Mathematical
Inequalities
& Applications
Volume 18, Number 1 (2015), 267–293 doi:10.7153/mia-18-20

ON STRONG DELTA–CONVEXITY AND HERMITE–HADAMARD TYPE INEQUALITIES FOR DELTA–CONVEX FUNCTIONS OF HIGHER ORDER

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(*Communicated by Z. Páles*)

Abstract. In our previous paper [15], using s-convex stochastic ordering [4], we investigate Hermite-Hadamard-Fejér type inequalities in the case of higher order convex functions. In the present paper, our aim is to extend this investigation from convex to delta-convex functions of higher order [8]. We offer some useful tools for obtaining and proving of various forms of the Hermite-Hadamard-Fejér type inequalities for delta-convex functions of higher order, that generalizes results of Dragomir et al. [5]. These results are applied to derive some inequalities between quadrature operators. We define also and study strong delta-convexity of n-th order that generalizes strong n-convexity studied in [14] and [9].

1. Introduction

Delta-convex functions of n-th order are extensions of delta-convex functions, which are functions representable as a difference of two convex functions (see [17]). The notion of a delta-convex function of n-th order is a particular case of the notion of a delta-convex mapping of n-th order between two normed linear spaces. The latter were introduced in R. Ger (1994) [8] as an extension of delta-convex mappings (see [19]). Delta-convex functions of n-th order are functions that are representable as a difference of two n-convex functions (see [8]).

In the following, let n be a fixed positive integer. Let $I \subseteq \mathbb{R}$ be an open interval. It is well known that continuous solutions $f\colon I \to \mathbb{R}$ of the functional inequality

$$\Delta_h^{n+1} f(x) \geqslant 0, \tag{1.1}$$

where $x \in I$, $h > 0$, $x + (n+1)h \in I$, and Δ_h^{n+1} stands for the $(n+1)$-th iterate of the difference operator $\Delta_h f(x) = f(x+h) - f(x)$, are just C^{n-1} – functions whose derivatives $f^{(n-1)}(x)$ are convex (see e.g. M. Kuczma [11, Chapter XV]). Therefore, the continuous solutions of (1.1) are used to be called *n-convex functions*.

If f is n-convex, then the right derivative of n-th order $f_R^{(n)}(x)$ exist for all $x \in I$. Henceforth we drop the subscript R, and $f^{(n)}(x)$ will be used to denote $f_R^{(n)}(x)$.

Mathematics subject classification (2010): 26A51, 39B62.
Keywords and phrases: higher-order convexity, delta-convexity of higher order, control function, strong convexity, Hermite-Hadamard-Fejér type inequality.

PROPOSITION 1.1. *A function* $f\colon I \to \mathbb{R}$ *is* n*-convex if and only if* $f^{(n)}$ *is non-decreasing.*

PROPOSITION 1.2. *Let* $f\colon I \to \mathbb{R}$ *be a function of the class* C^{n+1} *in* I. *Then* f *is* n*-convex if and only if* $f^{(n+1)}(x) \geqslant 0$ *for all* $x \in I$.

Let Π_n be the family of all polynomials of degree at most n. Recall that, for $x \in \mathbb{R}$, we have $x_+ = \max\{x,0\}$ and $x_+^n = (x_+)^n$. The integral representation of n-convex functions given in [14] (p.740, Theorem 2.10) can be written in the following form.

PROPOSITION 1.3. ([14]) *Let* $f\colon (a,b) \to \mathbb{R}$ *be a function and let* $\xi \in (a,b)$. *Then* f *is* n*-convex if and only if* f *has the representation*

$$f(x) = \int_{(a,\xi)} (-1)^{n+1} \frac{[-(x-u)]_+^n}{n!} \mu_{(n)}(du) + \int_{[\xi,b)} \frac{(x-u)_+^n}{n!} \mu_{(n)}(du) + Q_\xi(x), \quad (1.2)$$

where $\mu_{(n)}$ *is a Borel measure on* $\mathscr{B}((a,b))$ *such that* $\mu_{(n)}((c,d)) < \infty$ *for all* $a < c < d < b$, *and* $Q_\xi \in \Pi_n$. *Moreover, we have that*

$$\mu_{(n)}(du) = d f^{(n)}(u). \quad (1.3)$$

The measure $\mu_{(n)}$ *is unique, i.e. if* ξ_1, $\xi_2 \in (a,b)$ *and two triplets* $(\xi_1, \mu_{(n),1}, Q_{\xi_1})$ *and* $(\xi_2, \mu_{(n),2}, Q_{\xi_2})$ *correspond to* f *in the representation* (1.2), *then* $\mu_{(n),1} = \mu_{(n),2}$.

PROPOSITION 1.4. *Let* f_1 *and* f_2 *be two* n*-convex functions with the triplets* $(\xi_1, \mu_{(n),1}, Q_{\xi_1})$ *and* $(\xi_2, \mu_{(n),2}, Q_{\xi_2})$, *respectively, in the representation* (1.2). *Then* $f_1 - f_2 \in \Pi_n$ *if and only if* $\mu_{(n),1} = \mu_{(n),2}$.

DEFINITION 1.1. We will call the measure $\mu_{(n)}$, which we have introduced in Proposition 1.3, the n*-spectral measure* of the n-convex function f.

REMARK 1.1. ([14]) Note that, by (1.3), $f^{(n)}(x)$ is a distribution function corresponding to the n-spectral measure $\mu_{(n)}$. Furthermore, the measure $\mu_{(n)}$ can be regarded as the measure of n-th order convexity of the function f. Moreover, if f is of the class C^{n+1} in (a,b) then

$$\mu_{(n)}(du) = f^{(n+1)}(u)du. \quad (1.4)$$

Conversely, if $\mu_{(n)}$ is of the form (1.4), then f is of the class C^{n+1} in (a,b).

DEFINITION 1.2. ([8]) A function $f\colon (a,b) \to \mathbb{R}$ is called *delta-convex of* n*-th order*, if there exists an n-convex function g such that, for all $x, y \in (a,b)$,

$$x \leqslant y \Rightarrow \left| \Delta^{n+1}_{\frac{y-x}{n+1}} f(x) \right| \leqslant \Delta^{n+1}_{\frac{y-x}{n+1}} g(x). \quad (1.5)$$

PROPOSITION 1.5. ([8]) *A function* $f\colon (a,b) \to \mathbb{R}$ *is delta-convex of* n*-th order if and only if* f *is a difference of two* n*-convex functions on* (a,b).

PROPOSITION 1.6. ([8]) *Every* C^{n+1}*-function* $f\colon (a,b) \to \mathbb{R}$ *is delta-convex of* n*-th order.*

DEFINITION 1.3. Every function g satisfying (1.5) is called a *control function for* f, or we say that the function f is *a delta-convex function of* n*-th order* with the control function g, as well as we say that f is g*-convex dominated of* n*-th order* (briefly *delta-convex* or g*-convex dominated* when $n = 1$).

It is not difficult to prove the following lemma (see [8]).

LEMMA 1.1. *Let* $g\colon (a,b) \to \mathbb{R}$ *be an* n*-convex function and let* $f\colon (a,b) \to \mathbb{R}$ *be a function. Then the following statements are equivalent:*

(a) f *is delta-convex of* n*-th order with the control function* g,

(b) *the functions* $g - f$ *and* $g + f$ *are* n*-convex on* (a,b),

(c) *there exist two* n*-convex functions* $\varphi_1, \varphi_2\colon (a,b) \to \mathbb{R}$ *such that*

$$f = \varphi_1 - \varphi_2 \quad \text{and} \quad g = \varphi_1 + \varphi_2.$$

The following integral representation of a delta-convex function f (in the case of $f'(x)$ of bounded variation) can be found in [17].

PROPOSITION 1.7. ([17]) *A function* $f\colon [a,b] \to \mathbb{R}$ *is delta-convex having the decomposition* $f = \varphi_1 - \varphi_2$, *where* $\varphi_1, \varphi_2\colon [a,b] \to \mathbb{R}$ *are both convex and have finite endpoint derivatives, if and only if* $f(x) = f(a) + \int_a^x r(u)du$, *for some* $r\colon [a,b] \to \mathbb{R}$ *of bounded variation.*

In this paper we give an analogous integral representation for n-th order delta-convex functions f, in general case, without any additional assumptions on $f^{(n)}(x)$ (see Section 2). Our characterization is constructive. We give explicit formulas for an n-spectral signed measure corresponding to f in this representation. This integral representation will be applied to obtain a characterization of control functions corresponding to f, to define some canonical decomposition of f, and to show the existence and to study properties of a minimal control function for f (which generalizes results of Hartman [10] for convex functions). We find the minimum and maximum of two control functions corresponding to f (in the sense defined in the paper). We give also a simpler proof of Ger's theorem [8] (see Proposition 1.6) on delta-convexity of higher orders of functions of the class C^{n+1}. The strength of the representation developed in Section 2 is exploited in the rest of the paper. It is used to further study of n-th order delta-convexity, to define and study relative delta-convexity relation and strong delta-convexity of higher order, among others.

In Section 3 we define the relative delta-convexity relation of n-th order (the n-delta-convexity relation), which is a generalization of the n-convexity relation introduced in [14]). This relation induces a partial ordering on some equivalence classes of delta-convex functions of n-th order. We give a characterization of the n-delta-convexity relation in terms of minimal control functions, in terms of n-spectral signed measures, as well as in terms of derivatives of $(n+1)$-th order (which exist almost everywhere with respect to the Lebesgue measure). We define and study the notion of strong delta-convexity of n-th order that generalizes strong n-convexity studied in [14] and [9]. We give a characterization of strong delta-convexity of n-th order in general case, without any additional assumptions of differentiability of functions (which extend results in [14] concerning strong n-convexity).

In Section 4 we study Hermite-Hadamard-Fejér type inequalities concerning delta-convex functions of n-th order. We give a probabilistic characterization of 1-delta-convexity (i.e. usual delta-convexity), which is a generalization of the well known Jensen inequality concerning convex functions. Using this probabilistic characterization we obtain some Jensen-type inequalities for delta-convex functions. We give also an extension of very useful criterion for the verification of the s-convex order, which is given by Denuit, Lefèvre and Shaked in [4], from convex to delta-convex functions of higher order. Our theorem provides a useful tool for obtaining and proving of various forms of the Hermite-Hadamard-Fejér type inequalities concerning delta-convex functions of higher order. Then, considering some particular cases of random variables occurring in our criterion, we obtain a generalization of the well known results of Dragomir, Pearce and Pečarić [5] concerning delta-convex functions, and results obtained in [15] for convex functions of higher order.

Finally, in Section 5, our results are applied to obtain some inequalities between quadrature operators for delta-convex functions of n-th order.

2. Integral representation

In the following theorem we give an integral representation of a delta-convex function f of n-th order. This representation is a generalization and extension of Proposition 1.7 to the higher order convex functions.

