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A GENERALIZATION OF HILBERT'S INEQUALITY

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Abstract. In a generalization of the classical Hilbert inequality by Hardy, Littlewood and Pólya, the best constant for an inequality is determined provided that the generating function for the corresponding matrix satisfies certain monotonicity condition. In this paper, we determine the best constant for a class of inequalities when the monotonicity condition is no longer satisfied.

1. Introduction

Suppose throughout that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Let l^p be the Banach space of all complex sequences $\mathbf{x} = (x_n)_{n \geqslant 1}$ with norm

$$\|\mathbf{x}\|_p := (\sum_{n=1}^{\infty} |x_n|^p)^{1/p} < \infty.$$

Let $C = (c_{n,k})$ be a matrix acting on the l^p space, the l^p operator norm of C is defined as

$$\|C\|_{p,p} = \sup_{\|\mathbf{x}\|_p = 1} \Big|\Big| C \cdot \mathbf{x} \Big|\Big|_p .$$

The well-known Hilbert's inequality [7, Theorem 315] asserts that for $\mathbf{x} \in l^p$, $\mathbf{y} \in l^q$:

$$\Big| \sum_{i,j=1}^{\infty} \frac{x_i y_j}{i+j} \Big| \leqslant \frac{\pi}{\sin \pi/p} \|\mathbf{x}\|_p \|\mathbf{y}\|_q. \tag{1.1}$$

Let $H = (1/(i+j))_{i \geqslant 1, j \geqslant 1}$, then [7, Theorem 286] implies that inequality (1.1) is equivalent to $\|H\|_{p,p} \leqslant \pi/\sin(\pi/p)$. In fact, it is shown in [7, Theorem 317] that the constant $\pi/\sin(\pi/p)$ is best possible, hence $\|H\|_{p,p} = \pi/\sin(\pi/p)$.

There exists an extensive and rich literature on extensions and generalizations of Hilbert's inequality. For the recent developments in this area, we refer the reader to the articles [8], [4], [10] and the references therein. In [8], a unified treatment of Hilbert-type inequalities on a σ-finite measure space with a general kernel and with general weight functions is given. In [4], an extension to the case of general parameters p and q is provided. The monograph [10] provides a collection of recent results regarding Hilbert-type inequalities.

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A generalization of Hilbert's inequality is given in [7, Theorem 318]. For a matrix $K=(K(i,j))_{i\geqslant 1,j\geqslant 1}$ with $K(x,y)$ satisfying the following conditions:

$$\begin{aligned} &1. \quad K(x,y) \text{ is a non-negative, homogeneous function of degree } -1; \\ &2. \quad \int_0^\infty K(x,1)x^{-1/q}dx = \int_0^\infty K(1,y)y^{-1/p}dy = k. \end{aligned} \tag{1.2}$$

Then, with one more condition (in what follows, we shall refer to this assumption as the decreasing assumption) that $K(x,1)x^{-1/q}$, $K(1,y)y^{-1/p}$ are strictly decreasing functions of $x>0$, $y>0$ respectively, [7, Theorem 318] asserts that $\|K\|_{p,p}\leqslant k$. In fact, in this case $\|K\|_{p,p}=k$ (see the remark on [7, p. 229]).

The proof for [7, Theorem 318] given in [7] uses Schur's test to reduce the estimation of $\|K\|_{p,p}$ to the estimation of certain series and the decreasing assumption is to ensure that the series are bounded above by the corresponding integrals. In view of this, one sees that the decreasing assumption need not be necessary when determining $\|K\|_{p,p}$. It is therefore natural to study $\|K\|_{p,p}$ for a matrix $K=(K(i,j))_{i\geqslant 1,j\geqslant 1}$ with $K(x,y)$ satisfying the conditions in (1.2) only.

We now focus on a type of matrices of the form:

$$H(\alpha,\beta)=\Big(\frac{i^\alpha j^\beta}{(i+j)^{\alpha+\beta+1}}\Big)_{i\geqslant 1,j\geqslant 1}.$$

We note here that when either $\alpha>1/q$ or $\beta>1/p$, then $H(\alpha,\beta)$ satisfies the conditions in (1.2) but not the decreasing assumption.

Some results on $\|H(\alpha,\beta)\|_{p,p}$ can be deduced from a result of Bennett. Recall (see [1]) that the beta function $B(x,y)$, $x>0$, $y>0$ is defined as

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt. \tag{1.3}$$

Bennett's result (see [2, Proposition 2] and the discussions that follow) can be stated as follows:

THEOREM 1. *Let $p>1$ be fixed. Let μ be a positive measure on $(0,1)$. Let $K=(k_{i,j})_{i\geqslant 1,j\geqslant 1}$ be given by*

$$k_{i,j}=\int_0^1 \binom{i+j-2}{j-1} t^{i-1}(1-t)^j d\mu(t).$$

Then $\|K\|_{p,p}\leqslant \int_0^1 t^{-1/q}(1-t)^{1/q}d\mu(t)$.

