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SHARP ESTIMATES REGARDING THE REMAINDER OF THE ALTERNATING HARMONIC SERIES

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Abstract. In the present paper we obtain enhanced estimates regarding the remainder of the alternating harmonic series. More precisely, we show that

$$\frac{1}{4n^2+a} < \left|\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k} - (-1)^{n-1}\frac{1}{2n} - \ln 2\right| \leqslant \frac{1}{4n^2+b},$$

for all $n \in \mathbb{N}$, with $a = 2$ and $b = \frac{2(3-4\ln 2)}{2\ln 2 - 1} = 1.177398899\ldots$. In addition, the constants a and b are the best possible with the above-mentioned property.

1. Introduction

We denote by H_n the nth harmonic number, i.e. $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, and by γ the Euler–Mascheroni constant, i.e. $\gamma = \lim\limits_{n\to\infty}(H_n - \ln n)$.

A remarkable flurry of articles have been published on estimating the remainder of convergent series and sequences. We list below some of them (the best constants being provided in some of the cases):

for $\mathrm{e} - \left(1+\frac{1}{n}\right)^n$ in [11];

for $H_n - \ln n - \gamma$ in [14] ([13]), [1], [3];

for $H_n - \ln\left(n+\frac{1}{2}\right) - \gamma$ in [2];

for $\gamma - \left(H_n - \ln n - \frac{1}{2n}\right)$ in [8];

for $H_n - \ln\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3}\right) - \gamma$ in [4];

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for $-\frac{1}{2}-\gamma+\ln 2-\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{i+j}-2n\ln 2+\ln n\right)$ in [5], [6, problem 1.28, p. 5];

for $\left|\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{2k-1}-\frac{\pi}{4}\right|$ in [9], [15];

for $\left|\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}-\ln 2\right|$ in [16].

Now we detail the result from [16], where estimates for the remaider of the alternating harmonic series were given. L. Tóth and J. Bukor [16, Theorem] proved that

$$\frac{1}{2n+\frac{2\ln 2-1}{1-\ln 2}}\leqslant\left|\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}-\ln 2\right|<\frac{1}{2n+1},\quad n\in\mathbb{N},$$

and the constants $\frac{2\ln 2-1}{1-\ln 2}$ and 1 are the best possible with this property, i.e. $\frac{2\ln 2-1}{1-\ln 2}$ cannot be replaced by a smaller one and 1 cannot be replaced by a larger one. See also [12] for a different proof of this result.

Our theorem below improves the result from [16].

THEOREM 1. *We have*

$$\frac{1}{4n^2+2}<\left|\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}-(-1)^{n-1}\frac{1}{2n}-\ln 2\right|\leqslant\frac{1}{4n^2+\frac{2(3-4\ln 2)}{2\ln 2-1}},$$

for all $n\in\mathbb{N}$. Moreover, the constants 2 and $\frac{2(3-4\ln 2)}{2\ln 2-1}=1.177398899\ldots$ are the best possible with this property.

In order to obtain our estimates, we use the inequalities proved by B.-N. Guo and F. Qi in [8]

$$\frac{1}{12n^2+\frac{6}{5}}<\gamma-\left(H_n-\ln n-\frac{1}{2n}\right)\leqslant\frac{1}{12n^2+\frac{2(7-12\gamma)}{2\gamma-1}},\quad n\in\mathbb{N},\tag{1}$$

the constants $\frac{6}{5}$ and $\frac{2(7-12\gamma)}{2\gamma-1}$ being the best possible with this property.

Also, we will be using in our proof some formulae involving the digamma function ψ, which is the logarithmic derivative of the gamma function, i.e.

$$\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)},\quad x\in(0,+\infty).$$

We have ([7, Section 8.365, Entry 4, p. 904], [10, Section 5.4, Entry 5.4.14, p. 137])

$$\psi(n+1)=-\gamma+H_n,\quad n\in\mathbb{N}.\tag{2}$$

The asymptotic formula for the digamma function ([10, Section 5.11, Entry 5.11.2, p. 140], see also [7, Section 8.367, Entry 13, p. 906])

$$\psi(x) \sim \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \cdots \quad (x \to \infty),$$

along with the recurrence formula ([7, Section 8.365, Entry 1, p. 904], [10, Section 5.5, Entry 5.5.2, p. 138])

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \quad x \in (0, +\infty),$$

gives

$$\psi(x+1) \sim \ln x + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \cdots \quad (x \to \infty). \tag{3}$$

2. The proof

In this section we give the proof of Theorem 1.

