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WRIGHT TYPE MULTIPLICATIVELY CONVEX FUNCTIONS

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Abstract. The notion of Wright type multiplicatively convex functions is introduced. Relationships between such functions and multiplicatively convex functions are investigated, and a counterpart of the Ng representation theorem for Wright convex functions is presented. It is proved that the function $F(x_1,\dots,x_n) = f(x_1)\cdots f(x_n)$ is multiplicatively Schur-convex if and only if f is Wright type multiplicatively convex. A Hermite-Hadamard type inequality for Wright type multiplicatively convex functions is also given.

1. Introduction

Convexity plays a very important role in mathematical science and other applied fields, and hence convexity and generalized convexity have been investigating extensively [8,9].

Let I be an interval in R. Recall that a function $f: I \to R$ is said to be *convex* if

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y) \tag{1.1}$$

for all $x,y \in I$ and $\lambda \in [0,1]$; f is called *Jensen convex (or mid-convex)* if condition (1.1) is assumed only for $\lambda = \frac{1}{2}$, that is $f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2}$ for all $x,y \in I$. A function $f: I \to R$ is said to be *quasi-convex* if

$$f(\lambda x + (1-\lambda)y) \leqslant \max\{f(x), f(y)\}$$

for all $x,y \in I$ and $\lambda \in [0,1]$.

There are many papers giving conditions under which Jensen convex functions are convex functions. K. Nikodem [6] obtained the following characterization of convex functions.

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NIKODEM'S THEOREM. *Let f be a real-valued function defined on convex and open set $D(\subseteq R^n)$. Then f is a convex function on D if and only if it is Jensen convex and quasi-convex on D.*

Recall also that a function $f: I \to R$ is said to be *Wright-convex* (see [5, 10]) if

$$f(\lambda x+(1-\lambda)y)+f((1-\lambda)x+\lambda y) \leqslant f(x)+f(y) \tag{1.2}$$

for all $x,y \in I$ and $\lambda \in [0,1]$. A function $f: R \to R$ is *additive* if $f(x+y) = f(x)+f(y)$ for all $x,y \in R$.

It is known that both convex functions and additive functions are Wright-convex. In particular, C. T. Ng [5] presented the following interesting result that any Wright-convex function can be decomposed as the sum of such functions.

NG'S THEOREM. *Let $I \subseteq R$ be an open interval, and let $f: I \to R$ be a function. Then f is Wright-convex if and only if there exist a convex function $C: I \to R$ and an additive function $A: R \to R$ such that $f(x) = C(x)+A(x),\ x \in I$.*

It is also noted that the study of convex functions has evolved into a larger theory about functions which are adapted to other geometries of the domain and/or obey other laws of comparison of means. P. Montel [4] first considered the class of all multiplicatively (or geometrically) convex functions in a beautiful paper discussing the possible analogues of convex functions in n variable. C. P. Niculescu et al. [8] presented comprehensive survey on multiplicatively convex functions.

In what follows R_+ denotes the set of all positive real numbers and I is a nonvoid interval of R_+. A positive function $f: I \to R_+$ is said to be *multiplicatively (or geometrically) convex* (see [7] and [8, p. 66]) if

$$f(x^{\lambda}y^{1-\lambda}) \leqslant f^{\lambda}(x)f^{1-\lambda}(y) \tag{1.3}$$

for all $x,y \in I$ and $\lambda \in [0,1]$; f is called *multiplicatively (or geometrically) Jensen convex* if condition (1.3) is assumed only for $\lambda = 1/2$, that is

$$f(\sqrt{xy}) \leqslant \sqrt{f(x)f(y)} \quad \text{for all } x,y \in I. \tag{1.4}$$

For two $n-$tuples $x = (x_1,\dots,x_n), y = (y_1,\dots,y_n) \in I^n \ (n \geqslant 2)$, following Zhang [11] we say that x is *logarithmically majorized* by y, and write $\log x \prec \log y$, if

$$\prod_{i=1}^{m} x_{[i]} \leqslant \prod_{i=1}^{m} y_{[i]} \ (m = 1,\dots,n-1), \quad \text{and} \quad \prod_{i=1}^{n} x_{[i]} = \prod_{i=1}^{n} y_{[i]},$$

where $x_{[i]}$ denotes the ith largest component in x.