THEOREM 2.1. *Let $f\colon (a,b)\to\mathbb{R}$ be a function and let $\xi\in(a,b)$. Then f is delta-convex of n-th order if and only if f has the representation*

$$f(x)=\int_{(a,\xi)}(-1)^{n+1}\frac{[-(x-u)]_+^n}{n!}\tau_{(n)}(du)+\int_{[\xi,b)}\frac{(x-u)_+^n}{n!}\tau_{(n)}(du)+Q_\xi(x), \tag{2.1}$$

where $\tau_{(n)}$ is a signed measure on $\mathscr{B}((a,b))$ such that $-\infty<\tau_{(n)}((c,d))<\infty$ for all $a<c<d<b$, and $Q_\xi\in\Pi_n$. Moreover, the measure $\tau_{(n)}$ is unique, i.e. if ξ_1, $\xi_2\in(a,b)$ and two triplets $(\xi_1,\tau_{(n),1},Q_{\xi_1})$ and $(\xi_2,\tau_{(n),2},Q_{\xi_2})$ correspond to f in the representation (2.1), then $\tau_{(n)1}=\tau_{(n)2}$. If $f=\varphi_1-\varphi_2$, where $\varphi_1,\varphi_2\colon(a,b)\to\mathbb{R}$ are both n-convex, then $\tau_{(n)}=\mu_{(n)1}-\mu_{(n)2}$, $\mu_{(n)1}(du)=d\varphi_1^{(n)}(u)$ and $\mu_{(n)2}(du)=d\varphi_2^{(n)}(u)$.

Proof. $\Leftarrow$. Let f be of the form (2.1) with a signed measure $\tau_{(n)}$ and a polynomial $Q_\xi \in \Pi_n$. Taking any measures $\mu_{(n)1}$ and $\mu_{(n)2}$ such that

$$\tau_{(n)} = \mu_{(n)1} - \mu_{(n)2},$$

we obtain that f can be written in the form $f = \varphi_1 - \varphi_2$, where

$$\varphi_1(x) = \int_{(a,\xi)} (-1)^{n+1} \frac{[-(x-u)]_+^n}{n!} \mu_{(n)1}(du) + \int_{[\xi,b)} \frac{(x-u)_+^n}{n!} \mu_{(n)1}(du) + Q_\xi(x),$$
$$\varphi_2(x) = \int_{(a,\xi)} (-1)^{n+1} \frac{[-(x-u)]_+^n}{n!} \mu_{(n)2}(du) + \int_{[\xi,b)} \frac{(x-u)_+^n}{n!} \mu_{(n)2}(du).$$

By Proposition 1.3, φ_1 and φ_2 are both n-convex, consequently f is delta-convex of n-th order.

$\Rightarrow$. Let f be delta-convex of n-th order. Then $f = \varphi_1 - \varphi_2$, where $\varphi_1, \varphi_2 \colon (a,b) \to \mathbb{R}$ are both n-convex. By Proposition 1.3, the functions φ_i, are of the form (1.2) with φ_i, $\mu_{(n)i}(du) = d\varphi_i^{(n)}(u)$ and $Q_{\xi,i}$, in place of f, $\mu_{(n)}$ and Q_ξ, respectively, $i = 1,2$. Consequently, taking $\tau_{(n)} = \mu_{(n)1} - \mu_{(n)2}$ and $Q_\xi = Q_{\xi,1} - Q_{\xi,2}$, we obtain that f is of the form (2.1) with the signed measure $\tau_{(n)}$ and $Q_\xi \in \Pi_n$, which was to be proved.

The uniqueness of the measure $\tau_{(n)}$ in the representation (2.1) follows from the uniqueness of the n-spectral measure in the representation (1.2) of an n-convex function. Let ξ_1, $\xi_2 \in (a,b)$. Let the function f be delta-convex of n-th order. Then $f = \varphi_1 - \varphi_2$, where $\varphi_1, \varphi_2 \colon (a,b) \to \mathbb{R}$ are both n-convex. Let $(\xi_1, \mu_{(n),1}, Q_{\xi_1,1})$, $(\xi_2, \mu_{(n),1}, Q_{\xi_2,1})$ and $(\xi_1, \mu_{(n),2}, Q_{\xi_1,2})$, $(\xi_2, \mu_{(n),2}, Q_{\xi_2,2})$, be two triplets corresponding to φ_1 and φ_2, respectively, in the representation (1.2). Then we obtain that to f there correspond two triplets in the representation (2.1): $(\xi_1, \tau_{(n),1}, Q_{\xi_1})$, $(\xi_2, \tau_{(n),2}, Q_{\xi_2})$, where $\tau_{(n),1} = \tau_{(n),2} = \mu_{(n),1} - \mu_{(n),2}$, $Q_{\xi_1} = Q_{\xi_1,1} - Q_{\xi_1,2}$, $Q_{\xi_2} = Q_{\xi_2,1} - Q_{\xi_2,2}$. Thus the uniqueness of the measure $\tau_{(n)}$ in the representation (2.1) is proved. $\square$

DEFINITION 2.1. We will call $\tau_{(n)}$ the *n-spectral signed measure* of a delta convex function f of n-th order.

REMARK 2.1. Note that, by Remark 1.1, if the delta-convex function f of n-th order is of the class C^{n+1} in (a,b), and $\tau_{(n)}$ is the signed measure that appears in Theorem 2.1, then

$$\tau_{(n)}(du) = f^{(n+1)}(u)du.$$

In the following lemma we show that the set of n-convex functions of the class C^{n+1} in (a,b) is dense in the set of n-convex functions in (a,b).

LEMMA 2.1. *Let $f \colon (a,b) \to \mathbb{R}$ be an n-convex function. Then there exist a sequence $\{f_k\}$ of n-convex functions of the class C^{n+1} in (a,b), such that* $\lim_{k\to\infty} f_k(x) = f(x)$ $(x \in (a,b))$.

Proof. Let $\xi \in (a,b)$. Let $f \colon (a,b) \to \mathbb{R}$ be an n-convex function with the triplet $(\xi, \mu_{(n)}, Q_\xi)$ in the representation (1.2). Let $\{a_N\}$, $\{b_N\}$ be two sequences of real numbers such that $a < a_N < b_N < b$ $(N \in \mathbb{N})$, $a_N \downarrow a$ and $b_N \uparrow b$ as $N \to \infty$. It suffices to prove the lemma for the function $f_{(N)}\colon (a,b) \to \mathbb{R}$ with the spectral measure $\mu_{(n),(N)} = \mu_{(n)}|_{A_N}$ concentrated on the set $A_N = (a + a_N, b - b_N)$ $(N \in \mathbb{N})$, and then letting $N \to \infty$ we conclude the lemma for the function f with the spectral measure $\mu_{(n)}$. Similarly, it suffices to consider the case when both $a, b \in \mathbb{R}$. Note that all the measures $\mu_{(n),(N)}$ $(N \in \mathbb{N})$ are finite.

Fix $N \in \mathbb{N}$. Let $\{\mu_{(n),(N),k}\}$ be the sequence of measures absolutely continuous with respect to the Lebesgue measure such that they all are concentrated on the set A_N, $\mu_{(n),(N),k}(A_N) = \mu_{(n),(N)}(A_N)$ $(k \in \mathbb{N})$ and $\mu_{(n),(N),k} \Rightarrow \mu_{(n),(N)}$ as $k \to \infty$ ($\mu_{(n),(N),k}$ converges weakly to $\mu_{(n),(N)}$ as $k \to \infty$). Let $f_{(N),k}\colon (a,b) \to \mathbb{R}$ be the n-convex functions given by (1.2) with the triplet $(\xi, \mu_{(n),(N),k}, Q_\xi)$ in the representation (1.2). By Remark 1.1, the functions $f_{(N),k}$ $(k \in \mathbb{N})$ are of the class C^{n+1} in (a,b). Note that both the functions $f_{1,x}(u) = (-1)^{n+1}\frac{[-(x-u)]_+^n}{n!}\mu_{(n)}(du)$ and $f_{2,x}(u) = \frac{(x-u)_+^n}{n!}$ are real functions which are bounded and continuous. Then by the theorem on weak convergence of probability measures (the theorem is applied to the measures $\mu_{(n),(N),k}$ and $\mu_{(n),(N)}$ after norming), we obtain that $\lim_{k\to\infty} f_{(N),k}(x) = f_{(N)}(x)$ $(x \in (a,b))$ (see [2]). This completes the proof of the lemma for the function $f_{(N)}$ with the n-spectral measure $\mu_{(n),(N)}$. The lemma is proved. □

As a corollary we obtain that the set of delta-convex functions of n-th order of the class C^{n+1} in (a,b) is dense in the set of delta-convex functions of n-th order in (a,b).

COROLLARY 2.1. *Let $f\colon (a,b) \to \mathbb{R}$ be a delta-convex function of n-th order. Then there exist a sequence $\{f_k\}$ of delta-convex functions of n-th order of the class C^{n+1} in (a,b), such that $\lim_{k\to\infty} f_k(x) = f(x)$ $(x \in (a,b))$.*

Proof. Let $f\colon (a,b) \to \mathbb{R}$ be a delta-convex function of n-th order. Then $f = \varphi_1 - \varphi_2$, where $\varphi_1, \varphi_2\colon (a,b) \to \mathbb{R}$ are both n-convex. From Lemma 2.1 there exist two sequences $\{\varphi_{1,k}\}$, $\{\varphi_{2,k}\}$ of delta-convex functions of n-th order of the class C^{n+1} in (a,b), such that $\lim_{k\to\infty} \varphi_{1,k}(x) = \varphi_1(x)$ $(x \in (a,b))$ and $\lim_{k\to\infty} \varphi_{2,k}(x) = \varphi_2(x)$ $(x \in (a,b))$. Then it suffices to take $f_k = \varphi_{1,k} - \varphi_{2,k}$ $(k \in \mathbb{N})$. This completes the proof of the corollary. □

It is worth noting, that in Hartman (1959) [10] one can find a discussion on minimal control functions for delta-convex functions. We will extend Hartman's findings to delta-convexity of higher orders.

Note, that if $f\colon (a,b) \to \mathbb{R}$ is a delta-convex function of n-th order of the form $f = \varphi_1 - \varphi_2$ with the control function $g = \varphi_1 + \varphi_2$, where φ_1, φ_2 are both n-convex, then for any n-convex function φ, the function f can be trivially written as $f = (\varphi_1 + \varphi) - (\varphi_2 + \varphi)$ with the control function $g_\varphi = \varphi_1 + \varphi_2 + 2\varphi$. Since φ is n-convex, by (1.1), $\Delta_h^{n+1}\varphi(x) \geqslant 0$ $(x \in (a,b))$. Consequently, we have

$$\left|\Delta^{n+1}_{\frac{y-x}{n+1}} f(x)\right| \leqslant \Delta^{n+1}_{\frac{y-x}{n+1}} g(x) \leqslant \Delta^{n+1}_{\frac{y-x}{n+1}} g_\varphi(x)$$

for all $x, y \in (a,b)$ such that $x < y$.

This observation suggests the following question: given a delta-convex function f of n-th order, does there exist a control function g for f such that the difference operator $\Delta^{n+1}_{\frac{y-x}{n+1}} g(x)$ is minimal? We give an affirmative answer to this question.

DEFINITION 2.2. Let $f\colon (a,b) \to \mathbb{R}$ be a delta-convex function of n-th order. Let f be of the form $f = \varphi_1^* - \varphi_2^*$, where φ_1^*, φ_2^* are both n-convex functions. We say that φ_1^* and φ_2^* are *minimal n-convex functions* in the representation of the function f as a difference of two n-convex functions (shortly minimal n-convex functions), if for any other two n-convex functions φ_1 and φ_2 such that $f = \varphi_1 - \varphi_2$, we have that $\varphi_1 - \varphi_1^*$ and $\varphi_2 - \varphi_2^*$ are both n-convex.

