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ORLICZ GEOMINIMAL SURFACE AREAS

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Abstract. In 1996, E. Lutwak extended the important concept of geominimal surface area to L_p version, which serves as a bridge connecting a number of areas of geometry: affine differential geometry, relative differential geometry, and Minkowskian geometry. In this paper, by using the concept of Orlicz mixed volume, we extend geominimal surface area to the Orlicz version and give some properties and an isoperimetric inequalities for the Orlicz geominimal surface areas.

1. Introduction

$\mathbb{R}^n$ denotes the usual n-dimensional Euclidean space with the canonical inner product $\langle\cdot,\cdot\rangle$. The letter B is reserved for the unit ball centered at the origin o, and the surface of B is S^{n-1}. A bounded closed convex set $C \subset \mathbb{R}^n$ is called a convex body if it has non-empty interior. Let $\mathscr{K}^n$ be the class of convex bodies of $\mathbb{R}^n$, let $\mathscr{K}_0^n$ be the class of members of $\mathscr{K}^n$ containing the origin, and let $\mathscr{K}_{00}^n$ be those sets in $\mathscr{K}^n$ containing the origin in their interiors. Let $\mathscr{K}_c^n$ denote the set of convex bodies whose centroids lie at the origin. In general, we refer the reader to [15] for standard notation.

The Orlicz-Brunn-Minkowski theory, introduced by Lutwak, Yang, and Zhang (see [4, 10, 11]), is a new extension of the classical Brunn-Minkowski theory. For the recent development see [1, 5, 6, 16]. Quite recently, in [3], Gardner, Hug and Weil constructed a general framework for Orlicz-Brunn-Minkowski theory that includes Orlicz addition and Orlicz mixed volume, established the new Orlicz-Brunn-Minkowsk inequality and the Orlicz-Minkowski mixed volumes inequality, and made clear for the first time the relation to Orlicz spaces and norms.

The important concept of geominimal surface area was introduced by Petty [14]. It serves as a bridge connecting a number of areas of geometry: affine differential geometry, relative differential geometry, and Minkowskian geometry. The geominimal surface area, $G(K)$, of $K \in \mathscr{K}^n$, could be defined by

$$\omega_n^{1/n} G(K) = \inf\{nV_1(K,Q)V(Q^*)^{1/n} : Q \in \mathscr{K}_{00}^n\},$$

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Where ω_n denotes the volume of n-dimensional unit ball, and $V_1(K,Q)$ is the mixed volume of K,Q.

In [8], Lutwak extended Petty's geominimal surface area to the L_p version. For $p \geqslant 1$, the p-geominimal surface area, $G_p(K)$, of $K \in \mathscr{K}_{00}^n$ is defined by

$$\omega_n^{p/n} G_p(K) = \inf\{nV_p(K,Q)V(Q^*)^{p/n} : Q \in \mathscr{K}_{00}^n\},$$

where $V_p(K,Q)$ means p-mixed volume of K,Q (for the definition see section 2).

By the homogeneous of volume and p-mixed volume, the p-geominimal surface area could also be defined by

$$G_p(K) = \inf\{nV_p(K,Q) : Q \in \mathscr{K}_{00}^n \text{ and } V(Q^*) = \omega_n\}.$$

In this paper, we will introduce the Orlicz geominimal surface area $G_\varphi(K)$ of $K \in \mathscr{K}_{00}^n$. Let Φ denote the set of convex functions $\varphi : [0,\infty) \to [0,\infty)$ that satisfy $\varphi(0) = 0$ and $\varphi(1) = 1$. For $K \in \mathscr{K}_{00}^n$, and $\varphi \in \Phi$,

$$G_\varphi(K) = \inf\{nV_\varphi(K,Q) : Q \in \mathscr{K}_{00}^n \text{ and } V(Q^*) = \omega_n\},$$

where $V_\varphi(K,Q)$ means Orlicz mixed volume of K,Q (for the definition see section 2). It will be shown that the infimum in the above definition is attained.

