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STEFFENSEN TYPE INEQUALITIES INVOLVING CONVEX FUNCTIONS

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Abstract. In this paper a new class of functions $\mathscr{M}_1^c[a,b]$ that extends the class of convex functions is introduced. Moreover, Steffensen type inequalities for the class of convex functions are proved as a consequence of more general inequalities for class $\mathscr{M}_1^c[a,b]$. Using the linear functionals constructed from the difference of the left and the right hand side of proved Steffensen type inequalities new families of exponentially convex functions and related results are obtained.

1. Introduction

Since its appearance in 1918 Steffensen's inequality [8] has been generalized, refined and applied by many mathematicians for various motivations. For detailed information interested reader may refer to relevant chapters of monographs [4], [5], [7] and references cited therein. Recent overview of connections between some generalizations obtained by Pečarić, Milovanović, Mercer, Wu, Srivastava and Liu can be found in [6].

The well-known Steffensen inequality reads:

THEOREM 1. *Suppose that f is nonincreasing and g is integrable on $[a,b]$ with $0 \leqslant g \leqslant 1$ and $\lambda = \int_a^b g(t)dt$. Then we have*

$$\int_{b-\lambda}^{b} f(t)dt \leqslant \int_a^b f(t)g(t)dt \leqslant \int_a^{a+\lambda} f(t)dt. \tag{1.1}$$

The inequalities are reversed for f nondecreasing.

Milovanović and Pečarić in their paper [3] obtained weaker conditions on function g. In [9] Vasić and Pečarić showed that these weaker conditions are necessary and sufficient. Hence, we have the following theorem.

THEOREM 2. *Let f and g be integrable functions on $[a,b]$ and let $\lambda = \int_a^b g(t)dt$.*

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a) *The second inequality in (1.1) holds for every nonincreasing function f if and only if*

$$\int_a^x g(t)dt \leqslant x-a \text{ and } \int_x^b g(t)dt \geqslant 0 \text{ for every } x \in [a,b].$$

b) *The first inequality in (1.1) holds for every nonincreasing function f if and only if*

$$\int_x^b g(t)dt \leqslant b-x \text{ and } \int_a^x g(t)dt \geqslant 0 \text{ for every } x \in [a,b].$$

In this paper we introduce a new class of functions $\mathscr{M}_1^c[a,b]$ that extends the class of convex functions. As we show, class $\mathscr{M}_1^c[a,b]$ can be interpreted as class of functions which are "convex at point c". Further, we prove that a function is convex on $[a,b]$ if and only if it is convex at every point of $[a,b]$. Steffensen's inequality assumes that the function f is monotonic and our aim is to extend it to more general types of functions such as class $\mathscr{M}_1^c[a,b]$ and the class of convex functions.

First, let us recall the k-th order divided difference of f at distinct points $x_0, x_1, \ldots, x_k$. Let f be a real-valued function defined on $[a,b]$. The k-th order divided difference of f at distinct points $x_0, x_1, \ldots, x_k$ in $[a,b]$ may be defined recursively by

$$[x_i; f] = f(x_i), \, i = 0, \ldots, k$$

and

$$[x_0, \ldots, x_k; f] = \frac{[x_1, \ldots, x_k; f] - [x_0, \ldots, x_{k-1}; f]}{x_k - x_0}.$$

2. Main results

Let us introduce a new class of functions that extends the class of convex functions.

DEFINITION 1. Let $f : [a,b] \to \mathbb{R}$ be a function and $c \in (a,b)$. We say that f belongs to class $\mathscr{M}_1^c[a,b]$ ($\mathscr{M}_2^c[a,b]$) if there exists a constant A such that the function $F(x) = f(x) - Ax$ is nonincreasing (nondecreasing) on $[a,c]$ and nondecreasing (nonincreasing) on $[c,b]$.

REMARK 1. If $f \in \mathscr{M}_1^c[a,b]$ or $f \in \mathscr{M}_2^c[a,b]$ and $f'(c)$ exists, then $f'(c) = A$.

Let us show this for $f \in \mathscr{M}_1^c[a,b]$. Since F is nonincreasing on $[a,c]$ and nondecreasing on $[c,b]$ for every distinct points $x_1, x_2 \in [a,c]$ and $y_1, y_2 \in [c,b]$ we have

$$[x_1, x_2; F] = [x_1, x_2; f] - A \leqslant 0 \leqslant [y_1, y_2; f] - A = [y_1, y_2; F].$$

Therefore, if $f'_-(c)$ and $f'_+(c)$ exist, letting $x_i \nearrow c$ and $y_i \searrow c$, $i = 1, 2$ we get

$$f'_-(c) \leqslant A \leqslant f'_+(c). \tag{2.1}$$

In the following lemma and theorem we show connection between class of functions $\mathscr{M}_1^c[a,b]$ and the class of convex functions.

LEMMA 1. *If $f:[a,b]\to\mathbb{R}$ is convex (concave), then $f\in\mathscr{M}_1^c[a,b]$ ($f\in\mathscr{M}_2^c[a,b]$) for every $c\in(a,b)$.*

Proof. If f is convex, then f'_- and f'_+ exist (see [7]). Hence, for every $x_1,x_2\in[a,c]$ and $y_1,y_2\in[c,b]$ it holds

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}\leqslant f'_-(c)\leqslant f'_+(c)\leqslant\frac{f(y_2)-f(y_1)}{y_2-y_1}.$$

Therefore, for every $A\in[f'_-(c),f'_+(c)]$ the function $F(x)=f(x)-Ax$ satisfies

$$\frac{F(x_2)-F(x_1)}{x_2-x_1}\leqslant 0\leqslant\frac{F(y_2)-F(y_1)}{y_2-y_1},$$

so F is nonincreasing on $[a,c]$ and nondecreasing on $[c,b]$. □

THEOREM 3. *If $f\in\mathscr{M}_1^c[a,b]$ ($f\in\mathscr{M}_2^c[a,b]$) for every $c\in(a,b)$, then f is convex (concave).*

Proof. We give the proof for $f\in\mathscr{M}_1^c[a,b]$. First, let us recall the characterization of convexity given in [7]: the function g is convex if and only if the function

$$(x,y)\mapsto[x,y;g]=\frac{g(x)-g(y)}{x-y}$$

is nondecreasing in both variables.

