sigma}(t) = \Lambda(t) = \int_a^t \lambda(s)ds$)

$$\int_1^{\infty} \frac{\lambda(t)}{(\Lambda(t))^c}\left(\int_t^{\infty}\lambda(s)g(s)ds\right)^p dt \geqslant \left(\frac{p}{1-c}\right)^p \int_1^{\infty}\lambda(t)\left(\Lambda(t)\right)^{p-c} g^p(t)dt.$$

REMARK 6. Assume that $\mathbb{T}=\mathbb{N}$ in Theorem 2.3, $c \leqslant 0<p<1$ and $a=1$. In this case inequality (2.23) becomes the following discrete Bennett-Leindler inequality

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{(\Lambda(n))^{c}}\left(\sum_{k=n}^{\infty} \lambda(k) g(k)\right)^{p} \geqslant\left(\frac{p}{1-c}\right)^{p} \sum_{n=1}^{\infty} \lambda(n)\left(\sum_{k=1}^{n} \lambda(k)\right)^{p-c} g^{p}(n),$$

where $\Lambda(n)=\sum_{k=1}^{n} \lambda(n)$.

In the following theorem, we will prove a time scale version of the Bennett-Leindler inequality (1.8) on time scales.

THEOREM 2.4. *Let $\mathbb{T}$ be a time scale with $a \in(0, \infty)_{\mathbb{T}}$ and $0<p<1<c$. Let $\bar{\Phi}(t):=\int_{a}^{t} \lambda(t) g(s) \Delta s$, and $\Lambda(t)=\int_{a}^{t} \lambda(s) \Delta s$, and assume that $\Lambda(\infty)=\infty$. Then*

$$\int_{a}^{\infty} \frac{\lambda(t)}{(\Lambda(t))^{c}}(\bar{\Phi}^{\sigma}(t))^{p} \Delta t \geqslant\left(\frac{p}{c-1}\right)^{p} \int_{a}^{\infty} \lambda(t)(\Lambda(t))^{p-c} g^{p}(t) \Delta t. \tag{2.29}$$

Proof. Integrating the left hand side of (2.23) by the parts formula (2.3) with $u^{\Delta}(t)=\frac{\lambda(t)}{(\Lambda^{\sigma}(t))^{c}}$, and $v^{\sigma}(t)=\left(\bar{\Phi}^{\sigma}(t)\right)^{p}$, we obtain

$$\int_{a}^{\infty} \frac{\lambda(t)}{(\Lambda(t))^{c}}(\bar{\Phi}^{\sigma}(t))^{p} \Delta t=u(t) \bar{\Phi}^{p}(t)\Big|_{a}^{\infty}+\int_{a}^{\infty}(-u(t))(\bar{\Phi}^{p}(t))^{\Delta} \Delta t,$$

where $u(t)=-\int_{t}^{\infty} \frac{\lambda(s)}{(\Lambda(s))^{c}} \Delta s$. From $\bar{\Phi}(a)=u(\infty)=0$, we have

$$\int_{a}^{\infty} \frac{\lambda(t)}{(\Lambda(t))^{c}}(\bar{\Phi}^{\sigma}(t))^{p} \Delta t=\int_{a}^{\infty}(-u(t))(\bar{\Phi}^{p}(t))^{\Delta} \Delta t. \tag{2.30}$$

Applying the chain rule $f^{\Delta}(\delta(t))=f^{'}(\delta(d)) \delta^{\Delta}(t)$, where $d \in[t, \sigma(t)]$, we see that there exists $d \in[t, \sigma(t)]$ such that

$$\left(\bar{\Phi}^{p}(t)\right)^{\Delta}=\frac{p}{\bar{\Phi}^{1-p}(d)}(\bar{\Phi}^{\Delta}(t))=\frac{p \lambda(t) g(t)}{\bar{\Phi}^{1-p}(d)}. \tag{2.31}$$

Since $\bar{\Phi}^{\Delta}(t)=\lambda(t) g(t) \geqslant 0$, and $\sigma(t) \geqslant d$, we see that $\bar{\Phi}^{\sigma}(t) \geqslant \bar{\Phi}(d)$, and then

$$\frac{p \lambda(t) g(t)}{\bar{\Phi}^{1-p}(d)} \geqslant \frac{p \lambda(t) g(t)}{(\bar{\Phi}^{\sigma}(t))^{1-p}} \quad (\text{note } 0<p<1).$$