By taking $d\mu(t)=t^{1-\alpha}(1-t)^{-\beta}dt$ in the above theorem, we readily deduce the following

THEOREM 2. *Let $p>1$ and $\alpha<1+1/p$, $\beta<1+1/q$ be fixed. Let $M(\alpha,\beta)$ be a matrix given by*

$$M(\alpha,\beta)_{i,j}=\binom{i+j-2}{j-1}B(i+1-\alpha,j+1-\beta), \quad i,j\geqslant 1.$$

Then $\|M(\alpha,\beta)\|_{p,p}\leqslant B(1+1/p-\alpha,1+1/q-\beta)$.

Note that the case $\alpha=\beta=1$ in the above theorem corresponds to Hilbert's inequality (a stronger form with the corresponding matrix being $(1/(i+j-1))_{i\geqslant 1, j\geqslant 1}$). The case $\alpha=1, \beta=0$ in the above theorem corresponds to the matrix studied explicitly by Bennett in [2, Proposition 2]. Note that $H(0,1)_{i,j} \leqslant M(1,0)_{i,j}$ when $i,j \geqslant 1$. Similarly, it is easy to check that $H(1,1)_{i,j} \leqslant M(0,0)_{i,j}$ when $i,j \geqslant 1$. It follows from Theorem 2 that $\|H(0,1)\|_{p,p} \leqslant \|M(1,0)\|_{p,p} \leqslant B(1/p, 1+1/q), \|H(1,1)\|_{p,p} \leqslant \|M(0,0)\|_{p,p} \leqslant B(1+1/p, 1+1/q)$. On the other hand, Lemma 1 in Section 2 implies that $\|H(0,1)\|_{p,p} \geqslant B(1/p, 1+1/q), \|H(1,1)\|_{p,p} \geqslant B(1+1/p, 1+1/q)$. We therefore deduce the following

THEOREM 3. *Let $p>1$ be fixed and let $H(0,1)=(j/(i+j)^2)_{i\geqslant 1, j\geqslant 1}$, $H(1,1)=(ij/(i+j)^3)_{i\geqslant 1, j\geqslant 1}$. Then $\|H(0,1)\|_{p,p}=\pi/(q\cdot\sin(\pi/p)), \|H(1,1)\|_{p,p}=\pi/(2pq\cdot\sin(\pi/p))$.*

To establish results analogue to that given in Theorem 3, one hopes to have that for $\alpha<1+1/p, \beta<1+1/q$, $i,j\geqslant 1$,

$$H(1-\alpha, 1-\beta)_{i,j} \leqslant M(\alpha,\beta)_{i,j},$$

where $M(\alpha,\beta)$ is defined as in Theorem 2. However, the above inequality does not always hold. For example, one checks directly that when $\alpha=\beta=-1$, $i=j$, the above inequality fails to hold when $i\to\infty$. Similarly, with the help of Stirling's approximation, one can show that the above inequality fails to hold when $\alpha=\beta=-1/2$, $i=j$, $i\to\infty$.

In this paper, we use the approach in the proof of [7, Theorem 318] to study $\|H(\alpha,\beta)\|_{p,p}$. It follows from [7, Theorem 286] and the approach on [7, p. 229] that for any matrix $K=(K(i,j))_{i\geqslant 1, j\geqslant 1}$ with $K(x,y)$ a non-negative, homogeneous function of degree -1, we have $\|K\|_{p,p}\leqslant k$ provided that

$$\sum_{i=1}^{\infty} K(i,j)\left(\frac{i}{j}\right)^{-1/q} \leqslant k, \quad \sum_{j=1}^{\infty} K(i,j)\left(\frac{j}{i}\right)^{-1/p} \leqslant k.$$

Apply the above argument to $K(x,y)=x^{\alpha}y^{\beta}/(x+y)^{\alpha+\beta+1}$, we see that (with the help of Lemma 1) $\|H(\alpha,\beta)\|_{p,p}=B(\alpha+1/p,\ \beta+1/q)$ for any $\alpha>-1/p$, $\beta>-1/q$ provided that we can show for any $\lambda>-1, s>\lambda+1$:

$$\sum_{m=1}^{\infty} \frac{m^{\lambda}}{(m+n)^{s}} \leqslant B(\lambda+1, s-\lambda-1)n^{1+\lambda-s}. \tag{1.4}$$