For every $c\in(a,b)$ there exists constant A_c such that the function $F_c(x)=f(x)-A_cx$ is nonincreasing on $[a,c]$ and nondecreasing on $[c,b]$. So for every $x_1\neq x_2\leqslant c\leqslant y_1\neq y_2$ we have

$$\frac{F_c(x_2)-F_c(x_1)}{x_2-x_1}=\frac{f(x_2)-f(x_1)}{x_2-x_1}-A_c\leqslant 0\leqslant\frac{f(y_2)-f(y_1)}{y_2-y_1}-A_c=\frac{F_c(y_2)-F_c(y_1)}{y_2-y_1}.$$

Particularly, for $u<v<w$ we have

$$\frac{f(v)-f(u)}{v-u}\leqslant A_v\leqslant\frac{f(w)-f(v)}{w-v}. \tag{2.2}$$

Now, let $x_1,x_2,y\in[a,b]$ be arbitrary. If $y<x_1<x_2$, applying (2.2) we get

$$\frac{f(x_1)-f(y)}{x_1-y}\leqslant A_{x_1}\leqslant\frac{f(x_2)-f(x_1)}{x_2-x_1}=\frac{f(x_2)-f(y)}{x_2-x_1}-\frac{f(x_1)-f(y)}{x_2-x_1}.$$

By multiplying the above inequality with $\frac{x_2-x_1}{x_2-y}>0$ and simplifying we get

$$\frac{f(x_1)-f(y)}{x_1-y}\leqslant\frac{f(x_2)-f(y)}{x_2-y}.$$

Similarly for the cases $x_1 < y < x_2$ and $x_1 < x_2 < y$. So we can conclude that the function $(x,y) \mapsto [x,y;f]$ is nondecreasing in variable x. By symmetry, the same thing holds for variable y, so the proof is completed. □

REMARK 2. Taking into account Lemma 1 and Theorem 3, we can describe the property from Definition 1 as "convexity at point c". Therefore, function f is convex on $[a,b]$ if and only if it is convex at every $c \in (a,b)$.

In the following theorems we give Steffensen type inequalities for class of functions that are convex at point c.

THEOREM 4. *Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leqslant g \leqslant 1$. Let $c \in (a,b)$, $\lambda_1 = \int_a^c g(t)dt$ and $\lambda_2 = \int_c^b g(t)dt$. If $f \in \mathscr{M}_1^c[a,b]$ and*

$$\int_a^b tg(t)dt = a\lambda_1 + b\lambda_2 + \frac{\lambda_1^2 - \lambda_2^2}{2}, \tag{2.3}$$

then

$$\int_a^b f(t)g(t)dt \leqslant \int_a^{a+\lambda_1} f(t)dt + \int_{b-\lambda_2}^b f(t)dt \tag{2.4}$$

holds.

If $f \in \mathscr{M}_2^c[a,b]$ and (2.3) holds, the inequality in (2.4) is reversed.

Proof. We give the proof for $f \in \mathscr{M}_1^c[a,b]$. Let $F(x) = f(x) - Ax$, where A is the constant from Definition 1. Since $F : [a,c] \to \mathbb{R}$ is nonincreasing we can apply the right-hand side of Steffensen's inequality on function F, so

$$\int_a^c F(t)g(t)dt \leqslant \int_a^{a+\lambda_1} F(t)dt.$$

Hence, we obtain

$$\begin{aligned} 0 &\leqslant \int_a^{a+\lambda_1} F(t)dt - \int_a^c F(t)g(t)dt \\ &= \int_a^{a+\lambda_1} f(t)dt - \int_a^c f(t)g(t)dt - A\left(a\lambda_1 + \frac{\lambda_1^2}{2} - \int_a^c tg(t)dt\right). \end{aligned} \tag{2.5}$$

Further, since $F : [c,b] \to \mathbb{R}$ is nondecreasing we can apply the left-hand side of Steffensen's inequality on function F, so

$$\int_c^b F(t)g(t)dt \leqslant \int_{b-\lambda_2}^b F(t)dt.$$

Hence, we obtain

$$\begin{aligned} 0 &\geqslant \int_c^b F(t)g(t)dt - \int_{b-\lambda_2}^b F(t)dt \\ &= \int_c^b f(t)g(t)dt - \int_{b-\lambda_2}^b f(t)dt - A\left(\int_c^b tg(t)dt - b\lambda_2 + \frac{\lambda_2^2}{2}\right). \end{aligned} \tag{2.6}$$

Now from (2.5) and (2.6) we obtain

$$\int_a^{a+\lambda_1} f(t)dt + \int_{b-\lambda_2}^b f(t)dt - \int_a^b f(t)g(t)dt \geqslant A\left(a\lambda_1 + b\lambda_2 + \frac{\lambda_1^2 - \lambda_2^2}{2} - \int_a^b tg(t)dt\right).$$

Hence, if $\int_a^b tg(t)dt = a\lambda_1 + b\lambda_2 + \frac{\lambda_1^2-\lambda_2^2}{2}$, then (2.4) holds.