A function $G: I^n \to R_+$ is said to be *multiplicatively (or geometrically) Schur-convex* if $G(x) \leqslant G(y)$ whenever $\log x \prec \log y,\ x,y \in I^n$. G is called *multiplicatively (or geometrically) Schur-concave* if and only if $1/G$ is multiplicatively (or geometrically) Schur-convex.

The theory of multiplicatively convex functions and multiplicatively schur-convex functions has been recently received considerable attention. For example, see [2, 4, 7, 8, 11] and the references cited therein. Our main purpose of this paper is to present complementary information on the theory of multiplicatively convex functions and multiplicatively Schur-convex functions by proposing the following new notion of convexity, which may be called Wright type multiplicative (or geometric) convexity.

DEFINITION. A function $f: I \to R_+$ is said to be *Wright type multiplicatively (or geometrically) convex* if

$$f(x^\lambda y^{1-\lambda})f(x^{1-\lambda}y^\lambda) \leqslant f(x)f(y)$$

for all $x,y \in I$ and $\lambda \in [0,1]$.

2. Some characterizations

In this section we investigate the relationships between multiplicative Jensen convexity, multiplicative convexity and Wright type multiplicative convexity. Some characterizations are also presented. In particular, a representation of Wright type multiplicatively convex functions is given.

It is known that every multiplicatively convex function is multiplicative Jensen convex, but not the converse. It is also obvious that multiplicatively convex functions are Wright type multiplicatively convex, but the converse implication is not true as shown the example below.

EXAMPLE 2.1. A function $m: R_+ \to R_+$ is said to be multiplicative if

$$m(xy) = m(x)m(y) \tag{2.1}$$

for all $x,y \in R_+$. By [3, Theorem 1.49], (2.1) has a discontinuous solution, say m. Taking $f(x) = m(x)(1+x)$, $x \in R_+$, and using the weighted arithmetic and geometric means inequality, we have

$$\begin{aligned} f(x^\lambda y^{1-\lambda}) \cdot f(x^{1-\lambda}y^\lambda) &= m(x^\lambda y^{1-\lambda})(1+x^\lambda y^{1-\lambda}) \cdot m(x^{1-\lambda}y^\lambda)(1+x^{1-\lambda}y^\lambda) \\ &= m(xy)(1+xy+x^\lambda y^{1-\lambda}+x^{1-\lambda}y^\lambda) \\ &\leqslant m(xy)(1+xy+x+y) = f(x)f(y). \end{aligned}$$

This implies that f is Wright type multiplicatively convex. On the other hand, it is showed [8, p. 78] that every multiplicatively convex function is continuous in the interior of its domain of definition. This implies that f is not multiplicatively convex because it is not continuous.

To reveal the relationships between these kinds of geometric convexity, we start with the following useful fact.

LEMMA 2.1. *Let I be a non-void interval of R_+. If $f: I \to R_+$ is a multiplicatively Jensen convex, then*

$$f\left(x^{\frac{k}{2^n}}y^{1-\frac{k}{2^n}}\right) \leqslant f^{\frac{k}{2^n}}(x)f^{1-\frac{k}{2^n}}(y), \tag{2.2}$$

for all $x, y \in I$ *and all* $k, n \in N$ *such that* $k < 2^n$.

Proof. This proof is by induction on n. For $n = 1$, (2.2) reduces to (1.4). Assuming (2.2) to hold for some $n \in N$ and all $k < 2^n$, we will prove it for $n+1$. Fix $x, y \in I$ and take $k < 2^{n+1}$, without loss of generality we may assume that $k < 2^n$. Then, by (1.4) and the induction assumption, we obtain