THEOREM 1. *If $K \in \mathscr{K}_{00}^n$ and $\varphi \in \Phi$, then there exists a body $\bar{K} \in \mathscr{K}_{00}^n$ such that*

$$G_\varphi(K) = nV_\varphi(K,\bar{K}) \text{ and } V(\bar{K}^*) = \omega_n.$$

We will show that Orlicz geominimal surface area of a body is invariant under unimodular centro-transformations of the body. Another important property of Orlicz geominimal surface area is the following:

THEOREM 2. *If $\varphi \in \Phi$, then the functional $G_\varphi : \mathscr{K}_{00}^n \to (0,\infty)$ is continuous.*

Petty [14] established the fundamental affine isoperimetric inequality for geominimal surface area. He showed that for $K \in \mathscr{K}^n$,

$$G(K)^n \leqslant n^n \omega_n V(K)^{n-1},$$

with equality if and only if K is an ellipsoid. This inequality is closely related to the Blaschke-Santaló inequality. In [8], Lutwak established the corresponding inequality for p-geominimal surface area: For $K \in \mathscr{K}_c^n$,

$$G_p(K)^n \leqslant n^n \omega_n^p V(K)^{n-p},$$

with equality if and only if K is an ellipsoid. Now, an Orlicz extension of Petty's geominimal surface area inequality will be obtained.

THEOREM 3. *If $K \in \mathscr{K}_c^n$ and $\varphi \in \Phi$, then*

$$G_\varphi(K) \leqslant nV(K)\varphi\Big(\Big(\frac{\omega_n}{V(K)}\Big)^{1/n}\Big),$$

with equality only if K is an ellipsoid.

In this paper, we also follow the principle. We will use the technique which is developed by Lutwak [8]. So our work is a natural extension of the work of Lutwak [8] and Petty [14]. It would be impossible to overstate our reliance on his work.

2. Notations and Orlicz mixed volumes

For $K \in \mathscr{K}_0^n$, let $h_K = h(K,\cdot) : S^{n-1} \to \mathbb{R}$ denote the support function of K; i.e., for $u \in S^{n-1}$, $h(K,u) = \max\{\langle u,x\rangle : x \in K\}$. The formula (See (0.27), P. 18 in [2])

$$h_{AK}(u) = h_K(A^t u) \tag{2.1}$$

for $x \in \mathbb{R}^n$ and a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$, gives the change in a support function under A, where A^t denotes the transpose of A. For $K, L \in \mathscr{K}_0^n$, the Hausdorff metirc $\delta(K,L) = \max_{u\in S^{n-1}} |h(K,u) - h(L,u)|$. If $o \in L$ and L is a star-shaped at o, its radial function ρ_L, for $x \in \mathbb{R}^n \backslash \{o\}$, is defined by $\rho_L(x) = \rho(L,x) = \max\{\lambda : \lambda x \in L\}$.

For $K \in \mathscr{K}_{00}^n$, let K^* denote the polar of the body K; i.e.

$$K^* = \{x \in \mathbb{R}^n : \langle x,y\rangle \leqslant 1, \text{ for all } y \in K\}.$$

The Blaschke-Santaló inequality (see [12, 13]) is one of the fundamental affine isoperimetric inequalities. It states that if $K \in \mathscr{K}_c^n$ then

$$V(K)V(K^*) \leqslant \omega_n^2, \tag{2.2}$$

with equality if and only if K is an ellipsoid.

For $x \in \mathbb{R}^n$, let $\langle x\rangle = x/|x|$, whenever $x \neq 0$. For surface area measure $S_K(u)$ of K, a $\mathrm{A} \in \mathrm{SL}(n)$, define the measure $S_K^*(\mathrm{A}u)$ on S^{n-1} by

$$\int_{S^{n-1}} f(u)dS_K^*(\mathrm{A}u) = \int_{S^{n-1}} |\mathrm{A}^{-1}u| f(\langle \mathrm{A}^{-1}u\rangle)dS_K(u), \tag{2.3}$$

for each $f \in \mathscr{C}(S^{n-1})$. In [9], the following equation is proved:

$$dS_{\mathrm{A}K}(u) = dS_K^*(\mathrm{A}^t u). \tag{2.4}$$

For $K \in \mathscr{K}^n$, we have

$$\int_{S^{n-1}} u dS_K(u) = 0. \tag{2.5}$$

In [3], Gardner, Hug and Weil introduced the definition of Orlicz mixed volumes. For $\varphi \in \Phi$, $K \in \mathscr{K}_{00}^n$, $L \in K_0^n$, the Orlicz mixed volume $V_\varphi(K,L)$ of K,L is defined by

$$V_\varphi(K,L) = \frac{1}{n}\int_{S^{n-1}} \varphi\Big(\frac{h_L(u)}{h_K(u)}\Big) h_K(u)dS_K(u). \tag{2.6}$$

The special case of $\varphi(t) = t^p, p \geqslant 1$ is p-mixed volume $V_p(K,L)$ of K,L (see [7]):

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h(L,u)^p dS_p(K,u),$$

where $S_p(K,\cdot)$ is the L_p-surface area measure of K.