Proof for $f \in \mathscr{M}_2^c[a,b]$ is similar so we omit the details. □

REMARK 3. It is obvious from the proof that the condition (2.3) can be weakened. That is, for $f \in \mathscr{M}_1^c[a,b]$ inequality (2.4) still holds if (2.3) is replaced by the weaker condition

$$A\left(a\lambda_1 + b\lambda_2 + \frac{\lambda_1^2 - \lambda_2^2}{2} - \int_a^b tg(t)dt\right) \geqslant 0, \tag{2.7}$$

where A is the constant from Definition 1. Also, for $f \in \mathscr{M}_2^c[a,b]$ the reverse inequality in (2.4) holds if (2.3) is replaced by (2.7) with the reverse inequality.

Additionaly, condition (2.3) can be further weakened if the function f is monotonic. Since (2.1) holds, for a nondecreasing function $f \in \mathscr{M}_1^c[a,b]$ or nonincreasing function $f \in \mathscr{M}_2^c[a,b]$, from (2.7) we obtain that (2.3) can be weakened to

$$\int_a^b tg(t)dt \leqslant a\lambda_1 + b\lambda_2 + \frac{\lambda_1^2 - \lambda_2^2}{2}. \tag{2.8}$$

Further, if $f \in \mathscr{M}_1^c[a,b]$ is nonincreasing or $f \in \mathscr{M}_2^c[a,b]$ is nondecreasing, (2.3) can be weakened to (2.8) with the reverse inequality.

THEOREM 5. *Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leqslant g \leqslant 1$. Let $c \in (a,b)$, $\lambda_1 = \int_a^c g(t)dt$ and $\lambda_2 = \int_c^b g(t)dt$. If $f \in \mathscr{M}_1^c[a,b]$ and*

$$\int_a^b tg(t)dt = c(\lambda_1 + \lambda_2) + \frac{\lambda_2^2 - \lambda_1^2}{2}, \tag{2.9}$$

then

$$\int_a^b f(t)g(t)dt \geqslant \int_{c-\lambda_1}^{c+\lambda_2} f(t)dt \tag{2.10}$$

holds.

If $f \in \mathscr{M}_2^c[a,b]$ and (2.9) holds, the inequality in (2.10) is reversed.

Proof. We give the proof for $f \in \mathscr{M}_1^c[a,b]$. Let $F(x) = f(x) - Ax$. Since $F : [a,c] \to \mathbb{R}$ is nonincreasing applying the left-hand side of Steffensen's inequality on function F we obtain

$$0 \leqslant \int_a^c f(t)g(t)dt - \int_{c-\lambda_1}^c f(t)dt - A\left(\int_a^c tg(t)dt - c\lambda_1 + \frac{\lambda_1^2}{2}\right). \tag{2.11}$$

Further, since $F:[c,b]\to\mathbb{R}$ is nondecreasing applying the right-hand side of Steffensen's inequality on function F we obtain

$$0 \geqslant \int_c^{c+\lambda_2} f(t)dt - \int_c^b f(t)g(t)dt - A\left(c\lambda_2 + \frac{\lambda_2^2}{2} - \int_c^b tg(t)dt\right). \tag{2.12}$$

Now from (2.11) and (2.12) we obtain

$$\int_a^b f(t)g(t)dt - \int_{c-\lambda_1}^{c+\lambda_2} f(t)dt \geqslant A\left(\int_a^b tg(t)dt - c(\lambda_1+\lambda_2) + \frac{\lambda_1^2-\lambda_2^2}{2}\right).$$

Hence, if $\int_a^b tg(t)dt = c(\lambda_1+\lambda_2)+\frac{\lambda_2^2-\lambda_1^2}{2}$, then (2.10) holds.

Proof for $f\in\mathscr{M}_2^c[a,b]$ is similar so we omit the details. □

REMARK 4. For $f\in\mathscr{M}_1^c[a,b]$ the inequality (2.10) still holds if the condition (2.9) is replaced by the weaker condition

$$A\left(\int_a^b tg(t)dt - c(\lambda_1+\lambda_2) + \frac{\lambda_1^2-\lambda_2^2}{2}\right) \geqslant 0, \tag{2.13}$$

where A is the constant from Definition 1. Also, for $f\in\mathscr{M}_2^c[a,b]$ the reverse inequality in (2.10) holds if (2.9) is replaced by (2.13) with the reverse inequality.

Additionaly, condition (2.9) can be further weakened if the function f is monotonic. Since (2.1) holds, for a nondecreasing function $f\in\mathscr{M}_1^c[a,b]$ or nonincreasing function $f\in\mathscr{M}_2^c[a,b]$, from (2.13) we obtain that (2.9) can be weakened to

$$\int_a^b tg(t)dt \geqslant c(\lambda_1+\lambda_2) + \frac{\lambda_2^2-\lambda_1^2}{2}. \tag{2.14}$$

Further, if $f\in\mathscr{M}_1^c[a,b]$ is nonincreasing or $f\in\mathscr{M}_2^c[a,b]$ is nondecreasing, (2.9) can be weakened to (2.14) with the reverse inequality.

As a consequence of Theorems 4 and 5 we obtain Steffensen type inequalities that involve convex functions.

COROLLARY 1. *Let $g:[a,b]\to\mathbb{R}$ be an integrable function such that $0\leqslant g\leqslant 1$. Let $c\in(a,b)$, $\lambda_1=\int_a^c g(t)dt$ and $\lambda_2=\int_c^b g(t)dt$. If $f:[a,b]\to\mathbb{R}$ is convex function and (2.3) holds, then (2.4) holds.*

If $f:[a,b]\to\mathbb{R}$ is concave function and (2.3) holds, the inequality in (2.4) is reversed.

Proof. Since f is convex, from Lemma 1 we have that $f\in\mathscr{M}_1^c[a,b]$ for every $c\in(a,b)$. So we can apply Theorem 4. □

COROLLARY 2. *Let $g:[a,b]\to\mathbb{R}$ be an integrable function such that $0\leqslant g\leqslant 1$. Let $c\in(a,b)$, $\lambda_1=\int_a^c g(t)dt$ and $\lambda_2=\int_c^b g(t)dt$. If $f:[a,b]\to\mathbb{R}$ is convex function and (2.9) holds, then (2.10) holds.*

If $f:[a,b]\to\mathbb{R}$ is concave function and (2.9) holds, the inequality in (2.10) is reversed.