$$f\left(x^{\frac{k}{2^{n+1}}} y^{1-\frac{k}{2^{n+1}}}\right) = f\left(\left(x^{\frac{k}{2^n}} y^{1-\frac{k}{2^n}}\right)^{\frac{1}{2}} y^{\frac{1}{2}}\right) \leqslant \left(f\left(x^{\frac{k}{2^n}} y^{1-\frac{k}{2^n}}\right)\right)^{\frac{1}{2}} (f(y))^{\frac{1}{2}}$$
$$\leqslant f^{\frac{k}{2^{n+1}}}(x) f^{\frac{1}{2}-\frac{k}{2^{n+1}}}(y)(f(y))^{\frac{1}{2}} = f^{\frac{k}{2^{n+1}}}(x) f^{1-\frac{k}{2^{n+1}}}(y),$$

which finishes the proof. $\square$

THEOREM 2.1. *Let* $I \subseteq R_+$ *be a non-void interval. Then* $f : I \to R_+$ *is multiplicatively convex if and only if* f *is multiplicatively Jensen convex and satisfies the condition*

$$f(x^\lambda y^{1-\lambda}) \leqslant \max\{f(x), f(y)\}, \text{ for all } x, y \in I \text{ and } \lambda \in [0,1]. \tag{2.3}$$

Proof. Necessity. Assume that f is multiplicatively convex. Obviously, it is multiplicatively Jensen convex. Again, the multiplicative convexity of f and the weighted arithmetic and geometric means inequality show that for all $x, y \in I$ and $\lambda \in [0,1]$,

$$f(x^\lambda y^{1-\lambda}) \leqslant f^\lambda(x) f^{1-\lambda}(y) \leqslant \lambda f(x) + (1-\lambda) f(y) \leqslant \max\{f(x), f(y)\}.$$

Sufficiency. Fix arbitrary $x, y \in I$, $x \neq y$. Since f is multiplicatively Jensen convex, then it follows from Lemma 2.1 that

$$f(x^q y^{1-q}) \leqslant f^q(x) f^{1-q}(y) \tag{2.4}$$

for all dyadic $q \in (0,1)$. Consider the function $g : [0,1] \to R_+$ defined by

$$g(s) = f(x^s y^{1-s}), \quad s \in [0,1].$$

Take arbitrary $a, b \in [0,1]$, and $\lambda \in (0,1)$. The multiplicative Jensen convexity of f, together with the arithmetic and geometric means inequality, implies that

$$g\left(\frac{a+b}{2}\right) = f\left(x^{\frac{a+b}{2}} y^{1-\frac{a+b}{2}}\right) = f\left(\left(x^a y^{1-a}\right)^{\frac{1}{2}} \cdot \left(x^b y^{1-b}\right)^{\frac{1}{2}}\right)$$
$$\leqslant \sqrt{f\left(x^a y^{1-a}\right) \cdot f\left(x^b y^{1-b}\right)} = \sqrt{g(a) g(b)} \leqslant \frac{g(a)+g(b)}{2},$$

which implies that the function g is Jensen convex on $[0,1]$.

By (2.3) and the definition of g, we can easily obtain

$$g\left(\lambda a + (1-\lambda) b\right) = f\left(\left(x^a y^{1-a}\right)^\lambda \left(x^b y^{1-b}\right)^{1-\lambda}\right)$$
$$\leqslant \max\{f\left(x^a y^{1-a}\right), f\left(x^b y^{1-b}\right)\}$$
$$= \max\{g(a), g(b)\},$$

which implies that the function g is quasiconvex on $[0,1]$.

Thus, by Nikodem's Theorem, the function g is convex, and so it is continuous on the open interval $(0,1)$. From (2.4) it follows that

$$g(q) \leqslant (g(1))^q (g(0))^{1-q} \tag{2.5}$$

for all dyadic $q \in (0,1)$. Take a sequence $\{q_n\}$ $(n = 1,2,...)$ of dyadic numbers in $(0,1)$ tending to λ. Using (2.5) for $q = q_n$ and the continuity of g at λ, we obtain

$$g(\lambda) \leqslant (g(1))^\lambda (g(0))^{1-\lambda}.$$

Now, by the definition of g, we get

$$f(x^\lambda y^{1-\lambda}) \leqslant f^\lambda(x) f^{1-\lambda}(y),$$

which implies that f is multiplicatively convex, and so the proof is completed. □

