Mathematical
Inequalities
& Applications
Volume 18, Number 1 (2015), 337–345 doi:10.7153/mia-18-23

IMPROVEMENTS IN THE UPPER BOUNDS FOR THE SPREAD OF A MATRIX

PINGPING ZHANG AND HU YANG

(*Communicated by I. Perić*)

Abstract. In this paper, we present some new upper bounds for the spread of a matrix. These bounds improve the previous results. In addition, one of these bounds can be up to the optimum. Finally, some numerical matrices are given to show the effectiveness of our results.

1. Introduction

Let $C^{n\times n}$ and I_n denote the set of $n\times n$ complex matrices and the identity matrix of order n, respectively. For $A\in C^{n\times n}$, we denote by $\rho(A)$, $r(A)$, $\|A\|_F$, $\operatorname{tr}A$, A^*, $\operatorname{spe}(A)$ the spectral radius, the rank, the Frobenius norm, the trace, the conjugate transpose and the spectrum of A, respectively. If $A=(a_{ij})\in C^{n\times n}$, we write $R_i(A)=\sum_{j=1}^n |a_{ij}|^2$ and $C_i(A)=\sum_{j=1}^n |a_{ji}|^2$. Moreover, $A,B\in C^{n\times n}$, we denote $[A,B]=AB-BA$.

Let $A\in C^{n\times n}$, $\lambda_i,\lambda_j\in\operatorname{spe}(A)$, the spread $s(A)$ of A is defined by

$$s(A)=\max_{i,j}|\lambda_i-\lambda_j|.$$

The conception of spread was firstly proposed by L. Mirsky in 1956, where upper bounds were obtained for $s(A)$, and since then the bound for $s(A)$ has been studied by several authors [1, 2, 3, 4, 6, 7, 8, 10, 11, 12]. Now we list some published upper bounds for the spread of a matrix.

Let $A=(a_{ij})\in C^{n\times n}$. A simple upper bound for the spread of A was given by Mirsky in [8]

$$s(A)\leqslant\sqrt{2}\Big(\|A\|_F^2-\frac{|\operatorname{tr}A|^2}{n}\Big)^{\frac12}, \tag{1}$$

which was improved by B. Tu in [11] as follows

$$s(A)\leqslant\sqrt{2\min_{1\leqslant k\leqslant n-1}\Big\{\|A_k\|_F^2+\|D_k\|_F^2+2\|B_k\|_F\|C_k\|_F-\frac{|\operatorname{tr}A|^2}{n}\Big\}}, \tag{2}$$

Mathematics subject classification (2010): 15A09, 15A18, 15A45, 65F35.
Keywords and phrases: The spread of a matrix, upper bounds, singular matrix, improvement, eigenvalue.

Paper MIA-18-23

where $A \in C^{n\times n}$ was partitioned as

$$A = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}, \text{ and } A_k \in C^{k\times k},\ 1 \leqslant k \leqslant n-1. \tag{3}$$

In 2012, Wu, Zhang and Liao [12] provided the following two upper bounds:

$$s(A) \leqslant 2\sqrt{\frac{n-1}{n}\left(\|A\|_F^2 - \frac{|\operatorname{tr} A|^2}{n}\right) - \sum_{i=1}^{n} \frac{(R_i(A)-C_i(A))^2}{4(R_i(A)+C_i(A)+2|a_{ii}|^2+4\|A\|_F^2)}}, \tag{4}$$

and

$$s(A) \leqslant 2\sqrt{\frac{n-1}{n}}\left(\left(\|A\|_F^2 - \frac{|\operatorname{tr} A|^2}{n}\right)^2 - \frac{1}{2}\|[A,A^*]\|_F^2\right)^{\frac{1}{4}}. \tag{5}$$

The bounds in (4) and (5) were improved, respectively, by Sharma and Kumar [10] as follows

$$s(A) \leqslant \sqrt{2\sum_{i=1}^{n}(R_i(A)C_i(A))^{\frac{1}{2}} - \frac{2|\operatorname{tr} A|^2}{n}}, \tag{6}$$

and

$$s(A) \leqslant \left(\left(2\|A\|_F^2 - \frac{2|\operatorname{tr} A|^2}{n}\right)^2 - 2\|[A,A^*]\|_F^2\right)^{\frac{1}{4}}. \tag{7}$$

The bounds in (6) and (7) are the refinements of the bound in (1) by the fact that $\sum_{i=1}^{n}(R_i(A)C_i(A))^{\frac{1}{2}} \leqslant \sum_{i=1}^{n}\frac{R_i(A)+C_i(A)}{2} = \|A\|_F^2$ and $\|[A,A^*]\|_F^2 \geqslant 0$. However, none of the bounds in (2), (6) and (7) is uniformly better than the others. This will be shown in our example.