The following result provides an Orlicz-Minkowski inequality: let $\varphi \in \Phi$, if $K \in \mathscr{K}_{00}^n$ and $L \in \mathscr{K}_0^n$, then

$$V_\varphi(K,L) \geqslant V(K)\varphi\Big(\Big(\frac{V(L)}{V(K)}\Big)^{1/n}\Big). \tag{2.7}$$

If φ is strictly convex, equality holds iff K and L are dilates or $L = \{0\}$. In (2.7), the special case when $\varphi(t) = t^p$, $p \geqslant 1$, reduces to the L_p-Minkowski inequality.

For the further research, the corresponding properties are studied in the following.

LEMMA 2.1. *Suppose $K \in \mathscr{K}_{00}^n$, and $L \in \mathscr{K}_0^n$. If $\varphi \in \Phi$ and $A \in SL(n)$, then*

$$V_\varphi(AK,AL) = V_\varphi(K,L).$$

Proof. By (2.6), (2.4), (2.3) and (2.1), we have

$$\begin{aligned} V_\varphi(AK,L) &= \frac{1}{n}\int_{S^{n-1}} \varphi\Big(\frac{h_L(u)}{h_{\mathrm{A}K}(u)}\Big) h_{\mathrm{A}K}(u)dS_{\mathrm{A}K}(u) \\ &= \frac{1}{n}\int_{S^{n-1}} \varphi\Big(\frac{h_L(u)}{h_K(\mathrm{A}^t u)}\Big) h_K(\mathrm{A}^t u)dS_K^*(\mathrm{A}^t u) \\ &= \frac{1}{n}\int_{S^{n-1}} |\mathrm{A}^{-t}u|\varphi\Big(\frac{h_L(\langle \mathrm{A}^{-t}u\rangle)}{h_K(\mathrm{A}^t\langle \mathrm{A}^{-t}u\rangle)}\Big) h_K(\mathrm{A}^t\langle \mathrm{A}^{-t}u\rangle)dS_K(u) \\ &= \frac{1}{n}\int_{S^{n-1}} \varphi\Big(\frac{h_L(\mathrm{A}^{-t}u)}{h_K(u)}\Big) h_K(u)dS_K(u) \\ &= \frac{1}{n}\int_{S^{n-1}} \varphi\Big(\frac{h_{\mathrm{A}^{-1}L}(u)}{h_K(u)}\Big) h_K(u)dS_K(u) \\ &= V_\varphi(K,A^{-1}L). \quad \square \end{aligned}$$

It is easy to check that $V_\varphi(\lambda K,\lambda L) = \lambda^n V_\varphi(K,L)$, for $\lambda > 0$. Therefore, we have

PROPOSITION 2.2. *Suppose $K \in \mathscr{K}_{00}^n$, and $L \in \mathscr{K}_0^n$. If $\varphi \in \Phi$ and $A \in GL(n)$, then*

$$V_\varphi(AK,AL) = |detA| V_\varphi(K,L).$$

The following shows that the Orlicz mixed volumes $V_\varphi(\cdot,\cdot)$ are continuous on $\mathscr{K}_{00}^n \times \mathscr{K}_0^n$.