Proof. Similar as in proof of Corollary 1 we have that $f \in \mathscr{M}_1^c[a,b]$ for every $c \in (a,b)$. So we can apply Theorem 5. $\square$

Motivated by Theorem 2, in the following theorems we give weaker conditions for Steffensen type inequalities for class $\mathscr{M}_1^c[a,b]$.

THEOREM 6. *Let $c \in (a,b)$ and let $g : [a,b] \to \mathbb{R}$ be an integrable function such that*

$$\int_a^x g(t)dt \leqslant x-a \quad \text{and} \quad \int_x^c g(t)dt \geqslant 0 \quad \text{for every } x \in [a,c] \tag{2.15}$$

and

$$\int_x^b g(t)dt \leqslant b-x \quad \text{and} \quad \int_c^x g(t)dt \geqslant 0 \quad \text{for every } x \in [c,b]. \tag{2.16}$$

Let $\lambda_1 = \int_a^c g(t)dt$ and $\lambda_2 = \int_c^b g(t)dt$. If $f \in \mathscr{M}_1^c[a,b]$ and (2.3) holds, then (2.4) holds.

If $f \in \mathscr{M}_2^c[a,b]$ and (2.3) holds, the inequality in (2.4) is reversed.

Proof. Let $f \in \mathscr{M}_1^c[a,b]$ and let $F(x) = f(x) - Ax$. Since $F : [a,c] \to \mathbb{R}$ is nonincreasing and (2.15) holds, from Theorem 2 a) we obtain

$$\begin{aligned} 0 &\leqslant \int_a^{a+\lambda_1} F(t)dt - \int_a^c F(t)g(t)dt \\ &= \int_a^{a+\lambda_1} f(t)dt - \int_a^c f(t)g(t)dt - A\left(a\lambda_1 + \frac{\lambda_1^2}{2} - \int_a^c tg(t)dt\right). \end{aligned} \tag{2.17}$$

Further, since $F : [c,b] \to \mathbb{R}$ is nondecreasing and (2.16) holds, from Theorem 2 b) we obtain

$$\begin{aligned} 0 &\geqslant \int_c^b F(t)g(t)dt - \int_{b-\lambda_2}^b F(t)dt \\ &= \int_c^b f(t)g(t)dt - \int_{b-\lambda_2}^b f(t)dt - A\left(\int_c^b tg(t)dt - b\lambda_2 + \frac{\lambda_2^2}{2}\right). \end{aligned} \tag{2.18}$$

Now from (2.17) and (2.18) we obtain

$$\begin{aligned} &\int_a^{a+\lambda_1} f(t)dt + \int_{b-\lambda_2}^b f(t)dt - \int_a^b f(t)g(t)dt \\ &\qquad \geqslant A\left(a\lambda_1 + b\lambda_2 + \frac{\lambda_1^2 - \lambda_2^2}{2} - \int_a^b tg(t)dt\right). \end{aligned}$$

Hence, if $\int_a^b tg(t)dt = a\lambda_1 + b\lambda_2 + \frac{\lambda_1^2 - \lambda_2^2}{2}$, then (2.4) holds.

Similarly for $f \in \mathscr{M}_2^c[a,b]$. $\square$

THEOREM 7. *Let $c \in (a,b)$ and let $g : [a,b] \to \mathbb{R}$ be an integrable function such that*

$$\int_x^c g(t)dt \leqslant c-x \quad \text{and} \quad \int_a^x g(t)dt \geqslant 0 \quad \text{for every } x \in [a,c] \tag{2.19}$$

and

$$\int_c^x g(t)dt \leqslant x-c \quad \text{and} \quad \int_x^b g(t)dt \geqslant 0 \quad \text{for every } x \in [c,b]. \tag{2.20}$$

Let $\lambda_1 = \int_a^c g(t)dt$ and $\lambda_2 = \int_c^b g(t)dt$. If $f \in \mathscr{M}_1^c[a,b]$ and (2.9) holds, then (2.10) holds.

If $f \in \mathscr{M}_2^c[a,b]$ and (2.9) holds, the inequality in (2.10) is reversed.

Proof. Similar to the proof of Theorem 6. □

As a consequence of previous theorems we obtain weaker conditions for Steffensen type inequalities that involve convex functions.

COROLLARY 3. *Let $c \in (a,b)$ and let $g : [a,b] \to \mathbb{R}$ be an integrable function such that (2.15) and (2.16) hold. Let $\lambda_1 = \int_a^c g(t)dt$ and $\lambda_2 = \int_c^b g(t)dt$. If $f : [a,b] \to \mathbb{R}$ is convex function and (2.3) holds, then (2.4) holds.*

If $f : [a,b] \to \mathbb{R}$ is concave function and (2.3) holds, the inequality in (2.4) is reversed.

COROLLARY 4. *Let $c \in (a,b)$ and let $g : [a,b] \to \mathbb{R}$ be an integrable function such that (2.19) and (2.20) hold. Let $\lambda_1 = \int_a^c g(t)dt$ and $\lambda_2 = \int_c^b g(t)dt$. If $f : [a,b] \to \mathbb{R}$ is convex function and (2.9) holds, then (2.10) holds.*

If $f : [a,b] \to \mathbb{R}$ is concave function and (2.9) holds, the inequality in (2.10) is reversed.

