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ON MIXED COMPLEX INTERSECTION BODIES

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Abstract. Complex intersection bodies were introduced by Koldobsky, Paouris and Zymonopoulou. In this paper some geometric inequalities for mixed complex intersection bodies which are dual forms of inequalities for mixed complex projection bodies are established.

1. Introduction

Real intersection bodies have attracted increased interest since they are introduced by Lutwak [20]. Intersection bodies were used to solve the important Busemann-Petty problem (see [4, 5, 14, 24, 25]). More results and applications on intersection bodies can be found in [2, 3, 7–9, 11–13, 16, 22, 23, 26].

Given a convex body $A \subset \mathbb{C}$ and convex bodies $K_1, \dots, K_{2n} \subset \mathbb{C}^n$, the mixed projection body $\Pi^A(K_1, \dots, K_{2n})$ in a complex vector space was defined by Abardia and Bernig [1]. Moreover, they established the following Minkowski inquality and Brunn-Minkowski inequality for mixed complex projection bodies.

THEOREM A. [1] *If K and L are convex bodies in $\mathbb{C}^n$, then*

$$V(\Pi_1^A(K,L))^{2n-1} \geqslant V(\Pi^A K)^{2n-2} V(\Pi^A L), \tag{1.1}$$

with equality if and only if K and L are homothetic. Here $\Pi^A K = \Pi^A(K, \dots, K)$ and $\Pi_1^A(K,L) = \Pi^A(K, \dots, K, L)$.

THEOREM B. [1] *If K and L are convex bodies in $\mathbb{C}^n$, then*

$$V(\Pi^A(K+L))^{\frac{1}{2n(2n-1)}} \geqslant V(\Pi^A K)^{\frac{1}{2n(2n-1)}} + V(\Pi^A L)^{\frac{1}{2n(2n-1)}}, \tag{1.2}$$

with equality if and only if K and L are homothetic.

As Lutwak [20] shows (see also [6]), there is a duality between projection bodies and intersection bodies (that at present is not yet understood). Koldobsky, Paouris and

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Zymonopoulou [15] firstly introduced the complex intersection body and considered the complex Busemann-Petty problem.

In this paper we shall introduce the mixed complex intersection body. Based on the standard proof of geometric inequalities which was developed by Lutwak [18, 19, 21], we establish the dual forms of inequalities (1.1) and (1.2) for mixed complex intersection bodies. Our main results can be stated as follows:

THEOREM 1.1. *If K and L are star bodies in $\mathbb{C}^n$, then*

$$V(\mathscr{I}_1^C(K,L))^{2n-2} \leqslant V(\mathscr{I}^C K)^{2n-3} V(\mathscr{I}^C L), \tag{1.3}$$

with equality if and only if K and L are dilates. The precise definitions of $\mathscr{I}^C$ and $\mathscr{I}_1^C$ are introduced in Section 2.

THEOREM 1.2. *If K and L are star bodies in $\mathbb{C}^n$, then*

$$V(\mathscr{I}^C(K \tilde{+}_2 L))^{\frac{1}{n(n-1)}} \leqslant V(\mathscr{I}^C K)^{\frac{1}{n(n-1)}} + V(\mathscr{I}^C L)^{\frac{1}{n(n-1)}}, \tag{1.4}$$

with equality if and only if K and L are dilates.

This paper is organized as follows: In Section 2 we introduce above interrelated notations and their background materials, and recall several needed Lemmas. Section 3 contains the proof of our main results.

2. Notation and background material

The real vector space $\mathbb{R}^n$ of real dimension n is replaced by a complex vector space $\mathbb{C}^n$ of dimension n. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ using the standard mapping

$$\xi = (\xi_1, \dots, \xi_n) = (\xi_{11} + i\xi_{12}, \dots, \xi_{n1} + i\xi_{n2}) \mapsto (\xi_{11}, \xi_{12}, \dots, \xi_{n1}, \xi_{n2}). \tag{2.1}$$

The unit ball B in $\mathbb{C}^n$ is given by

$$B = \{\xi \in \mathbb{C}^n : \ \sum_{i=1}^{n} (\xi_{i1}^2 + \xi_{i2}^2) \leqslant 1\}.$$