We note that the above inequality is valid when $\lambda<0$ by the integral test. In [9, Lemma 2], it is shown that the above inequality is valid when $0<s\leqslant 2$, $-1<\lambda<s-1$ and $2<s\leqslant 14$, $-1<\lambda\leqslant 1$. In the next section, we extend this result to the case of $1<\lambda\leqslant 2$, $\lambda+1<s\leqslant 5$ to prove the following

THEOREM 4. *Let $p>1$ be fixed and let $H(\alpha,\beta)=(i^{\alpha}j^{\beta}/(i+j)^{\alpha+\beta+1})_{i\geqslant 1, j\geqslant 1}$, then $\|H(\alpha,\beta)\|_{p,p}=B(\alpha+1/p, \beta+1/q)$ when $-1/p<\alpha\leqslant 2$, $-1/q<\beta\leqslant 2$.*

We note here that [7, Theorem 286] again implies that Theorem 4 is equivalent to the statement that with the best possible constant $C(\alpha,\beta,p) = B(\alpha+1/p,\beta+1/q)$, for $\mathbf{x} \in l^q, \mathbf{y} \in l^p$:

$$\Big|\sum_{i,j=1}^{\infty} \frac{i^{\alpha} j^{\beta}}{(i+j)^{\alpha+\beta+1}} x_i y_j\Big| \leqslant C(\alpha,\beta,p)\|\mathbf{x}\|_q\|\mathbf{y}\|_p. \tag{1.5}$$

We note that [11, Theorem 1] establishes discrete Hilbert-type inequalities with a general measurable homogeneous kernel of a negative degree. Using the notations as in [11, Theorem 1], on taking $\lambda = \alpha+\beta+1$, $K(x,y) = (x+y)^{-\lambda}$, $u(x) = v(x) = x$, $A_1 = (1-p\beta)/(pq)$, $A_2 = (1-q\alpha)/(pq)$, $a_i = i^{\beta} y_i$, $b_i = i^{\alpha} x_i$ there, we see that [11, Theorem 1] also implies inequality (1.5) with the best possible constant $C(\alpha,\beta,p) = B(\alpha+1/p,\beta+1/q)$ (taking into account [11, Theorem 2]) for $-1/p < \alpha < 1/q$, $-1/q < \beta < 1/p$. This case is also what one can derive directly from [7, Theorem 318].

The proof for Theorem 4 relies on the estimation of the series given in (1.4). When $-1 < \lambda \leqslant 0$, $s > \lambda+1$, this series is studied by Bennett and Jameson in [3, Proposition 14]. In the same paper, they raised an open question (see the remark below [3, Proposition 13]) on whether the following sequence is increasing with n when $\alpha > 1$:

$$\frac{1}{n^{\alpha+2}} \sum_{r=1}^{n-1} r^{\alpha}(n-r). \tag{1.6}$$

In this paper, we give an affirmative answer to the above open question of Bennett and Jameson. We prove in Section 3 the following:

THEOREM 5. *Let $\alpha > 1$ be fixed, the sequence defined in (1.6) is increasing with n.*

2. Proof of Theorem 4

Note that when $\alpha \leqslant 2$, $\beta \leqslant 2$, then $\alpha+\beta+1 \leqslant 5$. Thus, by our discussion in Section 1, we see that Theorem 4 follows from a combination of [9, Lemma 2] and the following two lemmas:

LEMMA 1. *Let $p > 1$, $\alpha > -1/p$, $\beta > -1/q$ be fixed, let $H(\alpha,\beta) = (i^{\alpha} j^{\beta}/(i+j)^{\alpha+\beta+1})_{i\geqslant 1, j\geqslant 1}$, then $\|H(\alpha,\beta)\|_{p,p} \geqslant B(\alpha+1/p,\beta+1/q)$.*

Proof. By [7, Theorem 286], it suffices to show that if there exists a constant $C(\alpha,\beta,p)$ such that inequality (1.5) holds for any $\mathbf{x} \in l^q$, $\mathbf{y} \in l^p$, then $C(\alpha,\beta,p) \geqslant B(\alpha+1/p,\beta+1/q)$. We follow the method given on [7, p. 233] by setting $x_i = i^{-1/q}$, $y_j = j^{-1/p}$ when $1 \leqslant i,j \leqslant N$ and $x_i = y_j = 0$ otherwise, to see that

$$\sum_{i,j=1}^{\infty} \frac{i^{\alpha} j^{\beta}}{(i+j)^{\alpha+\beta+1}} x_i y_j = \sum_{i=1}^{N} i^{\alpha-1/q} \sum_{j=1}^{N} \frac{j^{\beta-1/p}}{(i+j)^{\alpha+\beta+1}}, \quad \|\mathbf{x}\|_q\|\mathbf{y}\|_p = \sum_{i=1}^{N} \frac{1}{i}.$$