In this paper, we present three new upper bounds for the spread of a matrix. These bounds are the refinements of the bounds in (2), (6) and (7), respectively. In addition, one of these bounds can be up to the optimum. Finally, we use some numerical matrices to show the effectiveness of our results.

To facilitate our statements, if $A \in C^{n\times n}$ is partitioned as in (3), we denote

$$A(k) = \begin{cases} \begin{pmatrix} A_k & \sqrt{\frac{\|C_k\|_F}{\|B_k\|_F}}B_k \\ \sqrt{\frac{\|B_k\|_F}{\|C_k\|_F}}C_k & D_k \end{pmatrix} & if\ \ \|B_k\|_F\|C_k\|_F \neq 0, \\ \begin{pmatrix} A_k & \mathbf{0} \\ \mathbf{0} & D_k \end{pmatrix} & otherwise. \end{cases}$$

In addition, we write $A(n) = A$, $B_n = \mathbf{0}$ and $C_n = \mathbf{0}$.

2. Some lemmas

In this section, we list some useful lemmas. The first lemma was proposed by Kress in [5].

LEMMA 1. *Let $A \in C^{n\times n}$. If $\lambda_1,\cdots,\lambda_n$ are all the eigenvalues of A, then*

$$\sum_{i=1}^{n}|\lambda_i|^2 \leqslant \left\{\|A\|_F^4 - \frac{1}{2}\|[A,A^*]\|_F^2\right\}^{\frac{1}{2}}.$$

Next, we present a tighter upper bound for the sum of the squares of the magnitudes of all the eigenvalues of a matrix as follows.

LEMMA 2. *Let $A \in C^{n\times n}$ be partitioned as in (3). If $\lambda_1,\cdots,\lambda_n$ are all the eigenvalues of A, then*

$$\sum_{i=1}^{n}|\lambda_i|^2 \leqslant \min_{1\leqslant k\leqslant n}\left\{\left(\|A\|_F^2 - (\|B_k\|_F - \|C_k\|_F)^2\right)^2 - \frac{1}{2}\|[A(k),A^*(k)]\|_F^2\right\}^{\frac{1}{2}}. \quad (8)$$

Proof. There are two cases to consider.

Case 1. When $\|B_k\|_F\|C_k\|_F \neq 0$, we let $x = \sqrt{\frac{\|C_k\|_F}{\|B_k\|_F}} > 0$.

Since

$$\begin{pmatrix} xI_k & 0 \\ 0 & I_{n-k} \end{pmatrix}\begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}\begin{pmatrix} \frac{I_k}{x} & 0 \\ 0 & I_{n-k} \end{pmatrix} = \begin{pmatrix} A_k & xB_k \\ \frac{C_k}{x} & D_k \end{pmatrix} = A(k),$$

then $\lambda_1,\lambda_2,\cdots,\lambda_n$ are all the eigenvalues of $A(k)$.

Case 2. When $\|B_k\|_F\|C_k\|_F = 0$, without loss of generality, we suppose $\|C_k\|_F = 0$. By Schur decomposition, we know that there are two unitary matrices $U_1 \in C^{k\times k}$, $U_2 \in C^{(n-k)\times(n-k)}$ such that

$$U_1A_kU_1^* = \begin{pmatrix} \tilde{\lambda}_1 & * & * & * \\ 0 & \tilde{\lambda}_2 & * & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{\lambda}_k \end{pmatrix},$$

and

$$U_2D_kU_2^* = \begin{pmatrix} \tilde{\lambda}_{k+1} & * & * & * \\ 0 & \tilde{\lambda}_{k+2} & * & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{\lambda}_n \end{pmatrix},$$

where $\tilde{\lambda}_1,\tilde{\lambda}_2,\cdots,\tilde{\lambda}_n$ are all the eigenvalues of $A(k)$.