Its unit sphere can be denoted by S^{2n-1}. The volume of $B \subset \mathbb{C}^n$ is denoted by ω_{2n}. A compact set $K \subset \mathbb{C}^n$ is called a star body if its radial function $\rho(K,\cdot)$ defined by

$$\rho(K,\xi) = \max\{\lambda : \ \lambda\xi \in K\}, \quad \xi \in S^{2n-1} \tag{2.2}$$

is positive and continuous on S^{2n-1}. For $\xi \in S^{2n-1}$, the complex hyperplane H_ξ is denoted by

$$H_\xi = \{z \in \mathbb{C}^n : \ (z,\xi) = \sum_{k=1}^{n} z_k \overline{\xi_k} = 0\},$$

which is a $(2n-2)$-dimensional subspace of $\mathbb{R}^{2n}$ orthogonal to the vectors

$$\xi = (\xi_{11}, \xi_{12}, \dots, \xi_{n1}, \xi_{n2}) \quad \text{and} \quad \xi^{\perp} = (-\xi_{12}, \xi_{11}, \dots, -\xi_{n2}, \xi_{n1}).$$

Let K,L be star bodies in $\mathbb{C}^n$, and $\lambda_1,\lambda_2 \geqslant 0$ (not both 0), the L_2 radial sum $\lambda_1 \cdot K \widetilde{+}_2 \lambda_2 \cdot L$ is a star body whose radial function is given by

$$\rho(\lambda_1 \cdot K \widetilde{+}_2 \lambda_2 \cdot L, \cdot)^2 = \lambda_1 \rho(K,\cdot)^2 + \lambda_2 \rho(L,\cdot)^2. \tag{2.4}$$

Let $K_1,\dots,K_{2n}$ be star bodies in $\mathbb{C}^n$, the dual mixed volume $\widetilde{V}(K_1,\dots,K_{2n})$ has the following integral representation [25]:

$$\widetilde{V}(K_1,\dots,K_{2n}) = \frac{1}{2n}\int_{S^{2n-1}} \rho(K_1,u)\cdots\rho(K_{2n},u)du, \tag{2.5}$$

where du is the standard spherical Lebesgue measure on S^{2n-1}. We write $\widetilde{V}_2(K_1,\dots,K_{2n-2},L)$ for $\widetilde{V}(K_1,\dots,K_{2n-2},L,L)$, where the K_i $(i=1,\dots,2n-2)$ appear once and L appears twice. For $i \geqslant 0$, $j \geqslant 0$ and $i+j \leqslant 2n$, we write $\widetilde{W}_{i,j}(K,L)$ for the dual mixed volume $\widetilde{V}(K,\dots,K,B,\dots,B,L,\dots,L)$, where K appears $2n-i-j$ times, B appears i times and L appears j times. The dual mixed volume $\widetilde{W}_{i,j}(K,K)$ will be written as $\widetilde{W}_i(K)$ and is called the dual ith quermassintegral of K.

Due to the work of Lutwak [17], we have the dual Aleksandrov Fenchel inequality for dual mixed volumes in $\mathbb{C}^n$: If $K_1,\dots,K_{2n}$ are star bodies in $\mathbb{C}^n$ and $1 \leqslant m \leqslant 2n$, then

$$\widetilde{V}(K_1,\dots,K_{2n})^m \leqslant \prod_{j=1}^{m} \widetilde{V}(K_j,\dots,K_j,K_{m+1},\dots,K_{2n}), \tag{2.6}$$

with equality if and only if $K_1,\dots,K_m$ are dilates.

A special case of inequality (2.6) is the following dual Minkowski inequality:

LEMMA 2.1. *Let K,L be star bodies in $\mathbb{C}^n$. If $0 \leqslant i \leqslant 2n-3$ and $1 \leqslant j \leqslant 2n-i$, then*

$$\widetilde{W}_{i,j}(K,L)^{2n-i} \leqslant \widetilde{W}_i(K)^{2n-i-j}\widetilde{W}_i(L)^j,$$

with equality if and only if K and L are dilates.

From (2.6), we have

LEMMA 2.2. *If K is a star body in $\mathbb{C}^n$ and $0 \leqslant i < j \leqslant 2n-1$, then*

$$\omega_{2n}^{j-i}\widetilde{W}_i(K)^{2n-j} \geqslant \widetilde{W}_j(K)^{2n-i},$$

with equality if and only if K is a ball.