We say that $g^*\colon (a,b) \to \mathbb{R}$ is a *minimal control function* for f, if

$$\left| \Delta^{n+1}_{\frac{y-x}{n+1}} f(x) \right| \leqslant \Delta^{n+1}_{\frac{y-x}{n+1}} g^*(x),$$

and for any function $g\colon (a,b) \to \mathbb{R}$, which is a control function for f, we have

$$\Delta^{n+1}_{\frac{y-x}{n+1}} g^*(x) \leqslant \Delta^{n+1}_{\frac{y-x}{n+1}} g(x) \tag{2.2}$$

for all $x, y \in (a,b)$ such that $x < y$.

THEOREM 2.2. *Let $f\colon (a,b) \to \mathbb{R}$ be a delta-convex function of n-th order and let g^* be a minimal control function corresponding to f. Then for any other control function g corresponding to f, the function $g - g^*$ is n-convex.*

Proof. From (2.2) we obtain

$$\Delta^{n+1}_{\frac{y-x}{n+1}} (g(x) - g^*(x)) \geqslant 0$$

for all $x, y \in (a,b)$ such that $x < y$, which, by (1.1), yields that the function $g - g^*$ is n-convex. The theorem is proved. □

THEOREM 2.3. *Let $a < \xi < b$ and let $f\colon (a,b) \to \mathbb{R}$ be a delta-convex function of n-th order with a signed measure $\tau_{(n)}$ and a polynomial $Q_\xi \in \Pi_n$ in the representation (2.1). Let φ_1^*, φ_2^* be the n-convex functions given by the formulas (2.6) and (2.7), respectively. Then f can be written in the form*

$$f = \varphi_1^* - \varphi_2^*. \tag{2.3}$$

Moreover, φ_1^* and φ_2^* are minimal n-convex functions and the function g^* given by the formula

$$g^* = \varphi_1^* + \varphi_2^*$$

is a minimal control function corresponding to f.

Proof. Let $a < \xi < b$ and let $f\colon (a,b) \to \mathbb{R}$ be delta-convex of n-th order with a signed measure $\tau_{(n)}$ and a polynomial $Q_\xi \in \Pi_n$ in the representation (2.1). Consider the Hahn-Jordan decomposition of the signed measure $\tau_{(n)}$

$$\tau_{(n)} = \tau^+_{(n)} - \tau^-_{(n)}, \tag{2.4}$$

where $\tau^+_{(n)}$ and $\tau^-_{(n)}$ are (non-negative) measures on $\mathscr{B}((a,b))$ (see [2]), which are concentrated on two disjoint sets $P,\ N \in \mathscr{B}((a,b))$ $(P \cup N = (a,b))$, respectively, such that

$$\tau^+_{(n)} = \tau_{(n)}|_P \quad \textit{and} \quad -\tau^-_{(n)} = \tau_{(n)}|_N .$$

The decomposition (2.4) is called the *canonical decomposition* of $\tau_{(n)}$. The measure $var\left(\tau_{(n)}\right) = \tau^+_{(n)} + \tau^-_{(n)}$ is called the *variation* of $\tau_{(n)}$. Then, for any measures ν_1, ν_2 on $\mathscr{B}((a,b))$

$$\tau_{(n)} = \nu_1 - \nu_2 \Rightarrow (\nu_1 \geqslant \tau^+_{(n)} \quad \textit{and} \quad \nu_2 \geqslant \tau^-_{(n)}). \tag{2.5}$$

By (2.1) and (2.4), f can be written as $f = \varphi_1^* - \varphi_2^*$, with the control function $g^* = \varphi_1^* + \varphi_2^*$, where

$$\varphi_1^*(x) = \int_{(a,\xi)} (-1)^{n+1} \frac{[-(x-u)]^n_+}{n!} \tau^+_{(n)}(du) + \int_{[\xi,b)} \frac{(x-u)^n_+}{n!} \tau^+_{(n)}(du) + Q_\xi(x), \tag{2.6}$$

$$\varphi_2^*(x) = \int_{(a,\xi)} (-1)^{n+1} \frac{[-(x-u)]^n_+}{n!} \tau^-_{(n)}(du) + \int_{[\xi,b)} \frac{(x-u)^n_+}{n!} \tau^-_{(n)}(du). \tag{2.7}$$

Let f be of the form $f = \varphi_1 - \varphi_2$ with the control function $g = \varphi_1 + \varphi_2$, where φ_1, φ_2 are two n-convex functions with the measures $\mu_{(n)1}, \mu_{(n)2}$ and the polynomials $Q_{\xi,1}, Q_{\xi,2}$, respectively, in the representation (1.2). Then $\tau_{(n)} = \mu_{(n)1} - \mu_{(n)2}$ and $Q_\xi = Q_{\xi,1} - Q_{\xi,2}$. Taking into account (2.4) and (2.5), we obtain that there exists a measure ν on $\mathscr{B}((a,b))$ such that $\mu_{(n)1} = \tau^+_{(n)} + \nu$ and $\mu_{(n)2} = \tau^-_{(n)} + \nu$. Then we have that

$$\varphi_1 = \varphi_1^* + \varphi + Q_{\xi,1} - Q_\xi, \quad \varphi_2 = \varphi_2^* + \varphi + Q_{\xi,2}, \tag{2.8}$$

$$g = \varphi_1 + \varphi_2 = \varphi_1^* + \varphi_2^* + 2\varphi = g^* + 2\varphi, \tag{2.9}$$

where φ is the n-convex function of the form (1.2) with ν in place of $\mu_{(n)}$ and $Q_\xi = 0$. Taking into account that g^* is the control function for f and that φ is n-convex, by (2.9), we obtain

$$\left|\Delta^{n+1}_{\frac{y-x}{n+1}} f(x)\right| \leqslant \Delta^{n+1}_{\frac{y-x}{n+1}} g^*(x) \leqslant \Delta^{n+1}_{\frac{y-x}{n+1}} g(x)$$

for all $x, y \in (a,b)$ such that $x < y$. This implies that g^* is a minimal control function for f. Moreover, by (2.8), we obtain that φ_1^* and φ_2^* are minimal n-convex functions. The theorem is proved. $\square$

DEFINITION 2.3. We will say that the decomposition (2.3) is the *canonical decomposition* of a delta-convex function f of n-th order.

In the next three theorems we give a characterization of control functions and minimal control functions.

THEOREM 2.4. *Let $g\colon (a,b) \to \mathbb{R}$ be an n-convex function with the measure $\mu_{(n)}$ in the representation (1.2), and let $f\colon (a,b) \to \mathbb{R}$ be a delta-convex function of n-th order with a signed measure $\tau_{(n)}$ in the representation (2.1). Then the following statements are equivalent:*

a) *f is controlled by g,*

b) *$\mu_{(n)} - \tau_{(n)} \geqslant 0$ and $\mu_{(n)} + \tau_{(n)} \geqslant 0$,*

c) *$\left|\tau_{(n)}\right| \leqslant \mu_{(n)}$,*

d) *$\left|f^{(n+1)}(x)\right| \leqslant g^{(n+1)}(x)$ $(x \in (a,b))$ when f and g are both of the class C^{n+1} in (a,b).*

Proof. Let $\xi \in (a,b)$. Let f and g have the representations (1.2) and (2.1) with the triplets $(\xi, \tau_{(n)}, Q_\xi)$ and $(\xi, \mu_{(n)}, P_\xi)$, respectively. By Lemma 1.1, f is controlled by g if and only if $g-f$ and $g+f$ are n-convex. Note that $g-f$ and $g+f$ have the representation (2.1) with the triplets $(\xi, \mu_{(n)} - \tau_{(n)}, P_\xi - Q_\xi)$ and $(\xi, \mu_{(n)} + \tau_{(n)}, P_\xi + Q_\xi)$, respectively. Thus $g-f$ and $g+f$ are n-convex if and only if $\mu_{(n)} - \tau_{(n)} \geqslant 0$ and $\mu_{(n)} + \tau_{(n)} \geqslant 0$. This proves that the statement a) is equivalent to b). The statement b) is equivalent to c) obviously. The equivalence of c) and d) follows immediately from Remarks 1.1 and 2.1. This completes the proof. □

THEOREM 2.5. *Let $g\colon (a,b) \to \mathbb{R}$ be an n-convex function with the n-spectral measure $\mu_{(n)}$ in the representation (1.2), and let $f\colon (a,b) \to \mathbb{R}$ be a delta-convex function of n-th order with the n-spectral signed measure $\tau_{(n)}$ in the representation (2.1). Then the following statements are equivalent:*

a) *g is a minimal control function for f,*

b) *$var\left(\tau_{(n)}\right) = \mu_{(n)}$,*

c) *$\left|f^{(n+1)}(x)\right| = g^{(n+1)}(x)$ $(x \in (a,b))$ when f and g are both of the class C^{n+1} in (a,b).*

d) *$g = g^*$ up to a polynomial of degree at most n, where g^* is the minimal control function which we have introduced in Theorem 2.3.*

Proof. To prove that a) implies d), assume that g is a minimal control function for f. Since g and g^* are the control functions for f, and g^* is minimal, we have

$$\Delta^{n+1}_{\frac{y-x}{n+1}} g^*(x) \leqslant \Delta^{n+1}_{\frac{y-x}{n+1}} g(x),$$

and taking into account that g is minimal we obtain

$$\Delta^{n+1}_{\frac{y-x}{n+1}} g(x) \leqslant \Delta^{n+1}_{\frac{y-x}{n+1}} g^*(x),$$

which implies

$$\Delta^{n+1}_{\frac{y-x}{n+1}} g^*(x) = \Delta^{n+1}_{\frac{y-x}{n+1}} g(x) \tag{2.10}$$

for all $x,y \in (a,b)$ such that $x<y$. The equality (2.10) means that $g=g^*$ up to a polynomial of degree at most n. This proves that the statement a) implies d). Similarly can be proved that d) implies a).

Since $var(\tau_{(n)})$ is the n-spectral measure corresponding to g^*, from Proposition 1.4 we obtain the equivalence of b) and d). The equivalence of b) and c) follows immediately from Remarks 1.1 and 2.1. This completes the proof. □

COROLLARY 2.2. *Any two minimal control functions corresponding to a delta-convex function $f\colon (a,b) \to \mathbb{R}$ of n-th order, differ by a polynomial of degree at most n.*

THEOREM 2.6. *Let $f\colon (a,b) \to \mathbb{R}$ be a delta-convex function of n-th order and let g be a minimal control function. Then*

$$\left|f^{(n+1)}(x)\right| = g^{(n+1)}(x) \quad \text{for } x \in (a,b) \quad \lambda \text{ a.e.} \tag{2.11}$$

(i.e. almost everywhere with respect to the Lebesgue measure λ).