3. Exponential convexity

Notice that Steffensen type inequalities (2.4) and (2.10) are linear in f. This motivates us to define the following linear functionals:

$$L_1(f) = \int_a^{a+\lambda_1} f(t)dt + \int_{b-\lambda_2}^b f(t)dt - \int_a^b f(t)g(t)dt \tag{3.1}$$

and

$$L_2(f) = \int_a^b f(t)g(t)dt - \int_{c-\lambda_1}^{c+\lambda_2} f(t)dt. \tag{3.2}$$

Under assumptions of Theorems 4–7 we have that $L_1(f) \geqslant 0$ and $L_2(f) \geqslant 0$ for $f \in \mathscr{M}_1^c[a,b]$. Further, under assumptions of Corollaries 1–4 we have that $L_1(f) \geqslant 0$ and $L_2(f) \geqslant 0$ for any convex function f.

First, we give mean value theorems.

THEOREM 8. *Let $c \in (a,b)$ and let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leqslant g \leqslant 1$. Let $\lambda_1 = \int_a^c g(t)dt$ and $\lambda_2 = \int_c^b g(t)dt$. If (2.3) holds, then for any $f \in C^2[a,b]$ there exists $\xi \in [a,b]$ such that*

$$L_1(f) = \frac{f''(\xi)}{2}\left[a^2\lambda_1 + a\lambda_1^2 + b^2\lambda_2 - b\lambda_2^2 + \frac{\lambda_1^3+\lambda_2^3}{3} - \int_a^b t^2 g(t)dt\right]. \tag{3.3}$$

where L_1 is defined by (3.1).

Proof. Since $f \in C^2[a,b]$ there exist $m = \min_{x\in[a,b]} f''(x)$ and $M = \max_{x\in[a,b]} f''(x)$. The functions

$$\Psi_1(x) = f(x) - \frac{m}{2}x^2 \quad \text{and} \quad \Psi_2(x) = \frac{M}{2}x^2 - f(x)$$

are convex since $\Psi_i''(x) \geqslant 0$, $i = 1,2$. Hence, by Corollary 1 we have $L_1(\Psi_i) \geqslant 0$, $i = 1,2$ and we get

$$\frac{m}{2}L_1(x^2) \leqslant L_1(f) \leqslant \frac{M}{2}L_1(x^2), \tag{3.4}$$

where

$$L_1(x^2) = a^2\lambda_1 + a\lambda_1^2 + b^2\lambda_2 - b\lambda_2^2 + \frac{\lambda_1^3+\lambda_2^3}{3} - \int_a^b t^2 g(t)dt.$$

Since x^2 is convex, by Corollary 1 we have $L_1(x^2) \geqslant 0$.

If $L_1(x^2) = 0$, then (3.4) implies $L_1(f) = 0$ and (3.3) holds for every $\xi \in [a,b]$. Otherwise, dividing (3.4) by $L_1(x^2)/2 > 0$ we get

$$m \leqslant \frac{2L_1(f)}{L_1(x^2)} \leqslant M,$$

so continuinity of f'' ensures existence of $\xi \in [a,b]$ satisfying (3.3). □

THEOREM 9. *Let $c \in (a,b)$ and let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leqslant g \leqslant 1$. Let $\lambda_1 = \int_a^c g(t)dt$ and $\lambda_2 = \int_c^b g(t)dt$. If (2.9) holds, then for any $f \in C^2[a,b]$ there exists $\xi \in [a,b]$ such that*

$$L_2(f) = \frac{f''(\xi)}{2}\left[\int_a^b t^2 g(t)dt - c^2(\lambda_1+\lambda_2) - c(\lambda_2^2 - \lambda_1^2) - \frac{\lambda_1^3+\lambda_2^3}{3}\right].$$

where L_2 is defined by (3.2).

Proof. Similar to the proof of Theorem 8. □

THEOREM 10. *Let $c \in (a,b)$ and let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leqslant g \leqslant 1$. Let $\lambda_1 = \int_a^c g(t)dt$, $\lambda_2 = \int_c^b g(t)dt$ and $f,h \in C^2[a,b]$. If (2.3) holds and $h''(x) \neq 0$ for every $x \in [a,b]$, then there exists $\xi \in [a,b]$ such that*

$$\frac{L_1(f)}{L_1(h)} = \frac{f''(\xi)}{h''(\xi)}$$

holds, where L_1 is defined by (3.1).

Proof. Let us define $\Phi \in C^2[a,b]$ by $\Phi(x) = L_1(h)f(x) - L_1(f)h(x)$. Due to linearity of L_1 we have $L_1(\Phi) = 0$. Now, by Theorem 8 there exist $\xi, \xi_1 \in [a,b]$ such that

$$0 = L_1(\Phi) = \frac{\Phi''(\xi)}{2} L_1(x^2)$$
$$0 \neq L_1(h) = \frac{h''(\xi_1)}{2} L_1(x^2).$$

Therefore, $L_1(x^2) \neq 0$ and

$$0 = \Phi''(\xi) = L_1(h)f''(\xi) - L_1(f)h''(\xi),$$

which gives the claim of the theorem. □

THEOREM 11. *Let $c \in (a,b)$ and let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leqslant g \leqslant 1$. Let $\lambda_1 = \int_a^c g(t)dt$, $\lambda_2 = \int_c^b g(t)dt$ and $f,h \in C^2[a,b]$. If (2.9) holds and $h(x) \neq 0$ for every $x \in [a,b]$, then there exists $\xi \in [a,b]$ such that*

$$\frac{L_2(f)}{L_2(h)} = \frac{f''(\xi)}{h''(\xi)}$$

holds, where L_2 is defined by (3.2).

Proof. Similar to the proof of Theorem 10. □

REMARK 5. Condition $0 \leqslant g \leqslant 1$ in Theorems 8 and 10 can be replaced by weaker conditions (2.15) and (2.16). Further, condition $0 \leqslant g \leqslant 1$ in Theorems 9 and 11 can be replaced by weaker conditions (2.19) and (2.20).

Next, we recall some basic definitions and results on exponential convexity.