THEOREM 2.2. *Let $I \subseteq R_+$ be a non-void interval. Then $f : I \to R^+$ is Wright type multiplicatively convex if and only if it is multiplicatively Jensen convex and satisfies the condition*

$$f(x^\lambda y^{1-\lambda}) f(x^{1-\lambda} y^\lambda) \leqslant \max\{f^2(x), f^2(y)\}, \text{ for all } x,y \in I \text{ and } \lambda \in [0,1]. \tag{2.6}$$

Proof. Necessity. Assume that f is Wright type multiplicatively convex. Obviously, it is multiplicatively Jensen convex. The Wright type multiplicative convexity of f also shows that for all $x,y \in I$ and $\lambda \in [0,1]$,

$$f(x^\lambda y^{1-\lambda}) f(x^{1-\lambda} y^\lambda) \leqslant f(x) f(y) \leqslant \max\{f^2(x), f^2(y)\}.$$

Sufficiency. Fix arbitrary $x,y \in I$, $x \neq y$. Since f is multiplicatively Jensen convex, it follows from Lemma 2.1 that

$$f(x^q y^{1-q}) \leqslant f^q(x) f^{1-q}(y), \tag{2.7}$$

and

$$f(x^{1-q} y^q) \leqslant f^{1-q}(x) f^q(y) \tag{2.8}$$

for all dyadic $q \in (0,1)$. Consider the function $g : [0,1] \to R_+$ defined by

$$g(s) = f(x^s y^{1-s}) \cdot f(x^{1-s} y^s), \ \ s \in [0,1].$$

Take arbitrary $a,b \in [0,1]$, and $\lambda \in (0,1)$. The multiplicative Jensen convexity of f, together with the arithmetic and geometric means inequality, implies that

$$\begin{aligned} g\left(\frac{a+b}{2}\right) &= f\left(x^{\frac{a+b}{2}} y^{1-\frac{a+b}{2}}\right) f\left(x^{1-\frac{a+b}{2}} y^{\frac{a+b}{2}}\right) \\ &= f\left(\left(x^a y^{1-a}\right)^{\frac{1}{2}} \left(x^b y^{1-b}\right)^{\frac{1}{2}}\right) \cdot f\left(\left(x^{1-a} y^a\right)^{\frac{1}{2}} \left(x^{1-b} y^b\right)^{\frac{1}{2}}\right) \\ &\leqslant \sqrt{f\left(x^a y^{1-a}\right) f\left(x^b y^{1-b}\right)} \cdot \sqrt{f\left(x^{1-a} y^a\right) f\left(x^{1-b} y^b\right)} \\ &= \sqrt{g(a) g(b)} \leqslant \frac{g(a)+g(b)}{2}, \end{aligned}$$

which implies that the function g is Jensen convex on $[0,1]$.

Again, from (2.7) it follows that for any $s \in (0,1)$,

$$g(s) = f(x^s y^{1-s}) \cdot f(x^{1-s} y^s) \leqslant \max\{f^2(x), f^2(y)\},$$

which implies that g is bounded above on $(0,1)$. Therefore by the famous Jensen Theorem (see [9, p. 220]), g is continuous on $(0,1)$.

By (2.7) and (2.8) we have

$$g(q) \leqslant \sqrt{g(1)g(0)} \tag{2.9}$$

for all dyadic $q \in (0,1)$. Take a sequence $\{q_n\}$ $(n = 1,2,...)$ of dyadic numbers in $(0,1)$ tending to λ. Using (2.9) for $q = q_n$ and the continuity of g at λ, we obtain

$$g(\lambda) \leqslant \sqrt{g(1)g(0)}.$$

Now, by the definition of g, we get

$$f(x^\lambda y^{1-\lambda}) f(x^\lambda y^{1-\lambda}) \leqslant f(x) f(y),$$

which implies that f is Wright type multiplicatively convex, and so the proof is completed. □

Noting that Ng's Theorem for Wrigth-convex functions, here we present a similar representation theorem for Wright type multiplicatively convex functions.