Proof. Let $a<\xi<b$ and let $f\colon (a,b) \to \mathbb{R}$ be a delta-convex function of n-th order with a signed measure $\tau_{(n)}$ and a polynomial $Q_\xi \in \Pi_n$ in the representation (2.1). Then, by Theorem 2.3, f can be written in the form

$$f = \varphi_1^* - \varphi_2^*, \tag{2.12}$$

with the minimal control function g^* given by the formula

$$g^* = \varphi_1^* + \varphi_2^*, \tag{2.13}$$

where φ_1^* and φ_2^* are given by the formulas

$$\varphi_1^*(x) = \int_{(a,\xi)} (-1)^{n+1} \frac{[-(x-u)]_+^n}{n!} \tau^+_{(n)}(du) + \int_{[\xi,b)} \frac{(x-u)_+^n}{n!} \tau^+_{(n)}(du) + Q_\xi(x), \tag{2.14}$$

$$\varphi_2^*(x) = \int_{(a,\xi)} (-1)^{n+1} \frac{[-(x-u)]_+^n}{n!} \tau^-_{(n)}(du) + \int_{[\xi,b)} \frac{(x-u)_+^n}{n!} \tau^-_{(n)}(du), \tag{2.15}$$

the signed measure $\tau_{(n)}$ has the Hahn-Jordan decomposition $\tau_{(n)} = \tau^+_{(n)} - \tau^-_{(n)}$ with measures $\tau^+_{(n)}$ and $\tau^-_{(n)}$ on $\mathscr{B}((a,b))$), which are concentrated on two disjoint sets $P,\ N \in \mathscr{B}((a,b))$ $(P \cup N = (a,b))$, respectively, such that $\tau^+_{(n)} = \tau_{(n)}|_P$, $-\tau^-_{(n)} = \tau_{(n)}|_N$. Note that the measures $\tau^+_{(n)}$ and $\tau^-_{(n)}$ can be written in the form where $\tau^+_{(n)} = \tau^+_{(n)cont} + \tau^+_{(n)sing} + \tau^+_{(n)pp}$ and $\tau^-_{(n)} = \tau^-_{(n)cont} + \tau^-_{(n)sing} + \tau^-_{(n)pp}$, where $\tau^+_{(n)cont}$, $\tau^-_{(n)cont}$ are absolutely continuous (with respect to the Lebesgue measure), $\tau^+_{(n)sing}$, $\tau^-_{(n)sing}$ are singular continuous and $\tau^+_{(n)pp}, \tau^-_{(n)pp}$ are discrete. Let $P_1 \subseteq P$, $N_1 \subseteq N$ are the Borel sets of the Lebesgue measure zero such that $\tau^+_{(n)sing} + \tau^+_{(n)pp}$ and $\tau^-_{(n)sing} + \tau^-_{(n)pp}$ are concentrated on P_1 and N_1, respectively. Then φ_1^* and φ_2^* can be written in the form

$$\varphi_1^* = \varphi_{1c}^* + \varphi_{1s}^*, \quad \varphi_2^* = \varphi_{2c}^* + \varphi_{2s}^*, \tag{2.16}$$

where φ_{1c}^* and φ_{1s}^* are of the form (2.14) with $\tau^+_{(n)cont}$ and $\tau^+_{(n)sing} + \tau^+_{(n)pp}$ in place of $\tau^+_{(n)}$, respectively, and with $Q_\xi(x) = 0$ in the case of φ_{1s}^*, and similarly, φ_{2c}^* and φ_{2s}^* are of the form (2.15) with $\tau^-_{(n)cont}$ and $\tau^-_{(n)sing} + \tau^-_{(n)pp}$ in place of $\tau^-_{(n)}$, respectively. Then φ_{1c}^* and φ_{2c}^* are of the class C^{n+1} and by Remark 1.1 $\varphi_{1c}^{*(n+1)}(x)dx = \tau^+_{(n)cont}(dx)$ and $\varphi_{2c}^{*(n+1)}(x)dx = \tau^-_{(n)cont}(dx)$. By Remark 1.1, $\varphi_{1s}^{*(n)}(x)$ and $\varphi_{2s}^{*(n)}(x)$ can be regarded as distribution functions of the measures $\tau^+_{(n)sing} + \tau^+_{(n)pp}$ and $\tau^-_{(n)sing} + \tau^-_{(n)pp}$, respectively. Since these measures are concentrated on the sets P_1 and N_1, respectively, without loss of generality we may conclude that $\varphi_{1s}^{*(n+1)}(x) = 0$ for $x \in (a,b) \setminus P_1$ and $\varphi_{2s}^{*(n+1)}(x) = 0$ for $x \in (a,b) \setminus N_1$. Taking into account (2.16), we obtain that

$$\varphi_1^{*(n+1)}(x) = \varphi_{1c}^{*(n+1)}(x) \quad for\, x \in P \setminus P_1,$$

$$\varphi_2^{*(n+1)}(x) = \varphi_{2c}^{*(n+1)}(x) \quad for\, x \in N \setminus N_1,$$

Since the sets P and N are disjoint, by (2.12) and (2.13), we obtain that

$$f^{(n+1)}(x) = \varphi_1^{*(n+1)}(x)\chi_{P\setminus P_1}(x) - \varphi_2^{*(n+1)}(x)\chi_{N\setminus N_1}(x),$$

$$g^{*(n+1)}(x) = \varphi_1^{*(n+1)}(x)\chi_{P\setminus P_1}(x) + \varphi_2^{*(n+1)}(x)\chi_{N\setminus N_1}(x),$$

consequently we have

$$\left|f^{(n+1)}(x)\right| = g^{*(n+1)}(x) = \varphi_1^{*(n+1)}(x)\chi_{P\setminus P_1}(x) + \varphi_2^{*(n+1)}(x)\chi_{N\setminus N_1}(x),$$

where $\chi_B(x) = 1$ if $x \in B$ and $\chi_B(x) = 0$ if $x \notin B$ $(B \subset \mathbb{R})$. This implies that

$$\left|f^{(n+1)}(x)\right| = g^{*(n+1)}(x) \quad for\, x \in (a,b) \quad \lambda\ a.e. \tag{2.17}$$

From Theorem 2.5, for any other minimal control function g we have that $g = g^*$ up to a polynomial of degree at most n, thus (2.17) yields (2.11). The theorem is proved. □

Now we are going to give a new proof of Proposition 1.6.

Proof. Let $f\colon (a,b) \to \mathbb{R}$ be a C^{n+1}- function. Put $h(x) = f^{(n+1)}(x)$. Then, taking F to be a delta-convex function of order n that has the form (2.1) with some $\xi \in (a,b)$, $Q_\xi \in \Pi_n$ and $\tau_{(n)}(du) = h(u)du$, by Remark 2.1 we have $F^{(n+1)}(x) = h(x)$ $(x \in (a,b))$. Since also $f^{(n+1)}(x) = h(x)$ $(x \in (a,b))$, we obtain $f = F + p_n$, where $p_n \in \Pi_n$. Consequently f is delta-convex of n-th order, which was to be proved. □

3. Relative delta-convexity of n-th order. Strong delta-convexity of n-th order.

First recall the definition of the relative n-convexity relation (see [14]).

DEFINITION 3.1. ([14]) Let $h\colon (a,b) \to \mathbb{R}$ be an n-convex function. We say that a function $f\colon (a,b) \to \mathbb{R}$ is *n-convex with respect to h* if $f-h$ is n-convex, and denote it by $f \succeq_n h$.

PROPOSITION 3.1. ([14]) *Let $f,h\colon (a,b) \to \mathbb{R}$ be n-convex functions with n-spectral measures $\mu_{(n)}$ and $\nu_{(n)}$, respectively. Then f is n-convex with respect to h if and only if*

$$\mu_{(n)} \geqslant \nu_{(n)}.$$

DEFINITION 3.2. Let $f\colon (a,b) \to \mathbb{R}$ be an n-convex function with the n-spectral measure $\mu_{(n)}$. The measure $\mu^f_{(n)} = \mu_{(n)}$ is called the *measure of n-th order convexity* of f (or shortly the *n-convexity measure*).

PROPOSITION 3.2. ([14]) *Let $f,h\colon (a,b) \to \mathbb{R}$ be n-convex functions with the n-convexity measures $\mu^f_{(n)}$ and $\mu^h_{(n)}$, respectively. Then the following conditions are equivalent:*

a) $f \succeq_n h$,

b) $\mu^f_{(n)} \geqslant \mu^h_{(n)}$,

c) $f^{(n+1)}(x) \geqslant h^{(n+1)}(x)$ $(x \in (a,b))$ *when f and h are both of the class C^{n+1} in* (a,b).

COROLLARY 3.1. *Let $f,h\colon (a,b) \to \mathbb{R}$ be n-convex functions. Then if $f \succeq_n h$ then $f^{(n+1)}(x) \geqslant h^{(n+1)}(x)$ for $x \in (a,b)$ λ a.e.*

DEFINITION 3.3. We shall say that functions $f,g\colon (a,b) \to \mathbb{R}$ are of *modulo* Π_n, or that they are members of the same *modulo* Π_n *class*, if they differ by a polynomial $Q \in \Pi_n$ (see [14]) .

The relation modulo Π_n is an equivalence relation and hence it defines equivalence classes. For n-convex functions f and $g\colon (a,b) \to \mathbb{R}$ that are members of the same modulo Π_n class we therefore have that $f^{(n)}(x)$ and $g^{(n)}(x)$ differ by a constant.

PROPOSITION 3.3. ([14])

$$f = g \ (mod\, \Pi_n) \quad \text{if and only if} \quad \mu^f_{(n)} = \mu^g_{(n)}.$$

PROPOSITION 3.4. ([14]) *The relative n-convexity relation induces a partial ordering on modulo Π_n equivalence classes of n-convex functions.*

Now we are going to define the relative delta-convexity relation of higher order.

DEFINITION 3.4. Let $f,h\colon (a,b) \to \mathbb{R}$ be delta-convex functions of n-th order with the minimal control functions g_f^* and g_h^*, respectively. We say that a function $f\colon (a,b) \to \mathbb{R}$ is *delta-convex of n-th order with respect to h* (shortly n-delta-convex with respect to h), if

$$\Delta^{n+1}_{\frac{y-x}{n+1}} g_f^*(x) \geqslant \Delta^{n+1}_{\frac{y-x}{n+1}} g_h^*(x)$$

for all $x,y \in (a,b)$ such that $x<y$, and denote it by $f \succeq_{dcn} h$.

DEFINITION 3.5. We shall say that functions $f,h\colon (a,b) \to \mathbb{R}$ which are delta-convex functions of n-th order (with the minimal control functions g_f^* and g_h^*, respectively) are of *modulo M_{dcn}*, or that they are members of the same *modulo M_{dcn} class*, if

$$\Delta^{n+1}_{\frac{y-x}{n+1}} g_f^*(x) = \Delta^{n+1}_{\frac{y-x}{n+1}} g_h^*(x)$$

for all $x,y \in (a,b)$ such that $x<y$, and denote it by $f = h(mod M_{dcn})$.

The relation modulo M_{dcn} is an equivalence relation and hence it defines equivalence classes. By Proposition 3.4, it is not difficult to prove that this relation induces a partial ordering (we omit the proof).

THEOREM 3.1. *The relative n-delta-convexity relation induces a partial ordering on modulo M_{dcn} equivalence classes of delta-convex functions of n-th order.*

DEFINITION 3.6. Let $f\colon (a,b) \to \mathbb{R}$ be a delta -convex function of n-th order with the n-spectral signed measure $\tau_{(n)}$. We will call the measure $var\left(\tau_{(n)}\right)$ a *measure of delta-convexity of n-th order* of the function f (or shortly an *n-delta-convexity measure*).