DEFINITION 2. A function $\psi : I \to \mathbb{R}$ is *n-exponentially convex in the Jensen sense* on I if

$$\sum_{i,j=1}^{n} \xi_i \xi_j \, \psi\left(\frac{x_i + x_j}{2}\right) \geqslant 0,$$

holds for all choices $\xi_1, \ldots, \xi_n \in \mathbb{R}$ and all choices $x_1, \ldots, x_n \in I$.

A function $\psi : I \to \mathbb{R}$ is *n-exponentially convex* on I if it is n-exponentially convex in the Jensen sense and continuous on I.

REMARK 6. It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions.

Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \leqslant n$, $k \in \mathbb{N}$.

DEFINITION 3. A function $\psi : I \to \mathbb{R}$ is *exponentially convex in the Jensen sense* on I if it is n-exponentially convex in the Jensen sense on I for every $n \in \mathbb{N}$.

A function $\psi : I \to \mathbb{R}$ is *exponentially convex* on I if it is exponentially convex in the Jensen sense and continuous on I.

REMARK 7. A function $\psi : I \to \mathbb{R}$ is log-convex in the Jensen sense, i.e.

$$\psi\left(\frac{x+y}{2}\right)^2 \leqslant \psi(x)\psi(y), \quad \text{for all } x, y \in I, \tag{3.5}$$

if and only if

$$\alpha^2\psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2\psi(y) \geqslant 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$, i.e., if and only if ψ is 2-exponentially convex in the Jensen sense. By induction from (3.5) we have

$$\psi\left(\frac{1}{2^k}x + \left(1 - \frac{1}{2^k}\right)y\right) \leqslant \psi(x)^{\frac{1}{2^k}}\psi(y)^{1-\frac{1}{2^k}}.$$

Therefore, if ψ is continuous and $\psi(x) = 0$ for some $x \in I$, then from the last inequality and nonnegativity of ψ (see Remark 6) we get

$$\psi(y) = \lim_{k\to\infty} \psi\left(\frac{1}{2^k}x + \left(1 - \frac{1}{2^k}\right)y\right) = 0 \quad \text{for all } y \in I.$$

Hence, 2-exponentially convex function is either identically equal to zero or it is strictly positive and log-convex.

The following lemma is equivalent to the definition of convex functions (see [7]).

LEMMA 2. *A function $\psi : I \to \mathbb{R}$ is convex if and only if the inequality*

$$(x_3 - x_2)\psi(x_1) + (x_1 - x_3)\psi(x_2) + (x_2 - x_1)\psi(x_3) \geqslant 0$$

holds for all $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$.

We also use the following result (see [7]).

PROPOSITION 1. If f is a convex function on I and if $x_1 \leqslant y_1$, $x_2 \leqslant y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality holds

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leqslant \frac{f(y_2) - f(y_1)}{y_2 - y_1}.$$

If the function f is concave, the inequality is reversed.

We use previously defined functionals L_1 and L_2 to construct exponentially convex functions, a special type of convex functions invented by Bernstein in [1]. An elegant method of producing $n-$ exponentially convex and exponentially convex functions is given in [2]. We use this method to prove $n-$exponential convexity of functionals L_1 and L_2. In the sequel the notion log denotes the natural logarithm function and I, J denote intervals in $\mathbb{R}$.

THEOREM 12. *Let $\Omega = \{f_p : I \to \mathbb{R} \mid p \in J\}$ be a family of functions such that for every mutually different points $x_0, x_1, x_2 \in I$ the mapping $p \mapsto [x_0, x_1, x_2; f_p]$ is n-exponentially convex in the Jensen sense on J. Let L_i, $i = 1, 2$ be linear functionals defined by (3.1) and (3.2). Then the mapping $p \mapsto L_i(f_p)$ is n-exponentially convex in the Jensen sense on J.*

If the mapping $p \mapsto L_i(f_p)$ is continuous on J, then it is n-exponentially convex on J.

Proof. For $\xi_j \in \mathbb{R}$ and $p_j \in J$, $j = 1, \ldots, n$, we define the function

$$\Phi(x) = \sum_{j,k=1}^{n} \xi_j \xi_k f_{\frac{p_j + p_k}{2}}(x).$$

Using the assumption that the mapping $p \mapsto [x_0, x_1, x_2; f_p]$ is n-exponentially convex in the Jensen sense we have

$$[x_0, x_1, x_2; \Phi] = \sum_{j,k=1}^{n} \xi_j \xi_k [x_0, x_1, x_2; f_{\frac{p_j + p_k}{2}}] \geqslant 0.$$

This implies that Φ is a convex function. So by Corollaries 1 and 2,

$$0 \leqslant L_i(\Phi) = \sum_{j,k=1}^{n} \xi_j \xi_k L_i\left(f_{\frac{p_j + p_k}{2}}\right), \quad i = 1, 2.$$

Therefore, the mapping $p \mapsto L_i(f_p)$ is n-exponentially convex on J in the Jensen sense.

If the mapping $p \mapsto L_i(f_p)$ is also continuous on J, then $p \mapsto L_i(f_p)$ is n-exponentially convex by definition. □

If the assumptions of Theorem 12 hold for all $n \in \mathbb{N}$, then we have the following corollary.

COROLLARY 5. *Let $\Omega = \{f_p : I \to \mathbb{R} \mid p \in J\}$ be a family of functions such that for every mutually different points $x_0, x_1, x_2 \in I$ the mapping $p \mapsto [x_0, x_1, x_2; f_p]$ is exponentially convex in the Jensen sense on J. Let L_i, $i = 1, 2$ be linear functionals defined by (3.1) and (3.2). Then the mapping $p \mapsto L_i(f_p)$ is exponentially convex in the Jensen sense on J.*

If the mapping $p \mapsto L_i(f_p)$ is continuous on J, then it is exponentially convex on J.