LEMMA 2.3. *If K,L,Q are star bodies in $\mathbb{C}^n$ and $0 \leqslant i \leqslant 2n-3$, then*

$$\widetilde{W}_{i,2}(K\widetilde{+}_2 L,Q)^{\frac{2}{2n-i-2}} \leqslant \widetilde{W}_{i,2}(K,Q)^{\frac{2}{2n-i-2}} + \widetilde{W}_{i,2}(L,Q)^{\frac{2}{2n-i-2}},$$

with equality if and only if K and L are dilates.

Proof. By (2.5), (2.4) and Minkowski integral inequality [10], we have

$$\begin{aligned}
&\widetilde{W}_{i,2}(K\widetilde{+}_2L,Q)^{\frac{2}{2n-i-2}}\\
&=\Big(\frac{1}{2n}\int_{S^{2n-1}}(\rho(K,\xi)^2+\rho(L,\xi)^2)^{\frac{2n-i-2}{2}}\rho(Q,\xi)^2d\xi\Big)^{\frac{2}{2n-i-2}}\\
&\leqslant\Big(\frac{1}{2n}\int_{S^{2n-1}}\rho(K,\xi)^{2n-i-2}\rho(Q,\xi)^2d\xi\Big)^{\frac{2}{2n-i-2}}\\
&\quad+\Big(\frac{1}{2n}\int_{S^{2n-1}}\rho(L,\xi)^{2n-i-2}\rho(Q,\xi)^2d\xi\Big)^{\frac{2}{2n-i-2}}\\
&=\widetilde{W}_{i,2}(K,Q)^{\frac{2}{2n-i-2}}+\widetilde{W}_{i,2}(L,Q)^{\frac{2}{2n-i-2}},
\end{aligned}$$

with equality if and only if K and L are dilates.

Let $K_1,\dots,K_{2n-2}$ be star bodies in $\mathbb{C}^n$. The mixed complex intersection body $\mathscr{I}^C(K_1,\dots,K_{2n-2})$ is defined by

$$\rho(\mathscr{I}^C(K_1,\dots,K_{2n-2}),\xi)^2=\frac{1}{(2n-2)\pi}\int_{S^{2n-1}\cap H_\xi}\rho(K_1,\omega)\cdots\rho(K_{2n-2},\omega)d\omega, \quad (2.7)$$

where $d\omega$ is the standard spherical Lebesgue measure on S^{2n-1} and H_ξ is a $(2n-2)$-dimensional subspace of $\mathbb{R}^{2n}$ orthogonal to the vectors ξ and $\xi^\perp$.

Obviously, every mixed complex intersection body corresponds to an origin symmetric convex body $\mathbb{R}^{2n}$ which is invariant with respect to any coordinate-wise two-dimensional rotation.

If $K_1=\cdots=K_{2n-i-2}=K$ and $K_{2n-i-1}=\cdots=K_{2n-2}=L$, the mixed complex intersection body $\mathscr{I}^C(K_1,\dots,K_{2n-2})$ is written as $\mathscr{I}_i^C(K,L)$. If $L=B$, $\mathscr{I}_i^C(K,L)$ is written as $\mathscr{I}_i^C(K)$ and is called the ith complex intersection body of K. We simply write $\mathscr{I}^CK$ rather than $\mathscr{I}_0^CK$, which was first defined by Koldobsky, Paouris and Zymonopoulou [15]. In particular, $\mathscr{I}^CB=\sqrt{\frac{\omega_{2n-2}}{\pi}}B$, since for every $\xi\in S^{2n-1}$,

$$\rho(\mathscr{I}^CB,\xi)^2=\frac{1}{(2n-2)\pi}\int_{S^{2n-1}\cap H_\xi}d\omega=\frac{\omega_{2n-2}}{\pi}. \quad \square$$

3. Main results

LEMMA 3.1. *If $K_1,\dots,K_{2n-2},L_1,\dots,L_{2n-2}$ are star bodies in $\mathbb{C}^n$, then*

$$\widetilde{V}_2(K_1,\dots,K_{2n-2},\mathscr{I}^C(L_1,\dots,L_{2n-2}))=\widetilde{V}_2(L_1,\dots,L_{2n-2},\mathscr{I}^C(K_1,\dots,K_{2n-2})).$$