THEOREM 3.2. *Let $f,h\colon (a,b) \to \mathbb{R}$ be delta-convex functions of n-th order with the n-delta-convexity measures $var\left(\tau^f_{(n)}\right)$ and $var\left(\tau^h_{(n)}\right)$, the minimal control functions g_f^* and g_h^*, the n-convexity measures $\mu^{g_f^*}_{(n)}$ and $\mu^{g_h^*}_{(n)}$ corresponding to the minimal control functions g_f^*, g_h^*, respectively. Then the following statements are equivalent:*

a) $f \succeq_{dcn} h$,

b) $\Delta^{n+1}_{\frac{y-x}{n+1}} g_f^*(x) \geqslant \Delta^{n+1}_{\frac{y-x}{n+1}} g_h^*(x)$, *for all $x,y \in (a,b)$ such that $x<y$,*

c) $var\left(\tau^f_{(n)}\right) \geqslant var\left(\tau^h_{(n)}\right)$,

d) $\left|f^{(n+1)}(x)\right| \geqslant \left|h^{(n+1)}(x)\right|$ $(x \in (a,b))$ *when f and h are both of the class C^{n+1} in (a,b),*

e) $g_f^* \succeq_n g_h^*$,

f) $\mu^{g_f^*}_{(n)} \geqslant \mu^{g_h^*}_{(n)}$,

g) $g_f^{*(n+1)}(x) \geqslant g_h^{*(n+1)}(x) \quad (x \in (a,b))$ *when* f *and* h *are both of the class* C^{n+1} *in* (a,b).

Proof. The equivalence of a) and b) this is just definition. Note that the condition b) means the n-convexity of $g_f^* - g_h^*$, and by the definition of the relative relation $\succeq_n$ this is equivalent to the condition e). From Proposition 3.2 it follows the equivalence of e), f) and g). By Theorem 2.5, we have that $var\left(\tau_{(n)}^f\right) = \mu_{(n)}^{g_f^*}$ and $var\left(\tau_{(n)}^h\right) = \mu_{(n)}^{g_h^*}$, consequently the condition f) is equivalent to c). By Remark 2.1, we obtain the equivalence of c) and d). The theorem is proved. □

THEOREM 3.3. *Let* $f,h\colon (a,b) \to \mathbb{R}$ *be delta-convex functions of* n*-th order with the minimal control functions* g_f^* *and* g_h^*. *Then*

a) *there exists an* n*-convex function* g_{max}, *which is a control function for* f *and* h, *such that*

$$g_{max} \succeq_n g_f^*, \quad g_{max} \succeq_n g_h^*,$$

and for every n*-convex function* k

$$(k \succeq_n g_f^* \text{ and } k \succeq_n g_h^*) \Rightarrow k \succeq_n g_{max},$$

b) *there exists an* n*-convex function* g_{min} *such that*

$$g_f^* \succeq_n g_{min}, \quad g_h^* \succeq_n g_{min},$$

and for every n*-convex function* k

$$(g_f^* \succeq_n k \text{ and } g_h^* \succeq_n k) \Rightarrow g_{min} \succeq_n k,$$

c) *if* $g_f^* \succeq_n g_h^*$ *and* $g_f^* \neq g_h^* (mod\, \Pi_n)$, *then there exists an* n*-convex function* w *such that* $g_f^* \neq w (mod\, \Pi_n)$, $g_h^* \neq w (mod\, \Pi_n)$ *and*

$$g_f^* \succeq_n w \succeq_n g_h^*.$$

Proof. Let $\mu_{(n)}^{g_f^*}$ and $\mu_{(n)}^{g_h^*}$ be the n-convexity measures corresponding to the minimal control functions g_f^*, g_h^*, respectively. Consider the Radon-Nikodym derivatives

$$\alpha = d\mu_{(n)}^{g_f^*} / d(\mu_{(n)}^{g_f^*} + \mu_{(n)}^{g_h^*}), \tag{3.1}$$

$$\beta = d\mu_{(n)}^{g_h^*}) / d(\mu_{(n)}^{g_f^*} + \mu_{(n)}^{g_h^*}). \tag{3.2}$$

It is not difficult to see that it suffices to take the functions g_{max} and g_{min} of the form (1.2) with the measures

$$\mu_{(n)}^{max} = \max(\alpha, \beta)(\mu_{(n)}^{g_f^*} + \mu_{(n)}^{g_h^*}), \tag{3.3}$$

$$\mu_{(n)}^{min} = \min(\alpha,\beta)(\mu_{(n)}^{g_f^*} + \mu_{(n)}^{g_h^*}), \tag{3.4}$$

to prove parts a) and b). To prove c) assume $g_f^* \succeq_n g_h^*$ and $g_f^* \neq g_h^* (\mathrm{mod}\, \Pi_n)$. Then $g_f^* - g_h^*$ is n-convex and $g_f^* - g_h^* \neq 0 (\mathrm{mod}\, \Pi_n)$. Thus it suffices to take $w = g_h^* + \frac{1}{2}(g_f^* - g_h^*)$. The theorem is proved. $\square$

THEOREM 3.4. *Let $f,h\colon (a,b) \to \mathbb{R}$ be delta-convex functions of n-th order. Then*

a) *there exists a delta-convex function f_{max} of n-th order such that*

$$f_{max} \succeq_{dcn} f, \quad f_{max} \succeq_{dcn} h,$$

and for every delta-convex function k of n-th order

$$(k \succeq_{dcn} f \text{ and } k \succeq_{dcn} h) \Rightarrow k \succeq_{dcn} f_{max},$$

b) *there exists a delta-convex function f_{min} of n-th order such that*

$$f \succeq_{dcn} f_{min}, \quad h \succeq_{dcn} f_{min},$$

and for every delta-convex function k of n-th order

$$(f \succeq_{dcn} k \text{ and } h \succeq_{dcn} k) \Rightarrow f_{min} \succeq_{dcn} k,$$

c) *if $f \succeq_{dcn} h$ and $f \neq h (mod\, M_{dcn})$, then there exists a delta-convex function u of n-th order such that $f \neq u (mod\, M_{dcn})$, $g \neq u (mod\, M_{dcn})$ and*

$$f \succeq_{dcn} u \succeq_{dcn} h.$$

Proof. Let $f,h\colon (a,b) \to \mathbb{R}$ be delta-convex functions of n-th order with the n-spectral signed measures $\tau_{(n)}^f$ and $\tau_{(n)}^h$ in the representation (2.1), respectively, and with the n-convexity measures $\mu_{(n)}^{g_f^*}$ and $\mu_{(n)}^{g_h^*}$ corresponding to their minimal control functions g_f^* and g_h^*, respectively. By Theorems 3.2 and 3.3, it suffices to take the functions f_{max}, f_{min} and u of the form (2.1) with any signed measures $\tau_{(n)}^{f_{\max}}$, $\tau_{(n)}^{f_{\min}}$ and $\tau_{(n)}^u$ such that $var\left(\tau_{(n)}^{f_{\max}}\right) = \mu_{(n)}^{max}$, $var\left(\tau_{(n)}^{f_{\min}}\right) = \mu_{(n)}^{min}$ and $var\left(\tau_{(n)}^{u}\right) = \frac{1}{2}\left(\mu_{(n)}^{g_h^*} + \mu_{(n)}^{g_f^*}\right)$, respectively, where $\mu_{(n)}^{max}$ and $\mu_{(n)}^{min}$ are the measures given by (3.3) and (3.4), which we have introduced in the proof of Theorem 3.3. The theorem is proved. $\square$

We recall the definition of strong convexity of higher order (see [14], [9]).

DEFINITION 3.7. ([14]) Let n be a fixed positive integer and c be a positive (fixed) real number. We say that a function $f\colon (a,b) \to \mathbb{R}$ is *strongly n-convex with modulus c* if the function $f(x) - cx^{(n+1)}/(n+1)!$ is n-convex.

The following two propositions give a characterization of a strongly n-convex function f with modulus c without any additional assumptions of differentiability of f.

PROPOSITION 3.5. ([14]) *Let $f\colon (a,b) \to \mathbb{R}$ be an n-convex function and $c > 0$. Then f is strongly n-convex with modulus c if and only if*

$$f^{(n+1)}(x) \geqslant c \quad \textit{for } x \in (a,b) \quad \lambda \textit{ a.e.}$$

PROPOSITION 3.6. ([14]) *Let $f\colon (a,b) \to \mathbb{R}$ be an n-convex function and $c > 0$. Then the following statements are equivalent:*

a) *f is strongly n-convex with modulus c,*

b) *f is of the form $f(x) = cx^{(n+1)}/(n+1)! + h(x)$ $(x \in (a,b))$, where $h\colon (a,b) \to \mathbb{R}$ is an n-convex function,*

c) *f is of the form $f(x) = f_{cont}(x) + R(x)$ $(x \in (a,b))$, where $f_{cont}\colon (a,b) \to \mathbb{R}$ is a strongly n-convex function with modulus c of the class C^{n+1} in (a,b), and $R\colon (a,b) \to \mathbb{R}$ is an n-convex function such that $R^{(n+1)}(x) = 0$ for $x \in (a,b)$ λ a.e.*

COROLLARY 3.2. ([14]) *Let $c > 0$ and let $f\colon (a,b) \to \mathbb{R}$ be an n-convex with modulus c function, which is of the class C^{n+1} in (a,b). Then f is strongly n-convex with modulus c if and only if*

$$f^{(n+1)}(x) \geqslant c$$

for all $x \in (a,b)$.

We now turn to defining the strong delta-convexity of higher order.

DEFINITION 3.8. Let $c > 0$ and let $f\colon (a,b) \to \mathbb{R}$ be a delta -convex function of n-th order. We say that f is a *strongly delta-convex function of n-th order with modulus c* if all control functions corresponding to f are strongly n-convex with modulus c.

LEMMA 3.1. *Let $c > 0$ and let $f\colon (a,b) \to \mathbb{R}$ be a delta -convex function of n-th order. If there exists a minimal control function g corresponding to f, such that g is strongly n-convex with modulus c, then all minimal control functions corresponding to f are strongly n-convex with modulus c.*

Proof. Let g be a minimal control function corresponding to f, such that g is strongly n-convex with modulus c. By Corollary 2.2, any two minimal control functions corresponding to f, differ by a polynomial of degree at most n, then for any other control function h corresponding to f we have the equality: $g = h$ up to a polynomial of degree at most n. Consequently, h is also strongly n-convex with modulus c. The lemma is proved. □

THEOREM 3.5. *Let $c>0$. Let $f\colon (a,b)\to\mathbb{R}$ be a delta-convex function of n-th order and let g^* be a minimal control function corresponding to f. Then f is a strongly delta-convex function of n-th order with modulus c if and only if g^* is strongly n-convex with modulus c.*

Proof. By the definition of strong delta-convexity of higher order, the necessity is obvious. We now prove the sufficiency. Let g^* be a minimal control function corresponding to f, which is strongly n-convex with modulus c. Let g be any other control function for f. From Theorem 2.2 we obtain that the function $g-g^*$ is n-convex. Then we have that the function $g=g^*+(g-g^*)$ is a sum of two n-convex functions such that one of them is strongly n-convex with modulus c. This implies that g is strongly n-convex with modulus c. Consequently, f is a strongly delta-convex function of n-th order with modulus c. The theorem is proved. □

From Theorems 2.6, 3.5, Lemma 3.1 and Proposition 3.5 we obtain the following characterization of strong delta-convexity of n-th order.

THEOREM 3.6. *Let $c>0$ and let $f\colon (a,b)\to\mathbb{R}$ be a delta-convex function of n-th order. Then f is strongly delta-convex of n-th order with modulus c if and only if*

$$\left|f^{(n+1)}(x)\right|\geqslant c \quad \text{for } x\in(a,b) \quad \lambda \text{ a.e.} \tag{3.5}$$

Proof. Let $f\colon (a,b)\to\mathbb{R}$ be a delta -convex function of n-th order. Let g be a minimal control function corresponding to f. By Theorem 2.6

$$\left|f^{(n+1)}(x)\right| = g^{(n+1)}(x) \quad \text{for } x\in(a,b) \quad \lambda \text{ a.e.} \tag{3.6}$$

According to Theorem 3.5 and Lemma 3.1, the strong delta-convexity (with modulus c) of n-th order of the function f is equivalent to the strong n-convexity (with modulus c) of the function g, which is equivalent (by Proposition 3.5) to

$$g^{(n+1)}(x)\geqslant c \quad \text{for } x\in(a,b) \quad \lambda \text{ a.e.} \tag{3.7}$$

Taking into account (3.6), we obtain that (3.7) is equivalent to (3.5). The theorem is proved. □

From Proposition 1.6 and Theorem 3.6 we obtain the following corollary.