COROLLARY 6. *Let $\Omega = \{f_p : I \to \mathbb{R} \,|\, p \in J\}$ be a family of functions such that for every mutually different points $x_0, x_1, x_2 \in I$ the mapping $p \mapsto [x_0, x_1, x_2; f_p]$ is 2-exponentially convex in the Jensen sense on J. Let L_i, $i = 1,2$ be linear functionals defined by (3.1) and (3.2). Then the following statements hold:*

(i) If the mapping $p \mapsto L_i(f_p)$ is continuous on J, then for $r, s, t \in J$, such that $r < s < t$, we have

$$[L_i(f_s)]^{t-r} \leqslant [L_i(f_r)]^{t-s} [L_i(f_t)]^{s-r}, \quad i = 1,2. \tag{3.6}$$

(ii) If the mapping $p \mapsto L_i(f_p)$ is strictly positive and differentiable on J, then for every $p, q, u, v \in J$ such that $p \leqslant u$ and $q \leqslant v$ we have

$$\mu_{p,q}(L_i, \Omega) \leqslant \mu_{u,v}(L_i, \Omega),$$

where

$$\mu_{p,q}(L_i, \Omega) = \begin{cases} \left(\frac{L_i(f_p)}{L_i(f_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\frac{d}{dp} L_i(f_p)}{L_i(f_p)}\right), & p = q. \end{cases} \tag{3.7}$$

Proof.

(i) By Theorem 12 the mapping $p \mapsto L_i(f_p)$ is 2-exponentially convex. Hence, by Remark 7, this mapping is either identically equal to zero, in which case inequality (3.6) holds trivially with zeros on both sides, or it is strictly positive and log-convex. Therefore, for $r, s, t \in J$ such that $r < s < t$ Lemma 2 gives

$$(t-s) \log L_i(f_r) + (r-t) \log L_i(f_s) + (s-r) \log L_i(f_t) \geqslant 0,$$

which is equivalent to inequality (3.6).

(ii) By (i) we have that the mapping $p \mapsto L_i(f_p)$ is log-convex on J, that is, the function $p \mapsto \log L_i(f_p)$ is convex on J. Applying Proposition 1 with $p \leqslant u$, $q \leqslant v$, $p \neq q$, $u \neq v$, we obtain

$$\frac{\log L_i(f_p) - \log L_i(f_q)}{p-q} \leqslant \frac{\log L_i(f_u) - \log L_i(f_v)}{u-v},$$

that is

$$\mu_{p,q}(L_i, \Omega) \leqslant \mu_{u,v}(L_i, \Omega).$$

Finally, the limit cases $p = q$ and $u = v$ are obtained by taking the limits $p \to q$ and $u \to v$. $\square$

Next, we give an example of a family of functions which satisfies previous conditions.

Let

$$\Upsilon = \{f_p \colon I \subset (0,\infty) \to \mathbb{R} \,|\, p \in \mathbb{R}\}$$

be a family of functions defined by

$$f_p(x) = \begin{cases} \frac{x^p}{p(p-1)}, & p \neq 0,1, \\ -\log x, & p = 0, \\ x\log x, & p = 1. \end{cases} \tag{3.8}$$

Functions f_p are convex since $f_p''(x) = x^{p-2} \geqslant 0$. Moreover, the function

$$\Phi(x) = \sum_{j,k=1}^{n} \xi_j \xi_k f_{\frac{p_j+p_k}{2}}(x)$$

satisfies

$$\Phi''(x) = \sum_{j,k=1}^{n} \xi_j \xi_k f''_{\frac{p_j+p_k}{2}}(x) = \left(\sum_{j=1}^{n} \xi_j x^{\frac{p_j}{2}-1}\right)^2 \geqslant 0,$$

so Φ is convex. Therefore,

$$0 \leqslant [t_0,t_1,t_2;\Phi] = \sum_{j,k=1}^{n} \xi_j \xi_k [t_0,t_1,t_2; f_{\frac{p_j+p_k}{2}}].$$

Hence, the mapping $p \mapsto [t_0,t_1,t_2;f_p]$ is n-exponentially convex in the Jensen sense. Since this holds for all $n \in \mathbb{N}$, we see that the family Υ satisfies the assumptions of Corollary 5, so the mapping $p \mapsto L_i(f_p)$ is exponentially convex in the Jensen sense. Now, let us prove that the mapping $p \mapsto L_1(f_p)$ is continuous on $\mathbb{R}$. Obviously, it is continuous on $\mathbb{R} \setminus \{0,1\}$. First, suppose $p \to 0$, then

$$\begin{aligned} \lim_{p\to 0} L_1(f_p) &= \lim_{p\to 0}\left(\int_a^{a+\lambda_1} \frac{t^p}{p(p-1)}dt + \int_{b-\lambda_2}^{b} \frac{t^p}{p(p-1)}dt - \int_a^b \frac{t^p}{p(p-1)}g(t)dt\right) \\ &= \lim_{p\to 0} \frac{\int_a^{a+\lambda_1} t^p dt + \int_{b-\lambda_2}^{b} t^p dt - \int_a^b t^p g(t)dt}{p(p-1)}. \end{aligned}$$

Since

$$\lim_{p\to 0}\left(\int_a^{a+\lambda_1} t^p dt + \int_{b-\lambda_2}^{b} t^p dt - \int_a^b t^p g(t)dt\right) = 0,$$

from L'Hospital rule limit we have

$$\begin{aligned} \lim_{p\to 0} L_1(f_p) &= \lim_{p\to 0} \frac{\int_a^{a+\lambda_1} t^p \log t dt + \int_{b-\lambda_2}^{b} t^p \log t dt - \int_a^b t^p \log t g(t)dt}{2p-1} \\ &= -\int_a^{a+\lambda_1} \log t dt - \int_{b-\lambda_2}^{b} \log t dt + \int_a^b \log t g(t)dt \\ &= L_1(f_0). \end{aligned}$$

In the same way, suppose $p \to 1$, then we get

$$\lim_{p\to 1} L_1(f_p) = \int_a^{a+\lambda_1} t\log t\,dt + \int_{b-\lambda_2}^b t\log t\,dt - \int_a^b t\log t g(t)\,dt = L_1(f_1).$$

Similarly we can check that the mapping $p \mapsto L_2(f_p)$ is continuous on $\mathbb{R}$. Hence, the mapping $p \mapsto L_i(f_p)$ is exponentially convex.