This and (2.31) implies that

$$\left(\bar{\Phi}^{p}(t)\right)^{\Delta} \geqslant \frac{p \lambda(t) g(t)}{(\bar{\Phi}^{\sigma}(t))^{1-p}}. \tag{2.32}$$

By the chain rule (2.2) and the fact that $(\Lambda(t))^{\Delta} = \lambda(t) \geqslant 0$ and $c > 1$, we see that

$$\begin{aligned}
\left((\Lambda(t))^{1-c}\right)^{\Delta} &= (1-c)\int_0^1 \frac{\lambda(t)}{[h\Lambda^{\sigma}(t)+(1-h)\Lambda(t)]^c}dh \\
&= -(c-1)\int_0^1 \frac{\lambda(t)}{[h\Lambda^{\sigma}(t)+(1-h)\Lambda(t)]^c}dh \\
&\geqslant -(c-1)\int_0^1 \frac{\lambda(t)}{[h\Lambda(t)+(1-h)\Lambda(t)]^c}dh \\
&= -(c-1)\frac{\lambda(t)}{[\Lambda(t)]^c}.
\end{aligned}$$

This and the condition $\Lambda(\infty) = \infty$ imply that

$$\begin{aligned}
u(t) &= -\int_t^{\infty} \frac{\lambda(s)}{(\Lambda(s))^c}\Delta s \\
&\leqslant \left(\frac{1}{c-1}\right)\int_t^{\infty}\left(\Lambda^{1-c}(s)\right)^{\Delta}\Delta s \\
&= \left(\frac{1}{c-1}\right)\frac{1}{\Lambda^{c-1}(\infty)} - \left(\frac{1}{c-1}\right)\frac{1}{\Lambda^{c-1}(t)} \\
&= -\left(\frac{1}{c-1}\right)\frac{1}{\Lambda^{c-1}(t)}.
\end{aligned}$$

Then

$$-u(t) \geqslant \left(\frac{1}{c-1}\right)\frac{1}{\Lambda^{c-1}(t)}. \tag{2.33}$$

Substituting (2.33) and (2.32) into (2.30) yields

$$\left(\int_a^{\infty}\frac{\lambda(t)}{(\Lambda(t))^c}\left(\overline{\Phi}^{\sigma}(t)\right)^p\Delta t\right)^p \geqslant \left(\frac{p}{c-1}\right)^p\left[\int_a^{\infty}\left(\frac{g^p(t)\lambda^p(t)}{(\overline{\Phi}^{\sigma}(t))^{p(1-p)}(\Lambda(t))^{p(c-1)}}\right)^{1/p}\Delta t\right]^p.$$

The rest of the proof is similar to the proof of Theorem 2.1 and hence is omitted. The proof is complete. □

REMARK 7. Assume that $\mathbb{T} = \mathbb{R}$ in Theorem 2.4, $0 < p < 1 < c$ and $a = 1$. In this case, we have the following integral inequality of Leindler type (note that when $\mathbb{T} = \mathbb{R}$, we have $\Lambda(t) = \int_a^t \lambda(s)ds$)

$$\int_1^{\infty}\frac{\lambda(t)}{(\Lambda(t))^c}\left(\int_a^t \lambda(t)g(s)ds\right)^p dt \geqslant \left(\frac{p}{c-1}\right)^p\int_1^{\infty}\lambda(t)(\Lambda(t))^{p-c}g^p(t)dt. \tag{2.34}$$

REMARK 8. Assume that $\mathbb{T} = \mathbb{N}$ in Theorem 2.4, $0 < p < 1 < c$ and $a = 1$. In this case inequality (2.29) becomes the following discrete Leindler inequality,

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{(\Lambda(n))^c}\left(\sum_{k=1}^{n}\lambda(k)g(k)\right)^p\Delta t \geqslant \left(\frac{p}{c-1}\right)^p\sum_{n=1}^{\infty}\lambda(n)(\Lambda(n))^{p-c}g^p(n). \tag{2.35}$$

where $\Lambda(n) = \sum_{k=1}^{n-1} \lambda(n)$.

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