COROLLARY 3.3. *Let $c>0$ and let $f\colon (a,b)\to\mathbb{R}$ be a function of the class C^{n+1} in (a,b). Then f is strongly delta-convex of n-th order with modulus c if and only if*

$$\left|f^{(n+1)}(x)\right|\geqslant c$$

for all $x\in(a,b)$.

Note, that Theorem 3.6 is a generalization of Proposition 3.5 concerning strongly n-convex functions.

EXAMPLE 3.1. Let $c > 0$. For $f(x) = -cx^2/2\chi_{(-\infty,0)}(x) + cx^2/2\chi_{[0,\infty)}(x)$ $(x \in \mathbb{R})$ ($\chi_B(x) = 1$ if $x \in B$ and $\chi_B(x) = 0$ if $x \notin B$, $B \subset \mathbb{R}$), we have $\left|f''(x)\right| = c$ for $x \neq 0$. Consequently, from Theorem 3.6, taking into account that f is delta-convex, we obtain that the function f is strongly delta-convex with modulus c.

4. Hermite-Hadamard-Fejér type inequalities.

Recall that the function $f\colon (a,b) \to \mathbb{R}$ is *convex*, if the inequality

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y)$$

holds for all $x, y \in (a,b)$ and for all $t \in [0,1]$ (see [11], [17]). Convexity has a nice probabilistic characterization, known as Jensen's inequality (see [2]).

PROPOSITION 4.1. *A function $f\colon (a,b) \to \mathbb{R}$ is convex if and only if*

$$f(\mathbb{E}X) \leqslant \mathbb{E}f(X) \tag{4.1}$$

for all (a,b)-valued integrable random variables X.

In the following theorem we give a probabilistic characterization of delta-convexity.

THEOREM 4.1. *Let $f\colon (a,b) \to \mathbb{R}$ be a function and $g\colon (a,b) \to \mathbb{R}$ be a convex function. Then f is g-convex dominated (or delta-convex with the control function g) if and only if*

$$|\mathbb{E}f(X) - f(\mathbb{E}X)| \leqslant \mathbb{E}g(X) - g(\mathbb{E}X) \tag{4.2}$$

for all (a,b)-valued integrable random variables X.

Proof. By Lemma 1.1 f is g-convex dominated if and only if the functions $g - f$ and $g + f$ are convex on (a,b). From Proposition 4.1 $g - f$ and $g + f$ are convex on (a,b) if and only if

$$\mathbb{E}g(X) - \mathbb{E}f(X) \geqslant g(\mathbb{E}X) - f(\mathbb{E}X) \text{ and } \mathbb{E}g(X) + \mathbb{E}f(X) \geqslant g(\mathbb{E}X) + f(\mathbb{E}X) \tag{4.3}$$

for all (a,b)-valued integrable random variables X. Note that (4.3) is equivalent to

$$\mathbb{E}f(X) - f(\mathbb{E}X) \leqslant \mathbb{E}g(X) - g(\mathbb{E}X) \text{ and } \mathbb{E}f(X) - f(\mathbb{E}X) \geqslant -[(\mathbb{E}g(X) - g(\mathbb{E}X))],$$

which is equivalent to (4.2). The theorem is proved. □

Now let us turn our attention to some particular case of Theorem 4.1. For the arbitrary $t \in (0,1)$ and $x_1, x_2 \in (a,b)$ consider the random variable X such that $P(X = x_1) = t$ and $P(X = x_2) = 1 - t$. Then, by Theorem 4.1, we obtain the following corollary.

COROLLARY 4.1. *Let $f\colon (a,b) \to \mathbb{R}$ be a function and $g\colon (a,b) \to \mathbb{R}$ be a convex function. Then, if f is g-convex dominated then*

$$|tf(x_1)+(1-t)f(x_2)-f(tx_1+(1-t)x_2)| \leqslant tg(x_1)+(1-t)g(x_2)-g(tx_1+(1-t)x_2) \tag{4.4}$$

for any $x_1,x_2 \in (a,b)$ and $t \in (0,1)$.

REMARK 4.1. Note that, by the definition of a convex function, the function f is convex if and only if the condition (4.1) is satisfied for all random variables X such that $P(X=x_1)=t$ and $P(X=x_2)=1-t$, where $t \in (0,1)$ and $x_1,x_2 \in (a,b)$. Similarly, it is not difficult to prove that the condition (4.4) is also sufficient to guarantee delta-convexity of f (with the control function g). It should be noted, that in [5], this condition is used as the definition (of f to be g-convex dominated).

The next result we state concerns the Jensen-type inequality for delta-convex functions.

COROLLARY 4.2. *Let $f\colon (a,b) \to \mathbb{R}$ be a function and $g\colon (a,b) \to \mathbb{R}$ be a convex function. Then, if f is g-convex dominated then*

$$\left|\sum_{i=1}^{n} t_i f(x_i) - f\left(\sum_{i=1}^{n} t_i x_i\right)\right| \leqslant \sum_{i=1}^{n} t_i g(x_i) - g\left(\sum_{i=1}^{n} t_i x_i\right),$$

for any $x_1,\dots,x_n \in (a,b)$ and $t_1,\dots,t_n > 0$ summing up to 1.

Proof. Let X be a random variable such that $P(X=x_i)=t_i$, $i=1,2,\dots,n$. Now it is enough to use Theorem 4.1. □

THEOREM 4.2. *Let $f\colon (a,b) \to \mathbb{R}$ be a function and $g\colon (a,b) \to \mathbb{R}$ be a convex function.*

a) *If f is delta-convex, g is the control function and g^* is the minimal control function corresponding to f then*

$$|\mathbb{E}f(X)-f(\mathbb{E}X)| \leqslant \mathbb{E}g^*(X)-g^*(\mathbb{E}X), \tag{4.5}$$

$$\mathbb{E}g^*(X)-g^*(\mathbb{E}X) \leqslant \mathbb{E}g(X)-g(\mathbb{E}X), \tag{4.6}$$

for all (a,b)-valued integrable random variables X.

b) *Conversely, if the inequality (4.2) is satisfied, then for any minimal control function g^* the inequalities (4.5) and (4.6) are satisfied.*

Proof. a) Since g^* is the control function, from Theorem 4.2 we obtain (4.5). By Theorem 2.2 the function $g-g^*$ is convex, then from Proposition 4.1 it follows (4.6). To prove b) assume that the inequality (4.2) is satisfied. Then from Theorem 4.1 we obtain that f is g-convex dominated, by a) it follows that the inequalities (4.5) and (4.6) are satisfied. The theorem is proved. □

Let I be an open interval. Let $f\colon I\to\mathbb{R}$ be a convex function and $a,b\in I$ with $a<b$. The following double inequality

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\cdot\int_a^b f(x)\,dx\leqslant\frac{f(a)+f(b)}{2} \tag{4.7}$$

is known as the Hermite-Hadamard inequality for convex functions (see [12], [6]). In [7] Fejér gave a generalization of the inequality (4.7):

PROPOSITION 4.2. *Let $f\colon I\to\mathbb{R}$ be a convex function defined on a real interval I, $a,b\in I$ with $a<b$ and let $g\colon [a,b]\to\mathbb{R}$ be nonnegative and symmetric with respect to the point $(a+b)/2$ (the existence of integrals is assumed in all formulas). Then*

$$f\left(\frac{a+b}{2}\right)\cdot\int_a^b g(x)\,dx\leqslant\int_a^b f(x)g(x)\,dx\leqslant\frac{f(a)+f(b)}{2}\cdot\int_a^b g(x)\,dx. \tag{4.8}$$

The double inequality (4.8) is known in the literature as the Hermite-Hadamard-Fejér inequality (see [12], [6] and [13] for the historical background).

In the paper of Dragomir et al. (2002) [5] can be found the Hermite-Hadamard type inequalities for g-convex dominated functions.

PROPOSITION 4.3. ([5]) *Let $g\colon I\to\mathbb{R}$ be a convex function and $f\colon I\to\mathbb{R}$ be a g-convex dominated function. Then, for all $a,b\in I$ with $a<b$,*

$$\left|\frac{1}{b-a}\cdot\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right|\leqslant\frac{1}{b-a}\cdot\int_a^b g(x)\,dx-g\left(\frac{a+b}{2}\right) \tag{4.9}$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\cdot\int_a^b f(x)\,dx\right|\leqslant\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\cdot\int_a^b g(x)\,dx. \tag{4.10}$$

We will give a generalization of the inequalities (4.9) and (4.10) of Dragomir et al. (2002).

In the sequel we will to make use of the theory of s-convex stochastic ordering (see Denuit et al. (1998) [4]). Let us review some notations.

As usual, F_X denotes the cumulative distribution function (or the distribution function) of a random variable X and μ_X is the distribution corresponding to X. For real valued random variables X,Y and any integer $s\geqslant 1$, we say that X is dominated by Y in the $(s+1)$-*convex ordering* sense if $\mathbb{E}f(X)\leqslant\mathbb{E}f(Y)$ for all s-convex functions $f\colon\mathbb{R}\to\mathbb{R}$, for which the expectations exist. In that case we write $X\leqslant_{(s+1)-cx}Y$, or $\mu_X\leqslant_{(s+1)-cx}\mu_Y$. Then the order $\leqslant_{2-cx}$ is just the usual convex order $\leqslant_{cx}$.

A very useful criterion for the verification of the $(s+1)$-convex order is given by Denuit, Lefèvre and Shaked in 1998 [4]. For the statement of this criterion, we need introduce first the following notation. Define the number of sign changes of a function $\varphi\colon\mathbb{R}\to\mathbb{R}$ by

$$S^-(\varphi)=\sup\{S^-[\varphi(x_1),\varphi(x_2),\ldots,\varphi(x_n)]\colon x_1<x_2<\ldots<x_n\in\mathbb{R},\ n\in\mathbb{N}\},$$

where $S^-[y_1,y_2,\ldots,y_n]$ denotes the number of sign changes in the sequence $y_1,y_2,\ldots,y_n$ (zero terms are being discarded). Two real functions φ_1,φ_2 are said to have k crossing points (or cross each other k-times) if $S^-(\varphi_1-\varphi_2)=k$.

PROPOSITION 4.4. ([4]) *Let X and Y be two random variables such that $\mathbb{E}(X^j-Y^j)=0$, $j=1,2,\ldots,s$ $(s\geqslant 1)$. If $S^-(F_X-F_Y)=s$ and the last sign of F_X-F_Y is positive, then $X\leqslant_{(s+1)-cx}Y$.*

REMARK 4.2. a) Let X, Y, Z be three random variables such that $\mu_X=\delta_{(a+b)/2}$, $\mu_Y(dx)=\frac{1}{b-a}dx$, $\mu_Z=\frac{1}{2}(\delta_a+\delta_b)$. Then, by Proposition 4.4, we obtain that $X\leqslant_{2-cx}Y$ and $Y\leqslant_{2-cx}Z$, which implies (4.7).

b) Let f and g satisfy the assumptions of Proposition 4.2. Let X, Y, Z be three random variables such that $\mu_X=\delta_{(a+b)/2}$, $\mu_Y(dx)=(\int_a^b g(x)dx)^{-1}g(x)dx$, $\mu_Z=\frac{1}{2}(\delta_a+\delta_b)$. Then, by Proposition 4.4, we obtain that $X\leqslant_{2-cx}Y$ and $Y\leqslant_{2-cx}Z$, which implies (4.8).