Applying Theorems 10 and 11 for the functions $f = f_p$ and $h = f_q$ given by (3.8) and defined on segment $I = [a,b] \subset (0,\infty)$, we conclude that there exist $\xi_i \in [a,b]$, $i = 1,2$ such that

$$\xi_i = \left(\frac{f_p''}{f_q''}\right)^{-1}\left(\frac{L_i(f_p)}{L_i(f_q)}\right) = \left(\frac{L_i(f_p)}{L_i(f_q)}\right)^{\frac{1}{p-q}}, \quad p \neq q,\ i = 1,2.$$

Therefore, $\mu_{p,q}(L_i,\Upsilon)$ given by (3.7) is a mean of the segment $[a,b]$. For $p \neq q$ we have

$$\mu_{p,q}(L_1,\Upsilon) = \left(\frac{q(q-1)}{p(p-1)} \cdot \frac{\frac{(a+\lambda_1)^{p+1}-a^{p+1}}{p+1} + \frac{b^{p+1}-(b-\lambda_2)^{p+1}}{p+1} - \int_a^b t^p g(t)\,dt}{\frac{(a+\lambda_1)^{q+1}-a^{q+1}}{q+1} + \frac{b^{q+1}-(b-\lambda_2)^{q+1}}{q+1} - \int_a^b t^q g(t)\,dt}\right)^{\frac{1}{p-q}}.$$

The limiting cases can easily be calculated so we obtain

* for $p = q \neq 0,1$

$$\mu_{p,p}(L_1,\Upsilon) = \exp\left(\frac{\alpha - \int_a^b t^p \log t g(t)\,dt}{\frac{(a+\lambda_1)^{p+1}-a^{p+1}}{p+1} + \frac{b^{p+1}-(b-\lambda_2)^{p+1}}{p+1} - \int_a^b t^p g(t)\,dt} - \frac{2p-1}{p(p-1)}\right)$$

  where

$$\begin{aligned}\alpha = {} & \frac{(a+\lambda_1)^{p+1}\log(a+\lambda_1) - a^{p+1}\log a}{p+1} - \frac{(a+\lambda_1)^{p+1} - a^{p+1}}{(p+1)^2} \\ & + \frac{b^{p+1}\log b - (b-\lambda_2)^{p+1}\log(b-\lambda_2)}{p+1} - \frac{b^{p+1} - (b-\lambda_2)^{p+1}}{(p+1)^2},\end{aligned}$$

* for $p = q = 0$ $\quad \mu_{0,0}(L_1,\Upsilon) = \exp\left(\frac{1}{2} \cdot \frac{\beta_1 - \int_a^b \log^2 t g(t)\,dt}{\beta_2 - \int_a^b \log t g(t)\,dt} + 1\right)$

  where

$$\begin{aligned}\beta_1 = {} & (a+\lambda_1)\log^2(a+\lambda_1) - a\log^2 a + b\log^2 b - (b-\lambda_2)\log^2(b-\lambda_2) \\ & - 2(a+\lambda_1)\log(a+\lambda_1) + 2a\log a - 2b\log b + 2(b-\lambda_2)\log(b-\lambda_2) + 2(\lambda_1+\lambda_2) \\ \beta_2 = {} & (a+\lambda_1)\log(a+\lambda_1) - a\log a + b\log b - (b-\lambda_2)\log(b-\lambda_2) - \lambda_1 - \lambda_2\end{aligned}$$

* for $p=q=1$ $\quad \mu_{1,1}(L_1,\Upsilon)=\exp\left(\frac{1}{2}\cdot\frac{\gamma_1-\int_a^b t\log^2 t g(t)dt}{\gamma_2-\int_a^b t\log t g(t)dt}-1\right)$

where

$$\gamma_1=\frac{(a+\lambda_1)^2}{2}(\log^2(a+\lambda_1)-\log(a+\lambda_1))-\frac{a^2}{2}(\log^2 a-\log a)+\frac{b^2}{2}(\log^2 b-\log b)$$
$$-\frac{(b-\lambda_2)^2}{2}(\log^2(b-\lambda_2)-\log(b-\lambda_2))+\frac{a\lambda_1+b\lambda_2}{2}+\frac{\lambda_1^2-\lambda_2^2}{4}$$
$$\gamma_2=\frac{(a+\lambda_1)^2}{2}\log(a+\lambda_1)-\frac{a^2}{2}\log a+\frac{b^2}{2}\log b-\frac{(b-\lambda_2)^2}{2}\log(b-\lambda_2)$$
$$-\frac{a\lambda_1+b\lambda_2}{2}-\frac{\lambda_1^2-\lambda_2^2}{4}.$$

Similarly, we obtain that $\mu_{p,q}(L_2,\Upsilon)$ is given by

$$\mu_{p,q}(L_2,\Upsilon)=\begin{cases}\left(\frac{L_2(f_p)}{L_2(f_q)}\right)^{\frac{1}{p-q}}, & p\neq q,\\ \exp\left(\frac{-L_2(f_pf_0)}{L_2(f_p)}-\frac{2p-1}{p(p-1)}\right), & p=q\neq 0,1,\\ \exp\left(\frac{-L_2(f_0^2)}{2L_2(f_0)}+1\right), & p=q=0,\\ \exp\left(\frac{-L_2(f_0f_1)}{2L_2(f_1)}-1\right), & p=q=1.\end{cases}$$

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