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OPERATOR INEQUALITIES ASSOCIATED WITH TSALLIS RELATIVE OPERATOR ENTROPY

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Abstract. In this paper, we present some operator inequalities related to Tsallis relative operator entropy. Our results are refinements and generalizations of some existing inequalities.

1. Introduction

For two invertible positive operators A and B and $\lambda \in (0,\ 1]$, the Tsallis relative operator entropy $T_\lambda\,(A|B)$ and the relative operator entropy $S\,(A|B)$ are defined by

$$T_\lambda\,(A|B) = \frac{A\#_\lambda B - A}{\lambda},$$

$$S\,(A|B) = A^{1/2}\log\left(A^{-1/2}BA^{-1/2}\right)A^{1/2},$$

where $A\#_\lambda B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\lambda A^{1/2}$ is the wieghted geometric mean. When $\lambda = \frac{1}{2}$, this is the geometric mean, denoted by $A\#B$. Since

$$\lim_{\lambda\to 0} T_\lambda\,(A|B) = S\,(A|B),$$

the Tsallis relative operator entropy $T_\lambda\,(A|B)$ introduced by Yanagi, Kuriyama and Furuichi [10] is a generalization of the relative operator entropy $S\,(A|B)$ defined by Fujii and Kamei [1]. For more information on the Tsallis relative entropy the reader is referred to [3–4] and the references therein.

Furuichi, Yanagi and Kuriyama [5] obtained the following inequalities:

$$T_{-\lambda}\,(A|B) \leqslant S\,(A|B) \leqslant T_\lambda\,(A|B), \tag{1.1}$$

$$A - AB^{-1}A \leqslant T_\lambda\,(A|B) \leqslant B - A. \tag{1.2}$$

Meanwhile, they also proved that if $a > 0$, then

$$A\#_\lambda B - \frac{1}{a}A\#_{\lambda-1}B + \frac{1-a^\lambda}{\lambda a^\lambda}A \leqslant T_\lambda\,(A|B) \leqslant \frac{1}{a}B - \frac{1-a^\lambda}{\lambda a^\lambda}A\#_\lambda B - A. \tag{1.3}$$

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Putting $\lambda \to 0$ in (1.3), we have

$$(1-\log a)A - \frac{1}{a}AB^{-1}A \leqslant S(A|B) \leqslant (\log a - 1)A + \frac{1}{a}B, \tag{1.4}$$

which was due to Furuta [6] (see also [7]). The inequality (1.4) is a generalization of the following result:

$$A - AB^{-1}A \leqslant S(A|B) \leqslant B - A, \tag{1.5}$$

which was shown in [2]. For more inequalities on the Tsallis relative operator entropy the reader is referred to [8–9].

After seeing the inequalities (1.1), (1.2) and (1.5), it is hard not to be curious about the relationship between $T_{-\lambda}(A|B)$ and $A - AB^{-1}A$. This is a part of the motivation for the present paper.

In this paper, we first discuss the relationship between $T_{-\lambda}(A|B)$ and $A - AB^{-1}A$. After that, we obtain a generalization of (1.1) and (1.3), and present a refinement of (1.4).

2. Main results

We begin this section with the following result.

THEOREM 2.1. *Let $a > 0$ and $\lambda \in (0, 1]$. For any invertible positive operators A and B, we have*

$$A - \frac{1}{a}AB^{-1}A + \frac{1-a^{\lambda}}{\lambda a^{\lambda}}A \leqslant a^{-\lambda}T_{-\lambda}(A|B). \tag{2.1}$$

Proof. For $a > 0$ and $\lambda \in (0, 1]$, we have

$$1 - \frac{1}{ax} \leqslant a^{-\lambda}\frac{x^{-\lambda}-1}{-\lambda} - \frac{1-a^{\lambda}}{\lambda a^{\lambda}},\ x > 0,$$

and so

$$I - \frac{1}{a}A^{1/2}B^{-1}A^{1/2} \leqslant a^{-\lambda}\frac{\left(A^{-1/2}BA^{-1/2}\right)^{-\lambda} - I}{-\lambda} - \frac{1-a^{\lambda}}{\lambda a^{\lambda}}I.$$

Multiplying both sides by $A^{1/2}$, we have

$$A - \frac{1}{a}AB^{-1}A \leqslant a^{-\lambda}\frac{A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{-\lambda}A^{1/2} - A}{-\lambda} - \frac{1-a^{\lambda}}{\lambda a^{\lambda}}A.$$

This completes the proof. □

REMARK 2.1. Putting $a = 1$ in (2.1), we have

$$A - AB^{-1}A \leqslant T_{-\lambda}(A|B).$$

It follows from (1.1), (1.2) and this last inequality that

$$A - AB^{-1}A \leqslant T_{-\lambda}(A|B) \leqslant S(A|B) \leqslant T_{\lambda}(A|B) \leqslant B - A, \tag{2.2}$$

which is a refinement of (1.5).