PROPOSITION 2.3. *Suppose $K_i, K \in \mathscr{K}_{00}^n$, L_i, $L \in \mathscr{K}_0^n$ and $\varphi \in \Phi$, if $K_i \to K$ and $L_i \to L$, then*

$$V_\varphi(K_i,L_i) \to V_\varphi(K,L).$$

Proof. Since $K_i \to K$ and $L_i \to L$, we have $h_{K_i} \to h_K$ and $h_{L_i} \to h_L$, uniformly on S^{n-1}. Since the continuous function h_K is positive, the h_{K_i} are uniformly bounded away from 0. Thus

$$\varphi\Big(\frac{h_{L_i}}{h_{K_i}}\Big)h_{K_i} \to \varphi\Big(\frac{h_L}{h_K}\Big)h_K, \quad \text{uniformly on } S^{n-1}.$$

And $K_i \to K$ also implies that

$$S_{K_i}(\cdot) \to S_K(\cdot), \text{ weakly on } S^{n-1}.$$

Hence,

$$\frac{1}{n}\int_{S^{n-1}} \varphi\Big(\frac{h_{L_i}}{h_{K_i}}\Big)h_{K_i}S_{K_i}(u) \to \frac{1}{n}\int_{S^{n-1}} \varphi\Big(\frac{h_L}{h_K}\Big)h_K dS_K(u). \quad \square$$

By the similar argument of proposition 2.3, we have the following result.

PROPOSITION 2.4. *Let $K,L \in \mathscr{K}_0^n$, if φ_i, $\varphi \in \Phi$ and $\varphi_i \to \varphi$, i.e.,* $\max_{t\in I}|\varphi_i(t) - \varphi(t)| \to 0$, *for every compact interval $I \subset \mathbb{R}$, then $V_{\varphi_i}(K,L) \to V_\varphi(K,L)$.*

3. Orlicz geominimal surface areas

For $K \in \mathscr{K}_{00}^n$, and $\varphi \in \Phi$, define the Orlicz geominimal surface area, $G_\varphi(K)$, of K by

$$G_\varphi(K) = \inf\{nV_\varphi(K,Q) : Q \in \mathscr{K}_{00}^n \text{ and } V(Q^*) = \omega_n\}. \tag{3.1}$$

When $\varphi(t) = t^p$, $p \geqslant 1$, the Orlicz geominimal surface area becomes the $p-$ geominimal surface area. The next proposition shows that Orlicz geominimal surface area of a body is invariant under the special linear transformation.

PROPOSITION 3.1. *Suppose $K \in \mathscr{K}_{00}^n$. If $\varphi \in \Phi$, and $A \in SL(n)$, then*

$$G_\varphi(AK) = G_\varphi(K).$$

Proof. From (3.1) and Lemma 2.1, we get

$$\begin{aligned} G_\varphi(AK) &= \inf\{nV_\varphi(AK,Q) : Q \in \mathscr{K}_{00}^n \text{ and } V(Q^*) = \omega_n\} \\ &= \inf\{nV_\varphi(K,A^{-1}Q) : A^{-1}Q \in \mathscr{K}_{00}^n \text{ and } V((A^{-1}Q)^*) = \omega_n\} \\ &= G_\varphi(K). \quad \square \end{aligned}$$

Next, we will show that the infimum in the definition (3.1) of the Orlicz geominimal surface area can be attained. Let $\mathscr{C}^n$ denote the set of compact convex subsets of $\mathbb{R}^n$.

LEMMA 3.2. ([8] p. 264) *Suppose $K_i \in \mathscr{K}_{00}^n$ and $K_i \to L \in \mathscr{C}^n$. If the sequence $V(K_i^*)$ is bounded, then $L \in \mathscr{K}_{00}^n$.*

Proof of Theorem 1. By the definition of $G_\varphi(K)$, we can find a sequence $M_i \in \mathscr{K}_{00}^n$ with $V(M_i^*) = \omega_n$ such that $V_\varphi(K,B) \geqslant V_\varphi(K,M_i)$, for all i, and

$$nV_\varphi(K,M_i) \to G_\varphi(K).$$

Let $R_i = \max\{\rho(M_i,u) : u \in S^{n-1}\}$ denote the outer radius of M_i and the convex set $e_i = \{\lambda u_i : 0 \leqslant \lambda \leqslant R_i\} \subset M_i$, where u_i is any of the points in S^{n-1} such that $\rho(M_i,u_i) = R_i$. Then, we have

$$\begin{aligned} h(e_i,u) &= \max\{\langle 0,u\rangle, \langle R_i u_i,u\rangle\} \\ &= R_i \max\{\langle 0,u\rangle, \langle u_i,u\rangle\} \\ &= R_i \cdot \frac{1}{2}(|\langle u_i,u\rangle| + \langle u_i,u\rangle). \end{aligned}$$