For any given $\varepsilon > 0$, we choose N large enough such that (note that our assumption on α and β ensures that the infinite series in the following expression converges)

$$\sum_{j=1}^{N} \frac{j^{\beta-1/p}}{(i+j)^{\alpha+\beta+1}} > (1-\varepsilon)\sum_{j=1}^{\infty} \frac{j^{\beta-1/p}}{(i+j)^{\alpha+\beta+1}}.$$

Note that the function $x^{\beta-1/p}/(x+i)^{\alpha+\beta+1}$ is increasing when $x < (\beta-1/p)i/(\alpha+1+1/p)$ and decreasing when $x > (\beta-1/p)i/(\alpha+1+1/p)$. It follows that

$$\frac{j^{\beta-1/p}}{(i+j)^{\alpha+\beta+1}} \geqslant \int_{j-1}^{j} \frac{x^{\beta-1/p}}{(i+x)^{\alpha+\beta+1}}dx, \quad j \leqslant (\beta-1/p)i/(\alpha+1+1/p),$$
$$\frac{j^{\beta-1/p}}{(i+j)^{\alpha+\beta+1}} \geqslant \int_{j}^{j+1} \frac{x^{\beta-1/p}}{(i+x)^{\alpha+\beta+1}}dx, \quad j > (\beta-1/p)i/(\alpha+1+1/p).$$

We then deduce easily that

$$\begin{aligned}\sum_{j=1}^{\infty} \frac{j^{\beta-1/p}}{(i+j)^{\alpha+\beta+1}} &\geqslant \int_{0}^{\infty} \frac{x^{\beta-1/p}}{(i+x)^{\alpha+\beta+1}}dx - \frac{C}{i^{\alpha+1+1/p}} \\ &= B\Big(\alpha+\frac{1}{p},\beta+\frac{1}{q}\Big)i^{-(\alpha+1/p)} - \frac{C}{i^{\alpha+1+1/p}},\end{aligned}$$

where C is a constant depending on α,β,p and we used (1.3) to evaluate the integration above.

As $\sum_{i=1}^{\infty} i^{-2} < \infty$, we deduce that

$$\begin{aligned}\sum_{i=1}^{N} i^{\alpha-1/q}\sum_{j=1}^{N} \frac{j^{\beta-1/p}}{(i+j)^{\alpha+\beta+1}} &\geqslant B\Big(\alpha+\frac{1}{p},\beta+\frac{1}{q}\Big)(1-\varepsilon)\sum_{i=1}^{N}\frac{1}{i} + C' \\ &= B\Big(\alpha+\frac{1}{p},\beta+\frac{1}{q}\Big)(1-\varepsilon)\|\mathbf{x}\|_q\|\mathbf{y}\|_p + (1-\varepsilon)C',\end{aligned}$$

where C' is a constant depending on α,β,p. By letting $N \to \infty$, we see that the constant $C(\alpha,\beta,p)$ in (1.5) satisfies $C(\alpha,\beta,p) \geqslant B(\alpha+1/p,\beta+1/q)$ and this completes the proof. □

LEMMA 2. *Let* $1 < \lambda \leqslant 2$, *then inequality* (1.4) *holds for* $\lambda+1 < s \leqslant 5$.

Proof. We apply the following Euler-Maclaurin summation formula (see [9, p. 152]), which asserts that for $f(x) \in C^2[1,\infty)$ such that $\sum_{k=1}^{\infty} f(k) < \infty$, $\int_1^{\infty} f(t)dt < \infty$ and $\lim_{x\to\infty} f^{(r)}(x) = 0$, $0 \leqslant r \leqslant 1$, then the following equality holds:

$$\sum_{k=1}^{\infty} f(k) = \int_1^{\infty} f(t)dt + \frac{f(1)}{2} - \frac{1}{12}f'(1) - \frac{1}{2}\int_1^{\infty} B_2(\{t\})f''(t)dt, \tag{2.1}$$

where we denote by $\{x\}$ the fractional part of x, the unique real number in $[0,1)$ such that $x - \{x\} \in \mathbb{Z}$ and $B_2(x) = x^2 - x + 1/2$ the second Bernoulli polynomial.