Proof. By (2.5), (2.7) and Fubini's theorem, we have

$$
\begin{aligned}
&\widetilde{V}_2(K_1,\ldots,K_{2n-2},\mathscr{I}^C(L_1,\ldots,L_{2n-2}))\\
&=\frac{1}{2n}\int_{S^{2n-1}}\rho(\mathscr{I}^C(L_1,\ldots,L_{2n-2}),\xi)^2\rho(K_1,\xi)\cdots\rho(K_{2n-2},\xi)d\xi\\
&=\frac{1}{2n(2n-2)\pi}\int_{S^{2n-1}}\Big[\int_{S^{2n-1}\cap H_\xi}\rho(L_1,\omega)\cdots\rho(L_{2n-2},\omega)d\omega\Big]\rho(K_1,\xi)\cdots\rho(K_{2n-2},\xi)d\xi\\
&=\frac{1}{2n(2n-2)\pi}\int_{S^{2n-1}}\Big[\int_{S^{2n-1}\cap H_\omega}\rho(K_1,\xi)\cdots\rho(K_{2n-2},\xi)d\xi\Big]\rho(L_1,\omega)\cdots\rho(L_{2n-2},\omega)d\omega\\
&=\frac{1}{2n}\int_{S^{2n-1}}\rho(\mathscr{I}^C(K_1,\ldots,K_{2n-2}),\omega)^2\rho(L_1,\omega)\cdots\rho(L_{2n-2},\omega)d\omega\\
&=\widetilde{V}_2(L_1,\ldots,L_{2n-2},\mathscr{I}^C(K_1,\ldots,K_{2n-2})).\quad\square
\end{aligned}
$$

If $K_1=\cdots=K_{2n-i-2}=K$ and $K_{2n-i-1}=\cdots=K_{2n-2}=B$, then Lemma 3.1 reduces to

LEMMA 3.2. *If $K,L_1,\ldots,L_{2n-2}$ are star bodies in $\mathbb{C}^n$, then*

$$\widetilde{W}_{i,2}(K,\mathscr{I}^C(L_1,\ldots,L_{2n-2}))=\widetilde{V}_2(L_1,\ldots,L_{2n-2},\mathscr{I}_i^C K).$$

The special case of Lemma 3.2, where $L_1=\cdots=L_{2n-j-2}=L$ and $L_{2n-j-1}=\cdots=L_{2n-2}=B$, states as follows:

LEMMA 3.3. *Let K and L be star bodies in $\mathbb{C}^n$. If $0\leqslant i\leqslant 2n-2$ and $0\leqslant j\leqslant 2n-2$, then*

$$\widetilde{W}_{i,2}(K,\mathscr{I}_j^C L)=\widetilde{W}_{j,2}(L,\mathscr{I}_i^C K).$$

Take $K_1=\cdots=K_{2n-2}=B$ in Lemma 3.1 and note that $\mathscr{I}^C B=\sqrt{\frac{\omega_{2n-2}}{\pi}}B$ to get

LEMMA 3.4. *If $L_1,\ldots,L_{2n-2}$ are star bodies in $\mathbb{C}^n$, then*

$$\widetilde{W}_{2n-2}(\mathscr{I}^C(L_1,\ldots,L_{2n-2}))=\frac{\omega_{2n-2}}{\pi}\widetilde{V}_2(L_1,\ldots,L_{2n-2},B). \tag{3.1}$$

For $L_1=\cdots=L_{2n-i-2}=K$ and $L_{2n-i-1}=\cdots=L_{2n-2}=L$, identity (3.1) becomes

$$\widetilde{W}_{2n-2}(\mathscr{I}_i^C(K,L))=\frac{\omega_{2n-2}}{\pi}\widetilde{W}_{2,i}(K,L), \tag{3.2}$$

and for $L=B$,

$$\widetilde{W}_{2n-2}(\mathscr{I}_i^C K)=\frac{\omega_{2n-2}}{\pi}\widetilde{W}_{i+2}(K). \tag{3.3}$$

Next, we establish a generalization of Theorem 1.1.

THEOREM 3.5. *Let K and L be star bodies in $\mathbb{C}^n$. If $0 \leqslant i \leqslant 2n-2$ and $1 \leqslant j \leqslant 2n-3$, then*

$$\widetilde{W}_i(\mathscr{I}_j^C(K,L))^{2n-2} \leqslant \widetilde{W}_i(\mathscr{I}^C K)^{2n-j-2}\widetilde{W}_i(\mathscr{I}^C L)^j, \tag{3.4}$$

with equality if and only if K and L are dilates.