THEOREM 2.3. *Let I be a open interval of R_+. Then $f : I \to R_+$ is Wright type multiplicatively convex if and only if there exist a multiplicatively convex function $G : I \to R_+$ and a multiplicative function $M : R_+ \to R_+$ such that*

$$f(x) = G(x)M(x), \quad \text{for all } x \in I.$$

Proof. The sufficiency is clear. We now prove the necessity. Assume that $f : I \to R_+$ is Wright type multiplicatively convex on $I = (a,b)$. Take $g(x) = \log f(e^x)$, $x \in (\log a, \log b)$. Let $x, y \in (\log a, \log b)$, $\lambda \in [0,1]$, so $e^x, e^y \in (a,b)$, then we have

$$\begin{aligned} g(\lambda x + (1-\lambda)y) + g((1-\lambda)x + \lambda y) &= \log f(e^{\lambda x + (1-\lambda)y}) + \log f(e^{(1-\lambda)x + \lambda y}) \\ &= \log f\Big((e^x)^\lambda (e^y)^{1-\lambda}\Big) + \log f\Big((e^x)^{1-\lambda} (e^y)^\lambda\Big) \\ &\leqslant \log f(e^x) + \log f(e^y) \\ &= g(x) + g(y), \end{aligned}$$

which implies that $g(x) = \log f(e^x)$ is Wright-convex on $(\log a, \log b)$. By Ng's Theorem, there exist a convex function $C : (\log a, \log b) \to R$ and an additive function $A : R \to R$ such that

$$g(x) = C(x) + A(x), \quad \text{for all } x \in (\log a, \log b). \tag{2.10}$$

Set $M(x) = A(\log x)$ and $G(x) = C(\log x), x \in (a,b)$, one can verify that $G(x)$ is multiplicatively convex on (a,b) and $M(x)$ is multiplicative. From (2.10) it follows that

$$f(x) = G(x)M(x), \quad \text{for all } x \in (a,b).$$

The proof is completed. □

REMARK 2.1. It is known [3, Theorem 1.49] that the general solution $M : R_+ \to R_+$ of the multiplicative functional equation (2.1) is either $M(x) = 1$ or $M(x) = e^{A(\log x)}$, where $A : R \to R$ is an additive function. Further, the continuous solutions of (2.1) are $M(x) = x^{\mu} (\mu \in R)$. Thus, by Theorem 2.3, one can see that a positive function defined on an open interval I of R_+ is Wright type multiplicatively convex if and only if, either it is multiplicatively convex or it has the form $e^{A(\log x)}G(x)$, where G is multiplicatively convex on I. And furthermore, under the presence of continuity, a function $W : I \to R_+$ is Wright type multiplicatively convex if and only if it has the form $W(x) = x^{\mu}G(x)(\mu \in R)$, where $G(x)$ is multiplicatively convex. This also shows that a continuous function $f : W \to R_+$ is Wright type multiplicatively convex if and only if it is multiplicatively convex. Thus, from [8, p. 79–85], the functions $\frac{1-x}{x}$, $\frac{1+x}{1-x}$, $\Gamma(x)$, is continuous and multiplicatively convex on $(0, \frac{1}{2}]$, $(0,1)$, and $[1,\infty)$, respectively, and so they are Wright type multiplicatively convex on $(0, \frac{1}{2}]$, $(0,1)$, and $[1,\infty)$, respectively.

3. Functions generating multiplicatively Schur-convex products

It is showed [8, p. 80] that if a function $f : I \to R_+$ is multiplicatively convex, then it *generates multiplicatively Schur-convex products*, that is, the function $F : I^n \to R_+$ defined by

$$F(x) = F(x_1, ..., x_n) = f(x_1) \cdots f(x_n)$$

is multiplicatively Schur-convex.