Let $U=\operatorname{diag}(U_1,U_2)$. Obviously, U is a unitary matrix. Since

$$UAU^*=\begin{pmatrix} U_1A_kU_1^* & U_1B_kU_2^* \\ 0 & U_2D_kU_2^* \end{pmatrix}=\begin{pmatrix} \tilde{\lambda}_1 & * & * & * \\ 0 & \tilde{\lambda}_2 & * & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{\lambda}_n \end{pmatrix},$$

we know that A and $A(k)$ have the same eigenvalues.

It follows from Case 1 and Case 2 that $\lambda_1,\lambda_2,\cdots,\lambda_n$ are also the eigenvalues of $A(k)$. Applying Lemma 1 to $A(k)$ leads to

$$\begin{aligned}\sum_{i=1}^{n}|\lambda_i|^2 &\leqslant \left\{\|A(k)\|_F^4-\frac{1}{2}\|[A(k),A^*(k)]\|_F^2\right\}^{\frac{1}{2}}\\ &=\left\{\left(\|A\|_F^2-(\|B_k\|_F-\|C_k\|_F)^2\right)^2-\frac{1}{2}\|[A(k),A^*(k)]\|_F^2\right\}^{\frac{1}{2}}.\end{aligned}$$

The inequality (8) holds by the arbitrariness of k $(1\leqslant k\leqslant n-1)$ and Lemma 1. □

In 1980, Nowosad and Tovar gave the following lemma which can be found in [9].

LEMMA 3. *Let $A\in C^{n\times n}$. If $\lambda_1,\cdots,\lambda_n$ are all the eigenvalues of A, then*

$$\sum_{i=1}^{n}|\lambda_i|^2\leqslant\sum_{i=1}^{n}\left(R_i(A)C_i(A)\right)^{\frac{1}{2}}.$$

3. Main results

In this section, we will present three new upper bounds for the spread of a matrix. First, we give a refinement of the bound in (2).

THEOREM 1. *Let $A\in C^{n\times n}$ be partitioned as in (3). Then*

$$s(A)\leqslant\sqrt{2\left(\min_{1\leqslant k\leqslant n}\left\{\left(\|A\|_F^2-(\|B_k\|_F-\|C_k\|_F)^2\right)^2-\frac{1}{2}\|[A(k),A^*(k)]\|_F^2\right\}^{\frac{1}{2}}-\frac{|\operatorname{tr}A|^2}{l}\right)},$$

where $l=\min\{r(A)+1,\ n\}$.

Proof. Firstly, we prove the following inequality

$$s(A)^2\leqslant 2\Big(\sum_{i=1}^{n}|\lambda_i|^2-\frac{1}{l}|\sum_{i=1}^{n}\lambda_i|^2\Big). \tag{9}$$

There are two cases to consider.

Case 1. When $r(A)+1 \geqslant n$. Suppose that $\lambda_1, \lambda_2, \cdots, \lambda_n$ are all the eigenvalues of A and suppose

$$s(A) = \max_{1 \leqslant i,j \leqslant n} |\lambda_i - \lambda_j| = |\lambda_p - \lambda_q|,\ 1 \leqslant p < q \leqslant n.$$

By Lagrange's identity, we have

$$\begin{aligned} n\sum_{i=1}^{n}|\lambda_i|^2 - |\sum_{i=1}^{n}\lambda_i|^2 = & \sum_{1\leqslant i<j\leqslant n}|\lambda_i-\lambda_j|^2 \geqslant |\lambda_p-\lambda_q|^2 \\ & + \sum_{j=1, j\neq p,q}^{n}|\lambda_p-\lambda_j|^2 + \sum_{j=1, j\neq p,q}^{n}|\lambda_q-\lambda_j|^2 \\ & \geqslant |\lambda_p-\lambda_q|^2 + \frac{1}{2}\sum_{j=1, j\neq p,q}^{n}|\lambda_p-\lambda_q|^2 \\ & = \frac{n}{2}|\lambda_p-\lambda_q|^2 = \frac{n}{2}s(A)^2, \end{aligned}$$

which gives

$$s(A)^2 \leqslant 2\Big(\sum_{i=1}^{n}|\lambda_i|^2 - \frac{1}{n}|\sum_{i=1}^{n}\lambda_i|^2\Big) = 2\Big(\sum_{i=1}^{n}|\lambda_i|^2 - \frac{1}{l}|\sum_{i=1}^{n}\lambda_i|^2\Big).$$