The following theorem provides a useful tool for obtaining and proving of various forms of the Hermite-Hadamard-Fejér type inequality for delta-convex functions of higher order.

THEOREM 4.3. *Let $n\geqslant 1$. Let $g\colon I\to\mathbb{R}$ be an n-convex function and let $f\colon I\to\mathbb{R}$ be a function which is g-convex dominated of n-th order (or delta-convex of n-th order with the control function g). Let X and Y be two I valued random variables such that $\mathbb{E}(X^j-Y^j)=0$, $j=1,2,\ldots,n$, $S^-(F_X-F_Y)=n$ and the last sign of F_X-F_Y is positive. Then*

$$|\mathbb{E}f(Y)-\mathbb{E}f(X)|\leqslant\mathbb{E}g(Y)-\mathbb{E}g(X). \tag{4.11}$$

Proof. By Lemma 1.1 f is g-convex dominated of n-th order if and only if the functions $g-f$ and $g+f$ are n-convex on I. Let the random variables X and Y satisfy the assumptions of the theorem. Then by Proposition 4.4, for any n-convex function $h\colon I\to\mathbb{R}$ we have the inequality

$$\mathbb{E}h(X)\leqslant\mathbb{E}h(Y). \tag{4.12}$$

Since the functions $g-f$ and $g+f$ are n-convex on I, by (4.12) we obtain the following inequalities

$$\mathbb{E}(g-f)(X)\leqslant\mathbb{E}(g-f)(Y), \tag{4.13}$$

$$\mathbb{E}(g+f)(X)\leqslant\mathbb{E}(g+f)(Y). \tag{4.14}$$

From (4.13) and (4.14) we obtain

$$\mathbb{E}f(Y)-\mathbb{E}f(X)\leqslant\mathbb{E}g(Y)-\mathbb{E}g(X),$$

$$-(\mathbb{E}g(Y)-\mathbb{E}g(X))\leqslant\mathbb{E}f(Y)-\mathbb{E}f(X),$$

which are equivalent to (4.11). The theorem is proved. □

REMARK 4.3. Let $n = 1$. Let $f\colon I \to \mathbb{R}$ be a function which is g-convex dominated.

a) To prove (4.9), we consider random variables X and Y such that $\mu_X = \delta_{(a+b)/2}$ and $\mu_Y(dx) = \frac{1}{b-a}dx$, and then apply Theorem 4.3.

b) To prove (4.10), it suffices to consider random variables X and Y such that $\mu_X(dx) = \frac{1}{b-a}dx$ and $\mu_Y = \frac{1}{2}(\delta_a + \delta_b)$, and then apply Theorem 4.3.

Taking some particular cases of random variables X, Y, by Theorem 4.3, we obtain the following Hermite-Hadamard-Fejér type inequalities for delta-convex functions of higher order, that generalize the results of Dragomir et al. (2002) [5].

THEOREM 4.4. *Let $n \geqslant 1$. Let $g\colon I \to \mathbb{R}$ be an n-convex function and let $f\colon I \to \mathbb{R}$ be a g-convex dominated of n-th order function. Let $a, b \in I$ with $a < b$.*

Let $a(n) = \left[\frac{n}{2}\right] + 1$, $b(n) = \left[\frac{n+1}{2}\right] + 1$. Let $x_1, \ldots, x_{a(n)}, y_1, \ldots, y_{b(n)}$ be real numbers, and $\alpha_1, \ldots, \alpha_{a(n)}$, $\beta_1, \ldots, \beta_{b(n)}$ be positive numbers, such that $\alpha_1 + \ldots + \alpha_{a(n)} = 1$, $\beta_1 + \ldots + \beta_{b(n)} = 1$,

$$\frac{1}{b-a}\int_a^b x^k dx = \sum_{j=1}^{b(n)} y_j^k \beta_j = \sum_{i=1}^{a(n)} x_i^k \alpha_i \quad (k = 1, 2, \ldots, n),$$

$a \leqslant x_1 < x_2 < \ldots < x_{a(n)} \leqslant b$, $a \leqslant y_1 < y_2 < \ldots < y_{b(n)} < b$,

$$\begin{gathered} \tfrac{x_1-a}{b-a} < \alpha_1 < \tfrac{x_2-a}{b-a}, \\ \tfrac{x_2-a}{b-a} < \alpha_1 + \alpha_2 < \tfrac{x_3-a}{b-a}, \\ \ldots \\ \tfrac{x_{a(n)-1}-a}{b-a} < \alpha_1 + \ldots + \alpha_{a(n)-1} < \tfrac{x_{a(n)}-a}{b-a}, \end{gathered}$$

$$\begin{gathered} \tfrac{y_1-a}{b-a} < \beta_1 < \tfrac{y_2-a}{b-a}, \\ \tfrac{y_2-a}{b-a} < \beta_1 + \beta_2 < \tfrac{y_2-a}{b-a}, \\ \ldots \\ \tfrac{y_{b(n)-1}-a}{b-a} < \beta_1 + \ldots + \beta_{b(n)-1} < \tfrac{y_{b(n)}-a}{b-a}; \end{gathered}$$

if n is even then $y_1 = a$, $y_{b(n)} = b$, $x_1 > a$, $x_{a(n)} < b$;
if n is odd then $y_1 = a$, $y_{b(n)} < b$, $x_1 > a$, $x_{a(n)} = b$.

Then we have the following inequalities:

i) *if n is even then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \sum_{i=1}^{a(n)} \alpha_i f(x_i)\right| \leqslant \frac{1}{b-a}\int_a^b g(x)dx - \sum_{i=1}^{a(n)} \alpha_i g(x_i), \tag{4.15}$$

$$\left|\sum_{j=1}^{b(n)} \beta_j f(y_j) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \sum_{j=1}^{b(n)} \beta_j g(y_j) - \frac{1}{b-a}\int_a^b g(x)dx, \tag{4.16}$$

ii) *if n is odd then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \sum_{j=1}^{b(n)} \beta_j f(y_j)\right| \leqslant \frac{1}{b-a}\int_a^b g(x)dx - \sum_{j=1}^{b(n)} \beta_j g(y_j), \tag{4.17}$$

$$\left|\sum_{i=1}^{a(n)} \alpha_i f(x_i) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \sum_{i=1}^{a(n)} \alpha_i g(x_i) - \frac{1}{b-a}\int_a^b g(x)dx. \tag{4.18}$$

Proof. Let X,Y and Z be random variables such that

$$\mu_X = \sum_{i=1}^{a(n)} \alpha_i \delta_{x_i},$$

$$\mu_Y = \sum_{j=1}^{b(n)} \beta_j \delta_{y_j},$$

$$\mu_Z(dx) = \frac{1}{b-a}\chi_{[a,b]}(x)dx.$$

We now apply Theorem 4.3.

If n is even then we take the pairs (X,Z) and (Z,Y) in place of (X,Y) to obtain the inequalities (4.15) and (4.16), respectively.

If n is odd then we take the pairs (Y,Z) and (Z,X) in place of (X,Y) to obtain the inequalities (4.17) and (4.18), respectively. This completes the proof. □

THEOREM 4.5. *Let $n \geqslant 1$. Let $g : I \to \mathbb{R}$ be an n-convex function and let $f : I \to \mathbb{R}$ be a g-convex dominated of n-th order function. Let $a,b \in I$ with $a < b$.*

Let $a(n) = \left[\frac{n}{2}\right] + 1,\ b(n) = \left[\frac{n+1}{2}\right] + 1$.

Let $\alpha_1,\ldots,\alpha_{a(n)},\ x_1,\ldots,x_{a(n)},\ \beta_1,\ldots,\beta_{b(n)},\ y_1,\ldots,y_{b(n)}$ be real numbers such that

a) *if n is even then*

$$0 < \beta_1 < \alpha_1 < \beta_1 + \beta_2 < \alpha_1 + \alpha_2 < \ldots < \alpha_1 + \ldots + \alpha_{a(n)} = \beta_1 + \ldots + \beta_{b(n)} = 1,$$
$$a \leqslant y_1 < x_1 < y_2 < x_2 < \ldots < x_{a(n)} < y_{b(n)} \leqslant b,$$

b) *if n is odd then*

$$0 < \beta_1 < \alpha_1 < \beta_1 + \beta_2 < \alpha_1 + \alpha_2 < \ldots < \beta_1 + \ldots + \beta_{b(n)} < \alpha_1 + \ldots + \alpha_{a(n)} = 1$$
$$a \leqslant y_1 < x_1 < y_2 < x_2 < \ldots < y_{b(n)} < x_{a(n)} \leqslant b;$$

and

$$\sum_{i=1}^{a(n)} x_i^k \alpha_i = \sum_{j=1}^{b(n)} y_j^k \beta_j,$$

for any $k = 1,2,\ldots,n$.

Then we have the following inequalities:

i) *if n is even then*

$$\left|\sum_{j=1}^{b(n)} \beta_j f(y_j) - \sum_{i=1}^{a(n)} \alpha_i f(x_i)\right| \leqslant \sum_{j=1}^{b(n)} \beta_j g(y_j) - \sum_{i=1}^{a(n)} \alpha_i g(x_i), \tag{4.19}$$

ii) *if n is odd then*

$$\left|\sum_{i=1}^{a(n)} \alpha_i f(x_i) - \sum_{j=1}^{b(n)} \beta_j f(y_j)\right| \leqslant \sum_{i=1}^{a(n)} \alpha_i g(x_i) - \sum_{j=1}^{b(n)} \beta_j g(y_j). \tag{4.20}$$

Proof. We apply Theorem 4.3. Let X and Y be random variables such that:
if n is even, then we take

$$\mu_X = \sum_{i=1}^{a(n)} \alpha_i \delta_{x_i}, \quad \mu_Y = \sum_{j=1}^{b(n)} \beta_j \delta_{y_j},$$

to obtain the inequality (4.19),
if n is odd, then we take

$$\mu_X = \sum_{j=1}^{b(n)} \beta_j \delta_{y_j}, \quad \mu_Y = \sum_{i=1}^{a(n)} \alpha_i \delta_{x_i},$$

to obtain (4.20). The theorem is proved. □

5. Inequalities between quadrature operators

Many inequalities, which are connected with quadrature operators are known in the numerical analysis (cf. [1], [3] and the references therein).

The numerical analysts prove them using the suitable differentiability assumptions. As proved Wąsowicz in the papers [20], [21], [22], for convex functions of higher order some inequalities can be obtained without assumptions of this kind, using only the higher order convexity itself. The support-type properties play here the crucial role. As we will show in this paper, some inequalities can be obtained using a probabilistic characterization. In this paper we obtain new inequalities concerning delta-convex functions of higher order. Our method of proof using the convex stochastic ordering seems to be quite easy.