Here we point out that the multiplicative convexity of f is a sufficient but not necessary condition under which F is multiplicatively Schur-convex. For this purpose, let $m : R_+ \to R_+$ be a discontinuous multiplicative function, then $f : R_+ \to R_+$ given by $f(x) = m(x)\exp(x), x \in R_+$, is not geometrically convex (because it is not continuous). However, the function $F(x) = f(x_1) \cdots f(x_n)$ is multiplicatively Schur-convex in R^n_+. To see this, take $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$ such that $\log x \prec \log y$. Since $\exp(x)$ is multiplicatively convex in R_+, then $\prod_{i=1}^n \exp(x_i) \leqslant \prod_{i=1}^n \exp(y_i)$. Again, by the multiplicative of m, we have

$$m(x_1) \cdots m(x_n) = m(x_1 \cdots x_n) = m(y_1 \cdots y_n) = m(y_1) \cdots m(y_n).$$

Hence,

$$\begin{aligned} F(x) &= f(x_1) \cdots f(x_n) = m(x_1) \cdots m(x_n) \prod_{i=1}^n \exp(x_i) \\ &= m(y_1) \cdots m(y_n) \prod_{i=1}^n \exp(x_i) \\ &\leqslant m(y_1) \cdots m(y_n) \prod_{i=1}^n \exp(y_i) \\ &= f(y_1) \cdots f(y_n). \end{aligned}$$

This implies that $F(x) = f(x_1)\cdots f(x_n)$ is multiplicatively Schur-convex in R_+^n.

The following result shows that the multiplicatively Jensen convex is a necessary condition under which f generates multiplicatively Schur-convex products.

THEOREM 3.1. *Let $I \subseteq R_+$ be an interval. If the function $F : I^n \to R_+$ given by*

$$F(x) = F(x_1,...,x_n) = f(x_1)\cdots f(x_n), \quad (x_1,...,x_n) \in I^n \ (n \geqslant 2)$$

is multiplicatively Schur-convex, then f is multiplicatively Jensen convex.

Proof. Take arbitrary $x_1, x_2 \in I$, without loss of generality we may assume that $x_1 > x_2$, and put $y_1 = y_2 = \sqrt{x_1 x_2}$. Consider the points

$$x = (x_1, x_2, x_2, ..., x_2), \quad y = (y_1, y_2, x_2, ..., x_2)$$

(if $n = 2$, we then take $x = (x_1, x_2)$, $y = (y_1, y_2)$). Obviously, $\log y \prec \log x$. Therefore, the multiplicative Schur-convexity of F implies that $F(y) \leqslant F(x)$, and so $f(y_1)f(y_2) \leqslant f(x_1)f(x_2)$, i.e, $f(\sqrt{x_1x_2}) \leqslant \sqrt{f(x_1)f(x_2)}$, which means that f is multiplicatively Jensen convex. The proof is completed. □

THEOREM 3.2. *Let $I \subseteq R_+$ be an open and non-void interval. Then the function $F : I^n \to R_+$ given by*

$$F(x) = F(x_1,...,x_n) = f(x_1)\cdots f(x_n), \quad (x_1,...,x_n) \in I^n (n \geqslant 2)$$

is multiplicatively Schur-convex if and only if f is Wright type multiplicatively convex.

Proof. Necessity. Taking arbitrary $x_1, x_2 \in I$ with $x_1 > x_2$, and $0 < \lambda < 1$, one can easily see that $y = (x_1^{\lambda} x_2^{1-\lambda}, x_2^{\lambda} x_1^{1-\lambda}, x_2, ..., x_2)$ is logarithmically majorized by $x = (x_1, x_2, x_2, ..., x_2)$, i.e., $\log y \prec \log x$. If $n = 2$, we then take $y = (x_1^{\lambda} x_2^{1-\lambda}, x_2^{\lambda} x_1^{1-\lambda})$ and $x = (x_1, x_2)$. Since the function $F(x) = \prod_{i=1}^n f(x_i)$ is multiplicatively Schur-convex, then we have

$$f\big(x_1^{\lambda} x_2^{1-\lambda}\big) f\big(x_2^{\lambda} x_1^{1-\lambda}\big) \leqslant f(x_1) f(x_2),$$

which shows that the function f is Wright type multiplicatively convex on I.