Further, it follows from [6, Proposition 9.2.3] that if $f \in C^6[1,\infty)$, $\lim_{x\to\infty} f^{(r)}(x) = 0$, $0 \leqslant r \leqslant 5$, and $f^{(r)}(x) < 0$, $r = 4,6$, then the following inequality holds:

$$\frac{1}{720}f'(1) - \frac{1}{720\cdot 42}f^{(3)}(1) \leqslant -\frac{1}{2}\int_1^\infty B_2(\{t\})f(t)dt \leqslant \frac{1}{720}f'(1).$$

We use the notion on [9, p. 152] to define for real numbers λ, s and $t > 0$,

$$f_{s,\lambda,n}(t) = \frac{t^\lambda}{(t+n)^s}.$$

We note that

$$f'_{s,\lambda,n}(t) = \frac{nst^{\lambda-1}}{(t+n)^{s+1}} - \frac{(s-\lambda)t^{\lambda-1}}{(t+n)^s},$$
$$f''_{s,\lambda,n}(t) = g_{s,\lambda,n}(t) + h_{s,\lambda,n}(t),$$

where

$$g_{s,\lambda,n}(t) = \frac{(s+1-\lambda)(s-\lambda)t^{\lambda-2}}{(t+n)^s} + \frac{n^2s(s+1)t^{\lambda-2}}{(t+n)^{s+2}},$$
$$h_{s,\lambda,n}(t) = -\frac{2ns(s+1-\lambda)t^{\lambda-2}}{(t+n)^{s+1}}.$$

It follows from our discussion above that when $1 < \lambda \leqslant 2, s > \lambda$,

$$-\frac{1}{2}\int_1^\infty B_2(\{t\})g_{s,\lambda,n}(t)dt \leqslant \frac{1}{720}g'_{s,\lambda,n}(1) - \frac{1}{720\cdot 42}g^{(3)}_{s,\lambda,n}(1),$$
$$-\frac{1}{2}\int_1^\infty B_2(\{t\})h_{s,\lambda,n}(t)dt \leqslant \frac{1}{720}h'_{s,\lambda,n}(1).$$

It now follows from (2.1) that

$$\sum_{k=1}^\infty f_{s,\lambda,n}(k) - \int_0^\infty f_{s,\lambda,n}(t)dt$$
$$\leqslant -\int_0^1 f_{s,\lambda,n}(t)dt + \frac{f_{s,\lambda,n}(1)}{2} - \frac{f'_{s,\lambda,n}(1)}{12} + \frac{1}{720}f^{(3)}_{s,\lambda,n}(1) - \frac{1}{720\cdot 42}g^{(3)}_{s,\lambda,n}(1).$$

Thus, inequality (1.4) is valid provided that we show

$$D_{s,\lambda}(n) := \int_0^1 f_{s,\lambda,n}(t)dt - \frac{1}{2}f_{s,\lambda,n}(1) + \frac{f'_{s,\lambda,n}(1)}{12} - \frac{1}{720}f^{(3)}_{s,\lambda,n}(1) + \frac{1}{720\cdot 42}g^{(3)}_{s,\lambda,n}(1) \qquad (2.2)$$
$$\geqslant 0.$$

Integration by parts shows that (see [9, p. 153])

$$\int_0^1 f_{s,\lambda,n}(t)dt \geqslant \sum_{i=0}^5 \frac{1}{(n+1)^{s+i}} \cdot \frac{\prod_{j=1}^i (s+j-1)}{\prod_{j=1}^{i+1}(j+\lambda)},$$

where we define the empty product to be 1.

We also have

$$-\frac{1}{2}f_{s,\lambda,n}(1) = -\frac{1}{2(n+1)^s},$$

$$\frac{f'_{s,\lambda,n}(1)}{12} = \frac{\lambda}{12(n+1)^s} - \frac{s}{12(n+1)^{s+1}},$$

$$-\frac{1}{720}f^{(3)}_{s,\lambda,n}(1) = -\frac{1}{720}\left(\frac{(\lambda-1)(\lambda-2)(\lambda-3)}{(n+1)^s} - \frac{3s\lambda(\lambda-1)}{(n+1)^{s+1}} + \frac{3s(s+1)\lambda}{(n+1)^{s+2}} - \frac{s(s+1)(s+2)}{(n+1)^{s+3}}\right),$$

$$\begin{aligned} g^{(3)}_{s,\lambda,n}(1) &= \frac{(s+1-\lambda)(s-\lambda)(\lambda-2)(\lambda-3)(\lambda-4)}{(n+1)^s} - \frac{3s(s+1-\lambda)(s-\lambda)(\lambda-2)(\lambda-3)}{(n+1)^{s+1}} \\ &+ \frac{3s(s+1)(s+1-\lambda)(s-\lambda)(\lambda-2)}{(n+1)^{s+2}} - \frac{3s(s+1)(s+2)(s+1-\lambda)(s-\lambda)}{(n+1)^{s+3}} \\ &+ \frac{n^2s(s+1)(\lambda-2)(\lambda-3)(\lambda-4)}{(n+1)^{s+2}} - \frac{3n^2s(s+1)(s+2)(\lambda-2)(\lambda-3)}{(n+1)^{s+3}} \\ &+ \frac{3n^2s(s+1)(s+2)(s+3)(\lambda-2)}{(n+1)^{s+4}} - \frac{n^2s(s+1)(s+2)(s+3)(s+4)}{(n+1)^{s+5}}. \end{aligned}$$