THEOREM 2.2. *Let $a > 0$ and $\lambda \in (0, 1]$. For any invertible positive operatorss A and B, we have*

$$-\left(\log a + \frac{1-a^{\lambda}}{\lambda a^{\lambda}}\right)A + a^{-\lambda}T_{-\lambda}(A|B) \leqslant S(A|B) \leqslant T_{\lambda}(A|B) - \frac{1-a^{\lambda}}{\lambda}A\#_{\lambda}B - (\log a)A. \tag{2.3}$$

Proof. It is known [5] that for positive real number x,

$$\frac{x^{-\lambda}-1}{-\lambda} \leqslant \log x \leqslant \frac{x^{\lambda}-1}{\lambda}.$$

So, we have

$$a^{-\lambda}\frac{x^{-\lambda}-1}{-\lambda} - \frac{1-a^{\lambda}}{\lambda a^{\lambda}} \leqslant \log ax \leqslant \frac{x^{\lambda}-1}{\lambda} + \frac{a^{\lambda}-1}{\lambda}x^{\lambda}.$$

That is

$$-\log a + a^{-\lambda}\frac{x^{-\lambda}-1}{-\lambda} - \frac{1-a^{\lambda}}{\lambda a^{\lambda}} \leqslant \log x \leqslant \frac{x^{\lambda}-1}{\lambda} - \frac{1-a^{\lambda}}{\lambda}x^{\lambda} - \log a.$$

It follows that

$$-\left(\log a + \frac{1-a^{\lambda}}{\lambda a^{\lambda}}\right)A + a^{-\lambda}T_{-\lambda}(A|B) \leqslant S(A|B) \leqslant T_{\lambda}(A|B) - \frac{1-a^{\lambda}}{\lambda}A\#_{\lambda}B - (\log a)A.$$

This completes the proof. $\square$

REMARK 2.2. Putting $a = 1$ in (2.3), we obtain (1.1).

THEOREM 2.3. *Let $a > 0$, $\lambda \in (0, 1]$. For any invertible positive operators A and B, we have*

$$\begin{aligned}(1-\log a)A - \frac{1}{a}AB^{-1}A &\leqslant -\left(\frac{1-a^{\lambda}}{\lambda a^{\lambda}} + \log a\right)A + a^{-\lambda}T_{-\lambda}(A|B)\\ &\leqslant S(A|B)\\ &\leqslant (\log a)A + T_{\lambda}(A|B) + \frac{1-a^{\lambda}}{\lambda a^{\lambda}}A\#_{\lambda}B\\ &\leqslant (\log a - 1)A + \frac{1}{a}B.\end{aligned} \tag{2.4}$$

Proof. The first part of (2.4) follows from (2.1). The second part of (2.4) is the first part of (2.3). We can replace a by a^{-1} in (2.3) to give the third part of (2.4). The last part of (2.4) follows from the second part of (1.3). □

REMARK 2.3. The inequality (2.4) is a refinement of (1.4). Putting $a = 1$ in (2.4), we get (2.2).

THEOREM 2.4. *Let $a > 0$, $\lambda \in (0, 1]$ and $v \in [0, 1]$. For any invertible positive operators A and B, we have*

$$l_3 A\#_\lambda B - l_1 A\#_{\lambda-1} B + l_2 A \leqslant T_\lambda(A|B) \leqslant l_1 B - l_2 A\#_\lambda B - l_3 A, \tag{2.5}$$

where

$$l_1 = \frac{a^{\lambda-1}}{v\left(a^\lambda - 1\right) + 1}, \quad l_2 = \frac{v\left(1 - a^\lambda\right)}{v\lambda\left(a^\lambda - 1\right) + \lambda}, \quad l_3 = \frac{(\lambda - 1 + v)a^\lambda + 1 - v}{v\lambda\left(a^\lambda - 1\right) + \lambda}.$$

Proof. Note that

$$\frac{1}{\lambda}\left(\left(\frac{x}{a}\right)^\lambda - 1\right) = \frac{x^\lambda - 1}{\lambda} + x^\lambda \frac{a^{-\lambda} - 1}{\lambda}, \tag{2.6}$$

$$\frac{1}{\lambda}\left(\left(\frac{x}{a}\right)^\lambda - 1\right) = a^{-\lambda}\frac{x^\lambda - 1}{\lambda} + \frac{a^{-\lambda} - 1}{\lambda}. \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$\begin{aligned}\frac{1}{\lambda}\left(\left(\frac{x}{a}\right)^\lambda - 1\right) &= v\left(\frac{x^\lambda - 1}{\lambda} + x^\lambda \frac{a^{-\lambda} - 1}{\lambda}\right) + (1 - v)\left(a^{-\lambda}\frac{x^\lambda - 1}{\lambda} + \frac{a^{-\lambda} - 1}{\lambda}\right) \\ &= \frac{v\left(a^\lambda - 1\right) + 1}{a^\lambda} \cdot \frac{x^\lambda - 1}{\lambda} + \left(v\left(x^\lambda - 1\right) + 1\right)\frac{a^{-\lambda} - 1}{\lambda}.\end{aligned} \tag{2.8}$$