Sufficiency. Assume that f is Wright type geometrically convex on I. By Theorem 2.3, there exist a multiplicatively convex function g and a multiplicative function m such that $f(x) = g(x)m(x), x \in I$. Taking $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ such that $\log x \prec \log y$, we obtain $\prod_{i=1}^n g(x_i) \leqslant \prod_{i=1}^n g(y_i)$ since g is multiplicatively convex. We also have $\prod_{i=1}^n m(x_i) = \prod_{i=1}^n m(y_i)$ since m is multiplicative. Therefore, we obtain

$$F(x) = \prod_{i=1}^n f(x_i) = \prod_{i=1}^n g(x_i)m(x_i) \leqslant \prod_{i=1}^n g(y_i)m(y_i) = \prod_{i=1}^n f(y_i) = F(y),$$

which implies that F is multiplicatively Schur-convex in I^n. The proof is completed. □

By Theorems 2.1 and 2.2, one can easily see that for continuous functions the concepts of multiplicative convexity, Wright type multiplicative convexity, and multiplicative Jensen convexity are equivalent. Thus, we can obtain the following corollary which generalizes Proposition 2.3.5 in [8].

COROLLARY 3.1. *Let I be an open interval in R_+ and $f : I \to R_+$ be continuous. Then the function $F : I^n \to R_+$ given by $F(x) = F(x_1,...,x_n) = f(x_1)\cdots f(x_n), (x_1,...,x_n) \in I^n$, is multiplicatively Schur-convex if and only if f is multiplicatively convex.*

EXAMPLE 3.1. By Remark 2.1 and Corollary 3.1, one can see that the functions

$$\phi_1(x) = \prod_{i=1}^{n} \frac{1-x_i}{x_i}, \quad \phi_2(x) = \prod_{i=1}^{n} \frac{1+x_i}{1-x_i}, \quad \phi_3(x) = \prod_{i=1}^{n} \Gamma(x_i)$$

are multiplicatively Schur-convex on $(0,\frac{1}{2}]^n$, $(0,1)^n$ and $[1,\infty)^n$, respectively.

Put $G_n(x) = \left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}}$ and $A_n(x) = \frac{1}{n}\sum_{i=1}^n x_i$. Since $\log(G_n(x),...,G_n(x)) \prec \log(x_1,...,x_n)$, we can easily obtain the following inequalities:

(i) $\prod_{i=1}^n \frac{1+x_i}{1-x_i} \geqslant \left(\frac{1+G_n(x)}{1-G_n(x)}\right)^n$, $0 < x_i < 1$, $i = 1,...,n$.

(ii) $\prod_{i=1}^n \Gamma(x_i) \geqslant \left(\Gamma(G_n(x))\right)^n$, $x_i \geqslant 1$, $i = 1,...,n$.

(iii) $\prod_{i=1}^n \frac{1-x_i}{x_i} \geqslant \left(\frac{1-G_n(x)}{G_n(x)}\right)^n \geqslant \left(\frac{1-A_n(x)}{A_n(x)}\right)^n$, i.e., the Ky Fan type inequalities

$$\frac{G_n(x)}{G_n(1-x)} \leqslant \frac{G_n(x)}{1-G_n(x)} \leqslant \frac{A_n(x)}{A_n(1-x)}, \quad 0 < x_i \leqslant \frac{1}{2}, \quad i = 1,...,n.$$

EXAMPLE 3.2. Let $a,b > 0$, by Remark 2.1, the function $f(x) = a + \frac{b}{x} = x^{-1}(ax+b)$ is Wright type multiplicatively convex on R_+, and so the function $\varphi(x) = \prod_{i=1}^n \left(a + \frac{b}{x_i}\right)$ is multiplicatively Schur-convex on R_+^n. Noting that $\log(G_n(x),...,G_n(x)) \prec \log(x_1,...,x_n)$ and $G_n(x) \leqslant A_n(x)$, one can obtain the inequalities

$$\prod_{i=1}^{n}\left(a+\frac{b}{x_i}\right) \geqslant \left(a + \frac{b}{G_n(x)}\right)^n \geqslant \left(a+\frac{b}{A_n(x)}\right)^n, \ x_i > 0, i = 1,2,...,n.$$

REMARK 3.1. I. C. Draghicescu [1] proposed the inequality

$$\prod_{i=1}^{n}\left(a+\frac{b}{x_i}\right) \geqslant \left(a+\frac{b}{A_n(x)}\right)^n, \ x_i > 0, \quad i = 1,2,...,n.$$

Some proofs of this inequality can be found in "Crux Mathematicorum, 30 (1) (2004), p. 58–60". Here we give another new proof and refine it.