Apply this in (2.2), we see that

$$D_{s,\lambda}(n) \geqslant \sum_{i=0}^{5} \frac{1}{(n+1)^{s+i}} D_i(s,\lambda),$$

where

$$D_0(s,\lambda) = \frac{1}{1+\lambda} - \frac{1}{2} + \frac{\lambda}{12} - \frac{(\lambda-1)(\lambda-2)(\lambda-3)}{720} + \frac{(\lambda-2)(\lambda-3)(\lambda-4)}{720\cdot 42}((s+1-\lambda)(s-\lambda)+s(s+1)),$$

$$D_1(s,\lambda) = \frac{s}{(1+\lambda)(2+\lambda)} - \frac{s}{12} + \frac{3s\lambda(\lambda-1)}{720} - \frac{s(\lambda-2)(\lambda-3)}{720\cdot 42}(3(s+1-\lambda)(s-\lambda)+2(s+1)(\lambda-4)+3(s+1)(s+2)),$$

$$D_2(s,\lambda) = \frac{s(s+1)}{(1+\lambda)(2+\lambda)(3+\lambda)} - \frac{3s(s+1)\lambda}{720} + \frac{s(s+1)(\lambda-2)}{720\cdot 42}\cdot \Big(3(s+1-\lambda)(s-\lambda)+(\lambda-3)(\lambda-4)+6(s+2)(\lambda-3)+3(s+2)(s+3)\Big),$$

$$D_3(s,\lambda) = \frac{s(s+1)(s+2)}{(1+\lambda)(2+\lambda)(3+\lambda)(4+\lambda)} + \frac{s(s+1)(s+2)}{720} - \frac{s(s+1)(s+2)}{720\cdot 42}\cdot \Big(3(s+1-\lambda)(s-\lambda)+3(\lambda-2)(\lambda-3)+6(s+3)(\lambda-2)+(s+3)(s+4)\Big),$$

$$D_4(s,\lambda) = \frac{s(s+1)(s+2)(s+3)}{(1+\lambda)(2+\lambda)(3+\lambda)(4+\lambda)(5+\lambda)} + \frac{s(s+1)(s+2)(s+3)}{720\cdot 42}(3(\lambda-2)+2(s+4)),$$

$$D_5(s,\lambda) = s(s+1)(s+2)(s+3)(s+4)\cdot \cdot\left(\frac{1}{(1+\lambda)(2+\lambda)(3+\lambda)(4+\lambda)(5+\lambda)(6+\lambda)} - \frac{1}{720\cdot 42}\right).$$

We now assume that $s > \lambda+1$, $1 < \lambda \leqslant 2$, it is then easy to see that $D_4(s,\lambda) \geqslant 0$, $D_5(s,\lambda) \geqslant 0$ when $1 < \lambda \leqslant 2$. We apply the bounds $(1+\lambda)(2+\lambda)(3+\lambda) \leqslant 60$, $(1+\lambda)(2+\lambda)(3+\lambda)(4+\lambda) \leqslant 360$ when $1 < \lambda \leqslant 2$ to see that $D_2(s,\lambda) \geqslant 0$, $D_3(s,\lambda) \geqslant 0$ respectively, when

$$6\cdot 42 \geqslant 3(s+1-\lambda)(s-\lambda) + (\lambda-3)(\lambda-4) + 6(s+2)(\lambda-3) + 3(s+2)(s+3),$$

$$126 \geqslant 3(s+1-\lambda)(s-\lambda) + 3(\lambda-2)(\lambda-3) + 6(s+3)(\lambda-2) + (s+3)(s+4).$$

For fixed s, the right-hand expressions above are increasing functions of λ and hence are maximized when $\lambda = 2$. Thus, the above inequalities are valid for all $1 < \lambda \leqslant 2$, $s > \lambda+1$ provided that the following inequalities are valid:

$$6\cdot 42 \geqslant 3(s-1)(s-2) + 2 - 6(s+2) + 3(s+2)(s+3) = 6s^2 + 14,$$

$$126 \geqslant 3(s-1)(s-2) + (s+3)(s+4) = 4s^2 - 2s + 18.$$

One checks readily that the above inequalities are valid for $\lambda+1 < s \leqslant 5$.