Proof. Let $s_n = \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k}$, $n \in \mathbb{N}$. As one can easily see,

$$\begin{aligned} s_{2n} + \frac{1}{4n} - \ln 2 &= H_{2n} - H_n + \frac{1}{4n} - \ln 2 \\ &= \gamma - \left(H_n - \ln n - \frac{1}{2n} \right) - \left[\gamma - \left(H_{2n} - \ln(2n) - \frac{1}{4n} \right) \right], \end{aligned}$$

for all $n \in \mathbb{N}$.

Let $x_n = s_{2n} + \frac{1}{4n} - \frac{1}{4(2n)^2+2}$, $n \in \mathbb{N}$. We have

$$x_{n+1} - x_n = \frac{-9}{4n(n+1)(2n+1)(8n^2+1)(8n^2+16n+9)} < 0,$$

for all $n \in \mathbb{N}$, hence the sequence $(x_n)_{n\in\mathbb{N}}$ is strictly decreasing. Clearly, $\lim_{n\to\infty} x_n = \ln 2$. Now we can write that $\ln 2 < x_n$, for all $n \in \mathbb{N}$, so

$$\frac{1}{4(2n)^2+2} < s_{2n} + \frac{1}{4n} - \ln 2, \quad n \in \mathbb{N}. \tag{4}$$

Also, based on (2) and (3), we get that

$$\begin{aligned} \frac{1}{s_{2n} + \frac{1}{4n} - \ln 2} - 4(2n)^2 &= \frac{1}{\psi(2n+1) - \psi(n+1) + \frac{1}{4n} - \ln 2} - 16n^2 \\ &= \frac{1}{\frac{1}{16n^2} - \frac{1}{128n^4} + \frac{1}{256n^6} + O\left(\frac{1}{n^8}\right)} - 16n^2 \\ &= \frac{\frac{1}{8} - \frac{1}{16n^2} + O\left(\frac{1}{n^4}\right)}{\frac{1}{16} - \frac{1}{128n^2} + O\left(\frac{1}{n^4}\right)} \to 2 \quad (n \to \infty). \end{aligned} \tag{5}$$

Having in view (1), we are able to write that

$$s_{2n}+\frac{1}{4n}-\ln 2<\frac{1}{12n^2+\frac{2(7-12\gamma)}{2\gamma-1}}-\frac{1}{12(2n)^2+\frac{6}{5}}, \quad n\in\mathbb{N}.$$

A straightforward calculation shows that

$$\frac{1}{12n^2+\frac{2(7-12\gamma)}{2\gamma-1}}-\frac{1}{12(2n)^2+\frac{6}{5}}<\frac{1}{4(2n)^2+\frac{2(3-4\ln 2)}{2\ln 2-1}},$$

for all $n\in\mathbb{N}$, which means that

$$s_{2n}+\frac{1}{4n}-\ln 2<\frac{1}{4(2n)^2+\frac{2(3-4\ln 2)}{2\ln 2-1}}, \quad n\in\mathbb{N}. \tag{6}$$

On the other hand,

$$\begin{aligned}\ln 2-\left(s_{2n+1}-\frac{1}{2(2n+1)}\right)&=\ln 2-\left(H_{2n}-H_n+\frac{1}{2(2n+1)}\right)\\&=\gamma-\left(H_{2n}-\ln(2n)-\frac{1}{4n}\right)\\&\quad-\left[\gamma-\left(H_n-\ln n-\frac{1}{2n}\right)\right]\\&\quad+\frac{1}{4n}-\frac{1}{2(2n+1)},\end{aligned}$$

for all $n\in\mathbb{N}$.