Case 2. When $r(A)+1 < n$. Suppose that $\lambda_1, \lambda_2, \cdots, \lambda_q$ are all the nonzero eigenvalues of A. If $q \geqslant 2$, by Lagrange's identity, we have

$$\begin{aligned} q\sum_{i=1}^{n}|\lambda_i|^2 - |\sum_{i=1}^{n}\lambda_i|^2 &= q\sum_{i=1}^{q}|\lambda_i|^2 - |\sum_{i=1}^{q}\lambda_i|^2 \\ &= \sum_{1\leqslant i<j\leqslant q}|\lambda_i-\lambda_j|^2 \\ &\geqslant \frac{q}{2}\max_{1\leqslant i<j\leqslant q}|\lambda_i-\lambda_j|^2, \end{aligned}$$

which together with the fact that $q \leqslant r(A)$ gives

$$\max_{1\leqslant i<j\leqslant q}|\lambda_i-\lambda_j|^2 \leqslant 2\Big(\sum_{i=1}^{n}|\lambda_i|^2 - \frac{1}{r(A)}|\sum_{i=1}^{n}\lambda_i|^2\Big). \tag{10}$$

Let $\rho(A) = \max_{1\leqslant i\leqslant q}|\lambda_i| = |\lambda_{t_0}|,\ \ 1 \leqslant t_0 \leqslant q$.

Since

$$\frac{1}{2}|\lambda_{t_0}|^2 \leqslant |\lambda_{t_0}-\lambda_1|^2 + |\lambda_1|^2,$$

$$\vdots$$

$$\frac{1}{2}|\lambda_{t_0}|^2 \leqslant |\lambda_{t_0}-\lambda_{t_0-1}|^2 + |\lambda_{t_0-1}|^2,$$

$$\frac{1}{2}|\lambda_{t_0}|^2 \leqslant |\lambda_{t_0} - \lambda_{t_0+1}|^2 + |\lambda_{t_0+1}|^2,$$

$$\vdots$$

$$\frac{1}{2}|\lambda_{t_0}|^2 \leqslant |\lambda_{t_0} - \lambda_q|^2 + |\lambda_q|^2,$$

we have

$$\begin{aligned}(1+\frac{q-1}{2})|\lambda_{t_0}|^2 &\leqslant \sum_{j=1, j\neq t_0}^{q} |\lambda_{t_0} - \lambda_j|^2 + \sum_{j=1}^{q} |\lambda_j|^2 \\ &\leqslant \sum_{1\leqslant i<j\leqslant q} |\lambda_i - \lambda_j|^2 + \sum_{j=1}^{q} |\lambda_j|^2 \\ &= q\sum_{i=1}^{q} |\lambda_i|^2 - |\sum_{i=1}^{q} \lambda_i|^2 + \sum_{j=1}^{q} |\lambda_j|^2 \\ &= (q+1)\sum_{i=1}^{q} |\lambda_i|^2 - |\sum_{i=1}^{q} \lambda_i|^2,\end{aligned}$$

which leads to

$$\begin{aligned}\rho(A)^2 &\leqslant 2\sum_{i=1}^{q} |\lambda_i|^2 - \frac{2}{q+1}|\sum_{i=1}^{q} \lambda_i|^2 \\ &= 2\sum_{i=1}^{n} |\lambda_i|^2 - \frac{2}{q+1}|\sum_{i=1}^{n} \lambda_i|^2 \\ &\leqslant 2\sum_{i=1}^{n} |\lambda_i|^2 - \frac{2}{r(A)+1}|\sum_{i=1}^{n} \lambda_i|^2,\end{aligned}$$

which together with (10) gives

$$\begin{aligned}s(A)^2 = \max\{\max_{1\leqslant i<j\leqslant q} |\lambda_i - \lambda_j|^2,\ \rho(A)^2\} &\leqslant 2\sum_{i=1}^{n} |\lambda_i|^2 - \frac{2}{r(A)+1}|\sum_{i=1}^{n} \lambda_i|^2 \\ &= 2\sum_{i=1}^{n} |\lambda_i|^2 - \frac{2}{l}|\sum_{i=1}^{n} \lambda_i|^2.\end{aligned}$$