By Jessen's inequality, we have

$$\begin{aligned}
V_\varphi(K,B) &\geqslant V_\varphi(K,M_i)\\
&= \frac{1}{n}\int_{S^{n-1}} \varphi\Big(\frac{h(M_i,u)}{h(K,u)}\Big) h(K,u)dS_K(u)\\
&\geqslant V(K)\varphi\Big(\frac{1}{nV(K)}\int_{S^{n-1}} h(M_i,u)dS_K(u)\Big)\\
&\geqslant V(K)\varphi\Big(\frac{1}{nV(K)}\int_{S^{n-1}} h(e_i,u)dS_K(u)\Big)\\
&= V(K)\varphi\Big(\frac{1}{nV(K)}\int_{S^{n-1}} R_i\cdot\frac{1}{2}(|\langle u_i,u\rangle| + \langle u_i,u\rangle)dS_K(u)\Big)\\
&= V(K)\varphi\Big(\frac{R_i v(K|u_i^\perp)}{nV(K)}\Big),
\end{aligned}$$

where $v(K|u_i^\perp)$ denotes the area of the projection of K onto the hyperplane $u_i^\perp$ with normal vector u_i. Since K contains the origin in its interior, there exists a constant $c>0$ such that $v(K|u_i^\perp) \geqslant c$. Hence, $V_\varphi(K,B) \geqslant V(K)\varphi(\frac{R_i c}{nV(K)})$, which leads to that $R_i \leqslant \frac{nV(K)}{c}\varphi^{-1}(\frac{V_\varphi(K,B)}{V(K)})$. So, M_i are uniformly bounded. By the Blaschke selection theorem, there exist a convergent subsequence M_{i_j} of M_i and a compact convex body L such that the limit of M_{i_j} must be L. Since $M_{i_j} \to L$, and the sequence $V(M_{i_j}^*) = \omega_n$, we get that $M_{i_j}^* \to L^*$ and $V(L^*) = \omega_n$. Lemma 3.2 gives $L \in \mathscr{K}_{00}^n$. Since the Orlicz mixed volumes $V_\varphi(\cdot,\cdot)$ are continuous on on $\mathscr{K}_{00}^n \times \mathscr{K}_0^n$, we get that L is the desired body $\bar{K}$. □

REMARK 3.3. We conjecture that $\bar{K}$ is unique if $\varphi \in \Phi$ is strictly convex, and the uniqueness of $\bar{K}$ will bring us many interesting results.

Set

$$T_\varphi K = \{\bar{K} \in \mathscr{K}_{00}^n : G_\varphi(K) = nV_\varphi(K,\bar{K}) \text{ and } V(\bar{K}^*) = \omega_n\}. \tag{3.2}$$

Let $\varphi(t) = t^p$, $p \geqslant 1$, we have $T_\varphi K = T_p(K)$. In [8], Lutwak showed that T_pK is a singleton. Although we do not know whether $T_\varphi K$ is a singleton, we get the following property of the body $T_\varphi K$.

PROPOSITION 3.4. *If $K \in \mathscr{K}_{00}^n$, $\varphi \in \Phi$ and $A \in SL(n)$, then $T_\varphi(AK) = A(T_\varphi K)$.*

Proof. First, let $\bar{K} \in T_\varphi K$, by Proposition 3.1, (3.2) and Proposition 2.2, we have

$$G_\varphi(AK) = G_\varphi(K) = nV_\varphi(K,\bar{K}) = nV_\varphi(AK,A\bar{K}).$$

Combining with $V(\bar{K}^*) = V((A\bar{K})^*) = \omega_n$, we have $A\bar{K} \in T_\varphi(AK)$, which implies that $T_\varphi(AK) \supset A(T_\varphi K)$.

Second, let $\bar{K} \in T_\varphi(AK)$, we also have

$$G_\varphi(K) = G_\varphi(AK) = nV_\varphi(AK,\bar{K}) = nV_\varphi(K,A^{-1}\bar{K}).$$

Combining with $V(\bar{K}^*) = V((A^{-1}\bar{K})^*) = \omega_n$, we have $A^{-1}\bar{K} \in T_\varphi K$, which implies that $T_\varphi(AK) \subset A(T_\varphi K)$. $\square$

The following two lemmas are the corresponding Orlicz version of Lemma 3.6 and Lemma 3.7 of [8], which will be helpful in the sequel.