Let $y_n=s_{2n+1}-\frac{1}{2(2n+1)}+\frac{1}{4(2n+1)^2+2}$, $n\geqslant 0$. We have

$$y_{n+1}-y_n=\frac{9}{2(n+1)(2n+1)(2n+3)(8n^2+8n+3)(8n^2+24n+19)}>0,$$

for all $n\geqslant 0$, which means that the sequence $(y_n)_{n\geqslant 0}$ is strictly increasing. Also, we have $\lim\limits_{n\to\infty} y_n=\ln 2$. From these we deduce that $y_n<\ln 2$, for all $n\geqslant 0$, so

$$\frac{1}{4(2n+1)^2+2}<\ln 2-\left(s_{2n+1}-\frac{1}{2(2n+1)}\right), \quad n\geqslant 0. \tag{7}$$

Using (2) and (3), we get that

$$\begin{aligned}
&\frac{1}{\ln 2-\left(s_{2n+1}-\frac{1}{2(2n+1)}\right)}-4(2n+1)^2\\
&=\frac{1}{\ln 2-\psi(2n+1)+\psi(n+1)-\frac{1}{2(2n+1)}}-4(2n+1)^2\\
&=\frac{1}{\frac{1}{4n}-\frac{1}{16n^2}+\frac{1}{128n^4}-\frac{1}{256n^6}+O\left(\frac{1}{n^8}\right)-\frac{1}{2(2n+1)}}-4(2n+1)^2\\
&=\frac{\frac{1}{8n^2}-\frac{1}{8n^3}+O\left(\frac{1}{n^4}\right)}{\frac{1}{4n}-\frac{1}{16n^2}+\frac{1}{128n^4}-\frac{1}{256n^6}+O\left(\frac{1}{n^8}\right)-\frac{1}{4n}\left(1+\frac{1}{2n}\right)^{-1}}\\
&=\frac{\frac{1}{8}-\frac{1}{8n}+O\left(\frac{1}{n^2}\right)}{\frac{1}{16}-\frac{1}{16n}+O\left(\frac{1}{n^2}\right)}\to 2 \quad (n\to\infty).
\end{aligned} \tag{8}$$

Based on (1), the following inequality holds

$$\ln 2-\left(s_{2n+1}-\frac{1}{2(2n+1)}\right)<\frac{1}{12(2n)^2+\frac{2(7-12\gamma)}{2\gamma-1}}-\frac{1}{12n^2+\frac{6}{5}}+\frac{1}{4n}-\frac{1}{2(2n+1)},\quad n\in\mathbb{N}.$$

One can check that

$$\frac{1}{12(2n)^2+\frac{2(7-12\gamma)}{2\gamma-1}}-\frac{1}{12n^2+\frac{6}{5}}+\frac{1}{4n}-\frac{1}{2(2n+1)}<\frac{1}{4(2n+1)^2+\frac{2(3-4\ln 2)}{2\ln 2-1}},$$

for all $n\geqslant 2$, and

$$\ln 2-\left(s_3-\frac{1}{2\cdot 3}\right)=0.02648\ldots<0.02689\ldots=\frac{1}{4\cdot 3^2+\frac{2(3-4\ln 2)}{2\ln 2-1}}.$$

So,

$$\ln 2-\left(s_{2n+1}-\frac{1}{2(2n+1)}\right)\leqslant\frac{1}{4(2n+1)^2+\frac{2(3-4\ln 2)}{2\ln 2-1}},\quad n\geqslant 0 \tag{9}$$

(the equality holds only for $n=0$).

Let's conclude what we have proved. From (4) and (7) it follows that

$$\frac{1}{\left|s_n-\frac{(-1)^{n-1}}{2n}-\ln 2\right|}-4n^2<2,$$

for all $n\in\mathbb{N}$, and (5) and (8) imply that

$$\lim_{n\to\infty}\left(\frac{1}{\left|s_n-\frac{(-1)^{n-1}}{2n}-\ln 2\right|}-4n^2\right)=2,$$

i.e.

$$\frac{1}{4n^2+2} < \left| s_n - \frac{(-1)^{n-1}}{2n} - \ln 2 \right|,$$

for all $n \in \mathbb{N}$, the constant 2 being the best possible with this property. From (6) and (9) we obtain

$$\left| s_n - \frac{(-1)^{n-1}}{2n} - \ln 2 \right| \leqslant \frac{1}{4n^2 + \frac{2(3-4\ln 2)}{2\ln 2 - 1}},$$

for all $n \in \mathbb{N}$; clearly, the constant $\frac{2(3-4\ln 2)}{2\ln 2-1}$ is the best possible with this property (the equality holds only for $n = 1$). □

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