Hence, the inequality (9) holds from Case 1 and Case 2. Combining the inequality (9) and Lemma 2 leads to the desired result. □

REMARK 1. Since $\|A\|_F^2 - (\|B_k\| - \|C_k\|_F)^2 = \|A_k\|_F^2 + \|D_k\|_F^2 + 2\|B_k\|_F\|C_k\|_F$, $\|[A(k), A^*(k)]\|_F^2 \geqslant 0$ and $l \leqslant n$, our bound in Theorem 1 is sharper than that in (2). Further, our bound in Theorem 1 can be up to the optimum which will be shown in our subsequent example.

Next, we propose a refinement of the bound in (7).

THEOREM 2. *Let $A \in C^{n\times n}$ be partitioned as in (3). Then*

$$s(A) \leqslant \sqrt{2 \min_{1\leqslant k\leqslant n} \left\{ \left(\|A\|_F^2 - (\|B_k\|_F - \|C_k\|_F)^2 - \frac{|\operatorname{tr} A|^2}{n} \right)^2 - \frac{1}{2}\|[A(k), A^*(k)]\|_F^2 \right\}^{\frac{1}{2}}}. \tag{11}$$

Proof. Let $B = A - \frac{\operatorname{tr} A}{n} I$. Applying Theorem 1 to B leads to

$$s(B) \leqslant \sqrt{2\left(\min_{1\leqslant k\leqslant n} \left\{ \left(\|B\|_F^2 - (\|B_k\|_F - \|C_k\|_F)^2 \right)^2 - \frac{1}{2}\|[B(k), B^*(k)]\|_F^2 \right\}^{\frac{1}{2}} - \frac{|\operatorname{tr} B|^2}{l(B)} \right)}, \tag{12}$$

where $l(B) = \min\{r(B)+1,\ n\}$. By direct calculations, we have

$$\operatorname{tr} B = 0, \tag{13}$$

$$\|B\|_F^2 = \|A\|_F^2 - \frac{|\operatorname{tr} A|^2}{n}, \tag{14}$$

and

$$[B(k), B^*(k)] = [A(k), A^*(k)]. \tag{15}$$

Plugging (13)-(15) into the inequality (12) leads to

$$s(B) \leqslant \sqrt{2 \min_{1\leqslant k\leqslant n} \left\{ \left(\|A\|_F^2 - (\|B_k\|_F - \|C_k\|_F)^2 - \frac{|\operatorname{tr} A|^2}{n} \right)^2 - \frac{1}{2}\|[A(k), A^*(k)]\|_F^2 \right\}^{\frac{1}{2}}},$$

which proves the theorem by the fact that $s(A) = s(B)$. □

REMARK 2. Obviously, our bound in Theorem 2 is better than that in (7).

Next, we present a refinement of the bound in (6).

THEOREM 3. *Let $A \in C^{n\times n}$. Then*

$$s(A) \leqslant \sqrt{2\left(\sum_{i=1}^{n} \left(R_i(A) C_i(A) \right)^{\frac{1}{2}} - \frac{|\operatorname{tr} A|^2}{l} \right)},$$

where $l = \min\{r(A)+1,\ n\}$.

Proof. Since the theorem can be easily obtained from Lemma 3 and the inequality (9), we omit it here. □

REMARK 3. Since $l \leqslant n$, our bound in Theorem 3 is sharper than that in (6).

4. Numerical example

In this section, we will give some matrices to show the effectiveness of our results.

EXAMPLE 1. Consider four 3×3 matrices

$$A=\begin{pmatrix}2&0&1\\0&0&0\\0&0&0\end{pmatrix},\quad B=\begin{pmatrix}0&1&0\\5&0&4\\0&2&0\end{pmatrix},$$

$$C=\begin{pmatrix}1&1&2\\1&1&2\\1&1&2\end{pmatrix},\qquad D=\begin{pmatrix}10&15&20\\5&10&30\\10&10&10\end{pmatrix}.$$

In order to compare more intuitively, we list the upper bounds for the spreads of A, B, C and D given in Example 1 according to (1), (2), (6), (7), Theorem 1, Theorem 2 and Theorem 3, respectively in Table 1.