Since

$$\frac{x^\lambda - 1}{\lambda} \leqslant x - 1$$

for any $x > 0$ and $\lambda \in (0, 1]$, we have

$$\frac{1}{\lambda}\left(\left(\frac{x}{a}\right)^\lambda - 1\right) \leqslant \frac{x}{a} - 1,\ a > 0. \tag{2.9}$$

Combining (2.8) and (2.9), we have

$$\frac{x^\lambda - 1}{\lambda} \leqslant \frac{a^{\lambda-1}}{v\left(a^\lambda - 1\right) + 1}x - \frac{v\left(1 - a^\lambda\right)}{v\lambda\left(a^\lambda - 1\right) + \lambda}x^\lambda - \frac{(\lambda - 1 + v)a^\lambda + 1 - v}{v\lambda\left(a^\lambda - 1\right) + \lambda}. \tag{2.10}$$

Substituting x by x^{-1} in (2.10), we have

$$\frac{1}{\lambda}\left(\left(\frac{1}{x}\right)^{\lambda}-1\right) \leqslant \frac{a^{\lambda-1}}{v\left(a^{\lambda}-1\right)+1}\frac{1}{x} - \frac{v\left(1-a^{\lambda}\right)}{v\lambda\left(a^{\lambda}-1\right)+\lambda}\left(\frac{1}{x}\right)^{\lambda} - \frac{(\lambda-1+v)\,a^{\lambda}+1-v}{v\lambda\left(a^{\lambda}-1\right)+\lambda}.$$

By a small calculation we know that

$$\frac{1}{\lambda}\left(\left(\frac{1}{x}\right)^{\lambda}-1\right) = -x^{-\lambda}\cdot\frac{x^{\lambda}-1}{\lambda}$$

and so

$$\frac{(\lambda-1+v)\,a^{\lambda}+1-v}{v\lambda\left(a^{\lambda}-1\right)+\lambda}x^{\lambda} - \frac{a^{\lambda-1}}{v\left(a^{\lambda}-1\right)+1}x^{\lambda-1} + \frac{v\left(1-a^{\lambda}\right)}{v\lambda\left(a^{\lambda}-1\right)+\lambda} \leqslant \frac{x^{\lambda}-1}{\lambda}. \quad (2.11)$$

It follows from (2.10) and (2.11) that

$$\begin{aligned} l_3\left(A^{-1/2}BA^{-1/2}\right)^{\lambda} - l_1\left(A^{-1/2}BA^{-1/2}\right)^{\lambda-1} + l_2 I &\leqslant \frac{\left(A^{-1/2}BA^{-1/2}\right)^{\lambda}-I}{\lambda} \\ &\leqslant l_1 A^{-1/2}BA^{-1/2} - l_2\left(A^{-1/2}BA^{-1/2}\right)^{\lambda} - l_3 I. \end{aligned}$$

Multiplying $A^{1/2}$ from both sides, we have

$$l_3 A\#_{\lambda}B - l_1 A\#_{\lambda-1}B + l_2 A \leqslant T_{\lambda}\left(A|B\right) \leqslant l_1 B - l_2 A\#_{\lambda}B - l_3 A.$$

This completes the proof. □

REMARK 2.4. Putting $v=1$ in (2.5), we get (1.3). Putting $v=1$ and $\lambda\to 0$ in (2.5), we obtain (1.4).

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REFERENCES

[1] J. I. FUJII, E. KAMEI, *Relative operator entropy in noncommutative information theory*, Math. Japon., Vol. 34, **3** (1989), pp. 341–348.
[2] J. I. FUJII, E. KAMEI, *Uhlmann's interpolational method for operator means*, Math. Japon., Vol. 34, **4** (1989), pp. 541–547.
[3] S. FURUICHI, *Inequalities for Tsallis relative entropy and generalized skew information*, Linear Multilinear Algebra, Vol. 59, **10** (2011), pp. 1143–1158.

[4] S. FURUICHI, K. YANAGI, K. KURIYAMA, *Fundamental properties of Tsallis relative entropy*, J. Math. Phys., Vol. 45, **12** (2004), pp. 4868–4877.
[5] S. FURUICHI, K. YANAGI, K. KURIYAMA, *A note on operator inequalities of Tsallis relative operator entropy*, Linear Algebra Appl., Vol. 407, **1** (2005), pp. 19–31.
[6] T. FURUTA, *Furuta's inequality and its application to the relative operator entropy*, J. Operator Theory, Vol. 30, **1** (1993), pp. 21–30.
[7] T. FURUTA, *Invitation to Linear Operators*, Taylor & Francis, London and New York, (2001).
[8] T. FURUTA, *Reverse inequalities involving two relative operator entropies and two relative entropies*, Linear Algebra Appl., Vol. 403, **1** (2005), pp. 24–30.
[9] T. FURUTA, *Two reverse inequalities associated with Tsallis relative operator entropy via generalized Kantorovich constant and their applications*, Linear Algebra Appl., Vol. 412, **2–3** (2006), pp. 526–537.
[10] K. YANAGI, K. KURIYAMA, S. FURUICHI, *Generalized Shannon inequalities based on Tsallis relative operator entropy*, Linear Algebra Appl., Vol. 394, **1** (2005), pp. 109–118.

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