LEMMA 3.5. *Suppose $\varphi \in \Phi$, and $K \in \mathscr{K}^n_{00}$. If $r, R > 0$, are such that*

$$rB \subset K \subset RB,$$

then

$$h(\bar{K}, u) \leqslant \frac{nR^n \omega_n \varphi^{-1}(R^n \varphi(1/r)/r^n)}{r^{n-1}\omega_{n-1}},$$

for all $u \in S^{n-1}$ and $\bar{K} \in T_\varphi K$.

Proof. As the proof of Theorem 1, let the convex set $e_0 = \{\lambda u_0 : 0 \leqslant \lambda \leqslant R(\bar{K})\} \subset \bar{K}$ and u_0 be the point in S^{n-1} such that

$$\rho(\bar{K}, u_0) = \max\{\rho(\bar{K}, u) : u \in S^{n-1}\} = R(\bar{K}),$$

where $R(\bar{K})$ is the outer radii of $\bar{K}$. Then, we have

$$V_\varphi(K, \bar{K}) \geqslant V(K)\varphi\Big(\frac{R(\bar{K})v(K|u_0^\perp)}{nV(K)}\Big).$$

Since $r^n\omega_n \leqslant V(K) \leqslant R^n\omega_n$, and $v(K|u_0^\perp) \geqslant r^{n-1}\omega_{n-1}$, we have $V_\varphi(K, \bar{K}) \geqslant r^n\omega_n\varphi\Big(\frac{R(\bar{K})r^{n-1}\omega_{n-1}}{nR^n\omega_n}\Big)$.

From the minimality property of $\bar{K}$, it follows that

$$V_\varphi(K, \bar{K}) \leqslant V_\varphi(K, B).$$

By the integral representation (2.6) of Orlicz mixed volume, we have

$$V_\varphi(K, B) = \frac{1}{n}\int_{S^{n-1}} \varphi\Big(\frac{1}{h_K(u)}\Big) h_K(u) dS_K(u) \leqslant V(K)\varphi(1/r) \leqslant \omega_n R^n \varphi(1/r).$$

Therefore, $r^n\omega_n\varphi(\frac{R(\bar{K})r^{n-1}\omega_{n-1}}{nR^n\omega_n}) \leqslant \omega_n R^n\varphi(1/r)$, which leads to that

$$R(\bar{K}) \leqslant \frac{nR^n \omega_n \varphi^{-1}(R^n \varphi(1/r)/r^n)}{r^{n-1}\omega_{n-1}}.$$

We finished the proof. $\square$

LEMMA 3.6. *Suppose $\varphi \in \Phi$. If $K_i \in \mathcal{K}_0^n$ is a family of bodies for which there exist $r, R > 0$, such that*

$$rB \subset K_i \subset RB, \ \ for \ \ all \ \ i,$$

then there exist $r', R' > 0$, such that arbitrary $\bar{K_i} \in T_\varphi K_i$ satisfy

$$r'B \subset \bar{K_i} \subset R'B, \ \ for \ \ \ all \ \ i.$$

Proof. From Lemma 3.5, if the outer radii of a sequence of bodies are uniformly bounded and the inner radii of the sequence are bounded away from 0, then there exists $R' > 0$ such that $\bar{K_i} \subset R'B$.

Next, we will give the proof by contradiction to show that the inner radii of sequence $\bar{K_i}$ is also bounded away from 0. For arbitrary $\bar{K_i} \in T_\varphi K_i$, let $r_i = r(\bar{K_i})$ denote the inner radius of $\bar{K_i}$, and let u_i is any point in S^{n-1} which satisfies that

$$r_i = \min_{u \in S^{n-1}} h(\bar{K_i}, u) = h(\bar{K_i}, u_i).$$

Suppose that the infimum of the r_i is 0. Thus, there exists a subsequence of the $\bar{K_i}$, which will not be relabeled, such that

$$h(\bar{K_i}, u_i) \to 0.$$

By the Blaschke selection theorem, there exist a subsequence of the $\bar{K_i}$, which will also not be relabeled, and a convex body M, such that

$$\bar{K_i} \to M.$$

Since $V(\bar{K_i}^*) = \omega_n$, by Lemma 3.2, we get $M \in \mathcal{K}_{00}^n$.