For a function $f : [-1,1] \to \mathbb{R}$ we consider six operators approximating the integral mean value

$$\mathscr{I}(f) := \frac{1}{2}\int_{-1}^{1} f(x)dx.$$

They are

$$C(f) := \frac{1}{3}\Big(f\big(-\tfrac{\sqrt{2}}{2}\big) + f(0) + f\big(\tfrac{\sqrt{2}}{2}\big)\Big),$$

$$\begin{aligned}
\mathscr{G}_2(f) &:= \tfrac{1}{2}\Big(f\big(-\tfrac{\sqrt{3}}{3}\big)+f\big(\tfrac{\sqrt{3}}{3}\big)\Big),\\
\mathscr{G}_3(f) &:= \tfrac{4}{9}f(0)+\tfrac{5}{18}\Big(f\big(-\tfrac{\sqrt{15}}{5}\big)+f\big(\tfrac{\sqrt{15}}{5}\big)\Big),\\
\mathscr{L}_4(f) &:= \tfrac{1}{12}\big(f(-1)+f(1)\big)+\tfrac{5}{12}\Big(f\big(-\tfrac{\sqrt{5}}{5}\big)+f\big(\tfrac{\sqrt{5}}{5}\big)\Big),\\
\mathscr{L}_5(f) &:= \tfrac{16}{45}f(0)+\tfrac{1}{20}\big(f(-1)+f(1)\big)+\tfrac{49}{180}\Big(f\big(-\tfrac{\sqrt{21}}{7}\big)+f\big(\tfrac{\sqrt{21}}{7}\big)\Big),\\
S(f) &:= \tfrac{1}{6}\big(f(-1)+f(1)\big)+\tfrac{2}{3}f(0).
\end{aligned}$$

The operators $\mathscr{G}_2$ and $\mathscr{G}_3$ are connected with Gauss-Legendre rules. The operators $\mathscr{L}_4$ and $\mathscr{L}_5$ are connected with Lobatto quadratures. The operators S and C concern Simpson and Chebyshev quadrature rules, respectively. The operator $\mathscr{I}$ stands for the integral mean value (see e.g. [16], [23], [24], [25], [26]).

We will establish all possible inequalities between these operators in the class of g convex dominated functions of higher order.

REMARK 5.1. Let X_2, X_3, Y_4, Y_5, U, V and Z be random variables such that

$$\begin{aligned}
\mu_{X_2} &= \frac{1}{2}\left(\delta_{-\frac{\sqrt{3}}{3}}+\delta_{\frac{\sqrt{3}}{3}}\right),\\
\mu_{X_3} &= \frac{4}{9}\delta_0+\frac{5}{18}\left(\delta_{-\frac{\sqrt{15}}{5}}+\delta_{\frac{\sqrt{15}}{5}}\right),\\
\mu_{Y_4} &= \frac{1}{12}(\delta_{-1}+\delta_1)+\frac{5}{12}\left(\delta_{-\frac{\sqrt{5}}{5}}+\delta_{\frac{\sqrt{5}}{5}}\right),\\
\mu_{Y_5} &= \frac{16}{45}\delta_0+\frac{1}{20}(\delta_{-1}+\delta_1)+\frac{49}{180}\left(\delta_{-\frac{\sqrt{21}}{7}}+\delta_{\frac{\sqrt{21}}{7}}\right),\\
\mu_U &= \frac{2}{3}\delta_0+\frac{1}{6}(\delta_{-1}+\delta_1),\\
\mu_V &= \frac{1}{3}\left(\delta_{-\frac{\sqrt{2}}{2}}+\delta_0+\delta_{\frac{\sqrt{2}}{2}}\right),\\
\mu_Z(dx) &= \frac{1}{2}\chi_{[-1,1]}(x)dx.
\end{aligned}$$

Then we have

$$\mathscr{G}_2(f)=\mathbb{E}[f(X_2)],\quad \mathscr{G}_3(f)=\mathbb{E}[f(X_3)],$$

$$\mathscr{L}_4(f)=\mathbb{E}[f(Y_4)],\quad \mathscr{L}_5(f)=\mathbb{E}[f(Y_5)],$$

$$S(f)=\mathbb{E}[f(U)],\quad C(f)=\mathbb{E}[f(V)],\quad \mathscr{I}(f)=\mathbb{E}[f(Z)].$$

THEOREM 5.1. *Let $f\colon[-1,1]\to\mathbb{R}$ be g-convex dominated of 3-order. Then*

$$|\mathscr{I}(f)-\mathscr{G}_2(f)|\leqslant\mathscr{I}(g)-\mathscr{G}_2(g), \tag{5.1}$$

$$|S(f)-\mathscr{I}(f)|\leqslant S(g)-\mathscr{I}(g),$$

$$|C(f)-\mathscr{G}_2(f)| \leqslant C(g)-\mathscr{G}_2(g),$$
$$|T(f)-C(f)| \leqslant T(g)-C(g),$$
$$|S(f)-T(f| \leqslant S(g)-T(g),$$

where $T \in \{\mathscr{G}_3, \mathscr{L}_5\}$.

Proof. Let $n=3$. Let $f\colon [-1,1] \to \mathbb{R}$ be g-convex dominated of 3-order. It is not difficult to prove that for $X := X_2$ and $Y := Z$ the assumptions of Theorem 4.3 are satisfied. Taking into account that, by Remark 5.1, we have $\mathscr{G}_2(f) = \mathbb{E}[f(X_2)]$ and $\mathscr{I}(f) = \mathbb{E}[f(Z)]$, from Theorem 4.3 it follows the inequality (5.1).

Similarly, considering the following pairs of random variables: (Z,U), (X_2,V), (V,X_3), (V,Y_5), (X_3,U) and (Y_5,U) in place of the pair (X,Y), respectively, from Theorem 4.3 and taking into account Remark 5.1, we obtain the other inequalities of this theorem. The theorem is proved. □

THEOREM 5.2. *Let* $f\colon [-1,1] \to \mathbb{R}$ *be* g*-convex dominated of 5-order. Then*

$$|\mathscr{I}(f)-\mathscr{G}_3(f)| \leqslant \mathscr{I}(g)-\mathscr{G}_3(g),$$
$$|\mathscr{L}_4(f)-\mathscr{I}(f)| \leqslant \mathscr{L}_4(g)-\mathscr{I}(g),$$
$$|\mathscr{L}_5(f)-\mathscr{G}_3(f)| \leqslant \mathscr{L}_5(g)-\mathscr{G}_3(g),$$
$$|\mathscr{L}_4(f)-\mathscr{L}_5(f)| \leqslant \mathscr{L}_4(g)-\mathscr{L}_5(g),$$

Proof. Let $n=5$. Let $f\colon [-1,1] \to \mathbb{R}$ be g-convex dominated of 5-order. The proof is similar to the proof of Theorem 5.1. It suffices to apply Theorem 4.3 and Remark 5.1, considering the following pairs of random variables: (X_3,Z), (Z,Y_4), (X_3,Y_5) and (Y_5,Y_4) in place of the pair (X,Y), respectively. The theorem is proved. □

REFERENCES

[1] M. BESSENYEI, *Hermite–Hadamard-type inequalities for generalized convex functions*, J. Inequal. Pure Appl. Math., **9** (2008), 1–51.

[2] P. BILLINGSLEY, *Probability and Measure*, John Wiley & Sons, New York, 1995.

[3] H. BRASS AND K. PETRAS, *Quadrature theory. The theory of numerical integration on a compact interval*, Mathematical Surveys and Monographs, **178**, American Mathematical Society, Providence, RI, 2011.

[4] M. DENUIT, C. LEFÈVRE AND M. SHAKED, *The s-convex orders among real random variables, with applications*, Math. Inequal. Appl., **1** (1998), 585–613.

[5] S. S. DRAGOMIR, C. E. M. PEARCE, J. PEČARIĆ, *Means, g-Convex Dominated Functions and Hadamard-Type Inequalities*, Tamsui Oxford Journal of Mathematical Sciences, **18** (2) (2002), 161–173.

[6] S. S. DRAGOMIR AND C. E. M. PEARCE, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.
(Online: `http://rgmia.vu.edu.au/monographs/`).

[7] L. FEJÉR, *Über die Fourierreihen, II*, Math. Naturwiss. Anz. Ungar. Akad. Wiss., **24** (1906), 369–390.

[8] R. GER, *Stability of polynomial mappings controlled by n-convex functionals*, World Sci. Publ. Company (WSSIAA), **3** (1994) 255–268.

[9] R. GER AND K. NIKODEM, *Strongly convex functions of higher order*, Nonlinear Anal., **74** (2011), 661–665.

[10] P. HARTMAN, *On functions representable as difference of convex functions*, Pacific J. Math., **9** (1959) 707–713.

[11] M. KUCZMA, *An Introduction to the Theory of Functional Equations and Inequalities*, Prace Naukowe Uniwersytetu Śląskiego w Katowicach, vol. 489, Państwowe Wydawnictwo Naukowe – Uniwersytet Śląski, Warszawa, Kraków, Katowice, 1985.

[12] D. S. MITRINOVIĆ AND I. B. LACKOVIĆ, *Hermite and convexity*, Aequationes Math., **28** (3) (1985), 229–232.

[13] J. E. PEČARIĆ, F. PROSCHAN, Y. L. TONG, *Convex functions*, Academic Press, Inc., 1992.

[14] T. RAJBA, *New integral representations of nth order convex functions*, J. Math. Anal. Appl., **379** (2) (2011), 736–747.

[15] T. RAJBA, *On the Ohlin lemma for Hermite-Hadamard-Fejér type inequalities*, Math. Inequal. Appl. **17**, 2 (2014) 557–571.

[16] A. RALSTON, *A First Course in Numerical Analysis*, McGraw-Hill Book Company, New York, St. Louis, San Francisco, Toronto, London, Sydney, 1965.

[17] A. W. ROBERTS, D. E. VARBERG, *Convex Functions*, Pure and Applied Mathematics, vol. **57**, Academic Press, New York-London, 1973.

[18] H. L. ROYDEN, *Real analysis*, Collier Macmillan, 1966.

[19] L. VESELÝ, L. ZAJIČEK, *Delta-convex mappings between Banach spaces and applications*, Dissertationes Math., **289**, Polish Scientific Publishers, Warszawa, 1989.

[20] SZ. WĄSOWICZ, *Inequalities between the quadrature operators and error bounds of quadrature rules*, J. Inequal. Pure Appl. Math., **8** (2007), no. 2., Article 42, 8 pp.

[21] SZ. WĄSOWICZ, *On quadrature rules, inequalities and error bounds*, J. Inequal. Pure Appl. Math., **9** (2008), no. 2, Article 36, 4 pp.

[22] SZ. WĄSOWICZ, *On some extremalities in the approximate integration*, Math. Inequal. Appl., **13** (2010), 165–174.

[23] E. W. WEISSTEIN, *Chebyshev Quadrature*, From MathWorld – A Wolfram Web Resource. (Online: `http://mathworld.wolfram.com/ChebyshevQuadrature.html`).

[24] E. W. WEISSTEIN, *Legendre-Gauss Quadrature*, From MathWorld – A Wolfram Web Resource. (Online: `http://mathworld.wolfram.com/Legendre-GaussQuadrature.html`).

[25] E. W. WEISSTEIN, *Lobatto Quadrature*, From MathWorld – A Wolfram Web Resource. (Online: `http://mathworld.wolfram.com/LobattoQuadrature.html`).

[26] E. W. WEISSTEIN, *Simpson's Rule*, From MathWorld – A Wolfram Web Resource. (Online: `http://mathworld.wolfram.com/SimpsonsRule.html`).

(Received September 23, 2013)

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