Table 1: *The upper bounds for the spreads of A, B, C and D given in Example 1.*

	(1)	(2)	(6)	(7)	Theo 1	Theo 2	Theo 3
A	2.7080	2.7080	2.5055	2.5723	2	2.3094	2.2236
B	9.5917	7.7460	7.3916	7.4833	7.4833	7.4833	7.3916
C	5.0332	4.6188	4.8244	4.7234	4	4.6188	4.2357
D	59.1608	50.3945	53.1410	52.0858	50.1167	50.0500	53.1410

From Table 1, we can see that all of the upper bounds in (2), (6), (7), Theorem 1, Theorem 2 and Theorem 3 are sharper than that in (1). We also see that the bounds in Theorem 1, Theorem 2 and Theorem 3 are the refinements of those in (2), (7) and (6), respectively. In addition, none of the upper bounds in (2), (6) and (7) is uniformly better than the others and neither is those in Theorem 1, Theorem 2 and Theorem 3. Further more, the bound in Theorem 1 is up to the optimum since $s(A)=2$ and $s(C)=4$.

5. Concluding remarks

From the proof of Theorem 1, we know that both of the upper bounds in Theorem 1 and Theorem 3 are also the upper bounds for the spectral radius. It is easily obtained that the spectral radius of a singular matrix is less than or equal to its spread. Therefore, for a singular matrix, the upper bound for the spread in Theorem 2 also bounds the spectral radius.

Acknowledgements. The authors would like to express their thankfulness to the anonymous referee for the valuable comments and suggestions for improving the paper.

REFERENCES

[1] E. B. BARNES AND A. J. HOFFMAN, *Bounds for the spectrum of normal matrices*, Linear Algebra Appl, Vol. 201, **1** (1994), pp. 79–90.

[2] P. R. BEESACK, *The spread of matrices and polynomials*, Linear Algebra Appl, Vol. 31, **1** (1980), pp. 145–149.

[3] E. DEUTSCH, *On the spread of matrices and polynomials*, Linear Algebra Appl, Vol. 22, **1** (1978), pp. 49–55.

[4] R. DRNOVŠEK, *The spread of the spectrum of a nonnegative matrix with a zero diagonal element*, Linear Algebra Appl, Vol. 439, **8** (2013), pp. 2381–2387.

[5] R. KRESS AND H. L. D. VRIES, *On nonnormal matrices*, Linear Algebra Appl, Vol. 8, **2** (1974), pp. 109–120.

[6] J. K. MERIKOSKI AND R. KUMAR, *Characterizations and lower bounds for the spread of a normal matrix*, Linear Algebra Appl, Vol. 364, **1** (2003), pp. 13–31.

[7] L. MIRSKY, *Inequalities for normal and Hermitian matrices*, Duke Math. J., Vol. 24, **4** (1957), pp. 591–599.

[8] L. MIRSKY, *The spread of a matrix*, Mathematika, Vol. 3, **2** (1956), pp. 127–130.

[9] P. NOWOSAD AND R. TOVAR, *Spectral inequalities and G-functions*, Linear Algebra Appl, Vol. 31, **1** (1980), pp. 179–197.

[10] R. SHARMA AND R. KUMAR, *Remark on upper bounds for the spread of a matrix*, Linear Algebra Appl, Vol. 438, **11** (2013), pp. 4359–4362.

[11] B. TU, *On the spread of a matrix*, Journal of Fudan University (Natural Science) (in Chinese), Vol. 23, **4** (1984), pp. 435–441.

[12] J. L. WU, P. P. ZHANG AND W. S. LIAO, *Upper bounds for the spread of a matrix*, Linear Algebra Appl, Vol. 437, **11** (2012), pp. 2813–2822.

(Received October 27, 2013)

Pingping Zhang
College of Mathematics and Physics
Chongqing University of Posts and Telecommunications
Chongqing, 400065, China
e-mail: zhpp04010248@126.com

Hu Yang
College of Mathematics and Statistics
Chongqing University
Chongqing, 401331, China
e-mail: yh@cqu.edu.cn