Note that

$$\frac{s}{(1+\lambda)(2+\lambda)} - \frac{s}{12} = \frac{s(2-\lambda)(5+\lambda)}{12(1+\lambda)(2+\lambda)}.$$

It is easy to see that $D_1(s,\lambda) \geqslant 0$ when $\lambda = 2$ for any $s > \lambda+1$. When $1 < \lambda < 2$, we see that $D_1(s,\lambda) \geqslant 0$ is equivalent to the following inequality:

$$\frac{60\cdot 42(5+\lambda)}{(1+\lambda)(2+\lambda)} + \frac{3\cdot 42\lambda(\lambda-1)}{(2-\lambda)} \geqslant (3-\lambda)\left(3(s+1-\lambda)(s-\lambda) + 2(s+1)(\lambda-4) + 3(s+1)(s+2)\right).$$

It is easy to see that the left-hand side expression above is $\geqslant 60\cdot 42\cdot 6/(3\cdot 4) = 30\cdot 42$ while the right-hand side expression above is $\leqslant 3(3(s+1-1)(s-1) + 3(s+1)(s+2)) = 18(s^2+s+1)$. It follows that the above inequality is valid for $\lambda+1 < s \leqslant 5$.

Note that

$$\frac{1}{1+\lambda} - \frac{1}{2} + \frac{\lambda}{12} = \frac{(\lambda-2)(\lambda-3)}{12(1+\lambda)}.$$

Thus, when $1 < \lambda \leqslant 2$ we see that $D_0(s,\lambda) \geqslant 0$ follows from

$$\frac{60\cdot 42}{1+\lambda} - 42\cdot(\lambda-1) \geqslant (4-\lambda)\left((s+1-\lambda)(s-\lambda) + s(s+1)\right).$$

It is easy to see that the left-hand side expression above is $\geqslant 19\cdot 42$. We apply the bound $4-\lambda \leqslant 3, (s+1-\lambda)(s-\lambda) \leqslant s(s-1)$ to see that $D_0(s,\lambda) \geqslant 0$ follows from

$$19\cdot 42 \geqslant 6s^2,$$

which is valid for $\lambda+1 < s \leqslant 5$. This completes the proof. □

3. Further Discussions

In this section, we first improve the result of [9, Lemma 2] by showing that inequality (1.4) is valid for any $s>2$ when $\lambda=1$.

PROPOSITION 1. *Let $s>2, C_s=((s-2)(s-1))^{-1}$, then for for any integer $n \geqslant 1$,*

$$\sum_{m=1}^{\infty} \frac{m}{(n+m)^s} \leqslant \frac{C_s}{n^{s-2}}. \tag{3.1}$$

Proof. Note first that the condition $s>2$ ensures that the infinite series in (3.1) converges. Let

$$f_s(n) := \frac{C_s}{n^{s-2}} - \left(\sum_{m=1}^{\infty} \frac{1}{(n+m)^{s-1}} - n\sum_{m=1}^{\infty} \frac{1}{(n+m)^s}\right).$$

As $\sum_{i=1}^{\infty} i^{1-s} < \infty$, it follows that

$$\lim_{n\to\infty} \sum_{m=1}^{\infty} \frac{1}{(n+m)^{s-1}} = 0.$$

Note also that

$$0 \leqslant \lim_{n\to\infty} n\sum_{m=1}^{\infty} \frac{1}{(n+m)^s} \leqslant \lim_{n\to\infty} n\int_0^{\infty} \frac{1}{(n+x)^s}dx = \lim_{n\to\infty} \frac{1}{(s-1)n^{s-2}} = 0.$$

We then deduce that $\lim_{n\to\infty} f_s(n) = 0$. Therefore, it suffices to show that $f_s(n) - f_s(n+1) \geqslant 0$. Calculation shows that

$$f_s(n) - f_s(n+1) = \frac{C_s}{n^{s-2}} - \frac{C_s}{(n+1)^{s-2}} - \sum_{m=1}^{\infty} \frac{1}{(n+m)^s}.$$

Note that [5, Lemma 3] asserts that for $s>1$,

$$\sum_{i=k}^{\infty} \frac{1}{i^s} \leqslant \frac{s}{s-1} \cdot \frac{1}{k^s-(k-1)^s}.$$

We apply this to see that it suffices to show that

$$\frac{C_s}{n^{s-2}} - \frac{C_s}{(n+1)^{s-2}} - \frac{s}{s-1} \cdot \frac{1}{(n+1)^s - n^s} \geqslant 0.$$

We can recast the above inequality as

$$\left(\int_n^{n+1} x^{s-1} dx\right)\left(\int_n^{n+1} x^{1-s} dx\right) \geqslant 1.$$

As the above inequality follows from the Cauchy-Schwarz inequality, this completes the proof. □

We end this paper by proving Theorem 5. It amounts to show that

$$\frac{1}{n^{\alpha+2}} \sum_{r=1}^{n} r^{\alpha}(n-r) \leqslant \frac{1}{(n+1)^{\alpha+2}} \sum_{r=1}^{n} r^{\alpha}(n+1-r).$$