4. A Hermite-Hadamard type inequality

Suppose that $0 < a < b$ and let $f : [a,b] \to R_+$ be a multiplicatively convex function, C. P. Niculescu et al. [8, p. 82] obtained the following analogue of Hermite-Hadamard inequality,

$$f(\sqrt{ab}) \leqslant \exp\left(\frac{1}{\log b - \log a}\int_a^b \frac{\log f(t)}{t}dt\right) \leqslant \sqrt{f(a)f(b)}. \tag{4.1}$$

We will prove that (4.1) still holds for Wright type multiplicatively convex functions. To this end, we first present the following lemma.

LEMMA 4.1. *Suppose that f is a Wright type multiplicatively convex function defined on $[a,b]$ $(0<a<b)$, and let $s,t,u,v\in[a,b]$ with $s\leqslant t\leqslant u\leqslant v$ and $tu=sv$, then*

$$f(t)f(u)\leqslant f(s)f(v). \tag{4.2}$$

Proof. Set $x=(t,u)$ and $y=(s,v)$, one can see that $\log x\prec \log y$. By Theorem 3.2, we have $f(t)f(u)\leqslant f(s)f(v)$, and so the proof is completed. □

THEOREM 4.1. *Suppose that f is a Wright type multiplicatively convex and integrable function defined on $[a,b]$, then (4.1) holds.*

Proof. From (4.2) it follows that

$$\begin{aligned}
f(\sqrt{ab}) &= \exp\left(\frac{1}{\log b-\log a}\int_a^{\sqrt{ab}}\frac{\log(f(\sqrt{ab})\cdot f(\sqrt{ab}))}{x}dx\right)\\
&\leqslant \exp\left(\frac{1}{\log b-\log a}\int_a^{\sqrt{ab}}\frac{\log f(x)+\log f\left(\frac{ab}{x}\right)}{x}dx\right)\left(x\leqslant\sqrt{ab}\leqslant\sqrt{ab}\leqslant\frac{ab}{x}\right)\\
&= \exp\left(\frac{1}{\log b-\log a}\int_a^{\sqrt{ab}}\left(\frac{\log f(x)}{x}+\frac{\log f\left(\frac{ab}{x}\right)}{x}\right)dx\right)\\
&= \exp\left(\frac{1}{\log b-\log a}\left(\int_a^{\sqrt{ab}}\frac{\log f(x)}{x}dx+\int_{\sqrt{ab}}^{b}\frac{\log f(x)}{x}dx\right)\right)\\
&= \exp\left(\frac{1}{\log b-\log a}\int_a^{b}\frac{\log f(x)}{x}dx\right),
\end{aligned}$$

and

$$\begin{aligned}
\sqrt{f(a)f(b)} &= \exp\left(\frac{1}{\log b-\log a}\int_a^{\sqrt{ab}}\frac{\log(f(a)\cdot f(b))}{x}dx\right)\\
&\geqslant \exp\left(\frac{1}{\log b-\log a}\int_a^{\sqrt{ab}}\frac{\log f(x)+\log f\left(\frac{ab}{x}\right)}{x}dx\right)\left(a\leqslant x\leqslant\frac{ab}{x}\leqslant b\right)\\
&= \exp\left(\frac{1}{\log b-\log a}\int_a^{\sqrt{ab}}\left(\frac{\log f(x)}{x}+\frac{\log f\left(\frac{ab}{x}\right)}{x}\right)dx\right)\\
&= \exp\left(\frac{1}{\log b-\log a}\left(\int_a^{\sqrt{ab}}\frac{\log f(x)}{x}dx+\int_{\sqrt{ab}}^{b}\frac{\log f(x)}{x}dx\right)\right)\\
&= \exp\left(\frac{1}{\log b-\log a}\int_a^{b}\frac{\log f(x)}{x}dx\right),
\end{aligned}$$

which finishes the proof. □

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