But $h(\bar{K_i}, u_i) \to 0$, and $\max|h_{\bar{K_i}} - h_M| \to 0$, implies that $h_M(u_i) \to 0$, which is impossible since the continuous function h_M is positive. □

THEOREM 2. *If $\varphi \in \Phi$, then the functional $G_\varphi : \mathcal{K}_{00}^n \to (0, \infty)$ is continuous.*

Proof. That G_φ is upper semicontinuous follows immediately from Proposition 2.3 and the definition of $G_\varphi(\cdot)$.

To see that G_φ is lower semicontinuous at $K_0 \in \mathcal{K}_0^n$, let $K_i \in \mathcal{K}_{00}^n$ be a sequence of bodies such that $K_i \to K_0$, with $G_\varphi(K_i) \to l \in R$. It will be shown that $l \geqslant G_\varphi(K_0)$, and thus

$$\liminf G_\varphi(K_i) \geqslant G_\varphi(K_0).$$

By Lemma 3.6, any $\bar{K_i} \in T_\varphi K_i$ are uniformly bounded. By the Blaschke selection theorem and Lemma 3.2, we get that there exists a body $M \in \mathcal{K}_{00}^n$, and a subsequence of the $\bar{K_i}$, which will not be relabeled, such that $\bar{K_i} \to M$, and $V(M^*) = \omega_n$. From Proposition 4.3, the facts that $K_i \to K_0$, and $\bar{K_i} \to M$, we conclude that $G_\varphi(K_i) = nV_\varphi(K_i, \bar{K_i}) \to nV_\varphi(K_0, M)$. Since $G_\varphi(K_i) \to l$, we have $nV_\varphi(K_0, M) = l$. But the definition of $G_\varphi(K_0)$ shows that

$$l = nV_\varphi(K_0, M) \geqslant G_\varphi(K_0),$$

and completes the argument. □

In [14] and [8], Petty and Lutwak proved that geominimal surface area $G : \mathcal{K}^n \to (0,\infty)$ and p-geominimal surface area $G_p : \mathcal{K}^n_{00} \to (0,\infty)$ are continuous respectively. When $\varphi(t) = t^p$, $p \geqslant 1$, Theorem 2 becomes the above results.

In [8], Lutwak showed that: If $p \geqslant 1$,and $K \in \mathcal{K}_c^n$, then

$$G_p(K)^n \leqslant n^n \omega_n^p V(K)^{n-p}, \tag{3.3}$$

with equality iff K is an ellipsoid. In the sequel, we give the following Orlicz version. Unfortunately, without the uniqueness of $T_\varphi K$, we do not know whether the equality holds if K is an ellipsoid.

THEOREM 3. *If $K \in \mathcal{K}_c^n$ and $\varphi \in \Phi$, then*

$$G_\varphi(K) \leqslant nV(K)\varphi\Big(\Big(\frac{\omega_n}{V(K)}\Big)^{1/n}\Big), \tag{3.4}$$

with equality only if K is an ellipsoid.

Proof. Let $Q = \lambda K$ such that $V(Q^*) = \omega_n$, then $\lambda = \Big(\frac{V(K^*)}{\omega_n}\Big)^{1/n}$. By the definitions of $G_\varphi(K)$ and $V_\varphi(K,Q)$, the Blaschke-Santalo inequality (2.2), we have

$$\begin{aligned}
G_\varphi(K) &\leqslant nV_\varphi(K,\lambda K)\\
&= \int_{S^{n-1}} \varphi\Big(\frac{h(\lambda K,u)}{h(K,u)}\Big)h(K,u)dS(K,u)\\
&= n\varphi(\lambda)V(K)\\
&= nV(K)\varphi\Big(\Big(\frac{V(K^*)}{\omega_n}\Big)^{1/n}\Big)\\
&\leqslant nV(K)\varphi\Big(\Big(\frac{\omega_n}{V(K)}\Big)^{1/n}\Big)
\end{aligned}$$

If the equality holds, by the equality case of the Blaschke-Santalo inequality, we get K is an ellipsoid. □

In Theorem 3, when $\varphi(t) = t^p$, $p \geqslant 1$, (3.4) reduces to (3.3).

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