The above inequality can be rewritten as

$$\left(n - \frac{n^{\alpha+2}}{(n+1)^{\alpha+2} - n^{\alpha+2}}\right) \sum_{r=1}^{n} r^{\alpha} \leqslant \sum_{r=1}^{n} r^{\alpha+1}.$$

The above inequality holds trivially for $n = 1$ and therefore by induction, it suffices to show that

$$\left(n+1 - \frac{(n+1)^{\alpha+2}}{(n+2)^{\alpha+2} - (n+1)^{\alpha+2}}\right) \sum_{r=1}^{n+1} r^{\alpha} - \left(n - \frac{n^{\alpha+2}}{(n+1)^{\alpha+2} - n^{\alpha+2}}\right) \sum_{r=1}^{n} r^{\alpha}$$
$$\leqslant \sum_{r=1}^{n+1} r^{\alpha+1} - \sum_{r=1}^{n} r^{\alpha+1} = (n+1)^{\alpha+1}.$$

After simplification, the above inequality becomes

$$\left(\frac{((n+1)/n)^{\alpha+2}}{((n+1)/n)^{\alpha+2} - 1} - \frac{1}{((n+2)/(n+1))^{\alpha+2} - 1}\right) \sum_{r=1}^{n} r^{\alpha}$$
$$\leqslant \frac{(n+1)^{\alpha}}{((n+2)/(n+1))^{\alpha+2} - 1}.$$

Further simplification yields

$$\left(\frac{(n+2)^{\alpha+2} - (n+1)^{\alpha+2}}{(n+1)^{\alpha+2} - n^{\alpha+2}} - 1\right) \sum_{r=1}^{n} r^{\alpha} \leqslant (n+1)^{\alpha}.$$

The above inequality is equivalent to the following inequality:

$$\frac{\sum_{r=1}^{n} r^{\alpha}}{\sum_{r=1}^{n+1} r^{\alpha}} \leqslant \frac{(n+1)^{\alpha+2} - n^{\alpha+2}}{(n+2)^{\alpha+2} - (n+1)^{\alpha+2}}. \tag{3.2}$$

We note the following lemma:

LEMMA 3. ([13, Lemma 2.1]) *Let* $\{B_n\}_{n=1}^{\infty}$ *and* $\{C_n\}_{n=1}^{\infty}$ *be strictly increasing positive sequences with* $B_1/B_2 \leqslant C_1/C_2$. *If for any integer* $n \geqslant 1$,

$$\frac{B_{n+1}-B_n}{B_{n+2}-B_{n+1}} \leqslant \frac{C_{n+1}-C_n}{C_{n+2}-C_{n+1}}.$$

Then $B_n/B_{n+1} \leqslant C_n/C_{n+1}$ *for any integer* $n \geqslant 1$.

Applying the above lemma with $B_n = \sum_{r=1}^{n} r^{\alpha}$, $C_n = (n+1)^{\alpha+2} - n^{\alpha+2}$ and observing that the sequence $\{C_n\}_{n=1}^{\infty}$ is strictly increasing by the Mean Value Theorem, we see that inequality (3.2) holds provided that we have

$$\frac{1}{1^{\alpha}+2^{\alpha}} \leqslant \frac{2^{\alpha+2}-1}{3^{\alpha+2}-2^{\alpha+2}},$$
$$\frac{(n+1)^{\alpha}}{(n+2)^{\alpha}} \leqslant \frac{(n+2)^{\alpha+2}-2(n+1)^{\alpha+2}+n^{\alpha+2}}{(n+3)^{\alpha+2}-2(n+2)^{\alpha+2}+(n+1)^{\alpha+2}}, \quad n \geqslant 1.$$

It is easy to see that the first inequality above follows from the case $n=0$ of the second inequality. It therefore remains to prove the second inequality above for $n \geqslant 0$. We recast the second inequality above as $f(1/(n+2)) \leqslant f(1/(n+1))$, where

$$f(x) = x^{-2}\Big((1+x)^{\alpha+2} + (1-x)^{\alpha+2} - 2\Big).$$

Note that

$$\frac{x^2}{2} \cdot f'(x) = \frac{g(x)+g(0)}{2} - \frac{1}{x}\int_0^x g(t)dt,$$

where

$$g(x) = (\alpha+2)(1+x)^{\alpha+1} - (\alpha+2)(1-x)^{\alpha+1}.$$

As $g(x)$ is convex when $0 < x \leqslant 1$, it follows from the Hermite-Hadamard inequality [12, p. 10] that $f'(x) \geqslant 0$ when $0 < x \leqslant 1$ and this completes the proof.

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