Proof. Case $i < 2n-2$. Suppose that Q is a star body in $\mathbb{C}^n$. From Lemma 3.2, the dual Aleksandrov-Fenchel inequality (2.6) and Lemma 2.1, it follows that

$$\begin{aligned}
&\widetilde{W}_{i,2}(Q,\mathscr{I}_j^C(K,L))^{2n-2}\\
&= \widetilde{V}_2(K,\ldots,K,L,\ldots,L,\mathscr{I}_i^C Q)^{2n-2}\\
&\leqslant \widetilde{V}_2(K,\mathscr{I}_i^C Q)^{2n-j-2}\widetilde{V}_2(L,\mathscr{I}_i^C Q)^j\\
&= \widetilde{W}_{i,2}(Q,\mathscr{I}^C K)^{2n-j-2}\widetilde{W}_{i,2}(Q,\mathscr{I}^C L)^j\\
&\leqslant \widetilde{W}_i(Q)^{\frac{(2n-i-2)(2n-2)}{2n-i}}\widetilde{W}_i(\mathscr{I}^C K)^{\frac{2(2n-j-2)}{2n-i}}\widetilde{W}_i(\mathscr{I}^C L)^{\frac{2j}{2n-i}}.
\end{aligned} \tag{3.5}$$

By the equality conditions of Lemma 2.1, equality in (3.5) holds if and only if $Q, \mathscr{I}^C K$ and $\mathscr{I}^C L$ are dilates.

Set $Q = \mathscr{I}_j^C(K,L)$ and note that $\widetilde{W}_{i,2}(Q,Q) = \widetilde{W}_i(Q)$ to obtain the desired inequality (3.4). If there is equality in (3.4), then there exist $\lambda_1, \lambda_2 > 0$ such that

$$\mathscr{I}_j^C(K,L) = \lambda_1 \mathscr{I}^C K = \lambda_2 \mathscr{I}^C L. \tag{3.6}$$

From equality in (3.4), it follows that

$$\lambda_1^{2n-j-2}\lambda_2^j = 1.$$

Moreover, (3.2), (3.3) and (3.6) imply

$$\lambda_1^2 = \frac{\widetilde{W}_{2,j}(K,L)}{\widetilde{W}_2(K)} \quad \text{and} \quad \lambda_2^2 = \frac{\widetilde{W}_{2,j}(K,L)}{\widetilde{W}_2(L)}.$$

Hence, we have

$$\widetilde{W}_{2,j}(K,L)^{2n-2} = \widetilde{W}_2(K)^{2n-j-2}\widetilde{W}_2(L)^j,$$

which implies, by Lemma 2.1, that K and L are dilates.

The case $i = 2n-2$ follows from (3.2), (3.3) and Lemma 2.1. □

REMARK 1. Taking for $j = 1$ in Theorem 3.5, it becomes

$$\widetilde{W}_i(\mathscr{I}_1^C(K,L))^{2n-2} \leqslant \widetilde{W}_i(\mathscr{I}^C K)^{2n-3}\widetilde{W}_i(\mathscr{I}^C L),$$

with equality if and only if K and L are dilates.

This is just a dual form of the following Minkowski inequality of mixed complex projection bodies for general volume which was given by Abardia and Bernig [1]:

$$W_i(\Pi_1^C(K,L))^{2n-1} \geqslant W_i(\Pi^C K)^{2n-2}W_i(\Pi^C L),$$

with equality if and only if K and L are homothetic.

An immediate consequence of Theorem 3.5 states as follows:

THEOREM 3.6. *Let K,L be star bodies in $\mathbb{C}^n$ and $\mathscr{M}$ be a subset of $\mathbb{C}^n$. Suppose $K,L \subset \mathscr{M}$, $0 \leqslant i \leqslant 2n-2$ and $1 \leqslant j \leqslant 2n-2$. If either*

$$\widetilde{W}_i(\mathscr{I}_j^C(K,Q)) = \widetilde{W}_i(\mathscr{I}_j^C(L,Q)), \quad \text{for all } Q \subset \mathscr{M}, \tag{3.7}$$

or

$$\widetilde{W}_i(\mathscr{I}_j^C(Q,K)) = \widetilde{W}_i(\mathscr{I}_j^C(Q,L)), \quad \text{for all } Q \subset \mathscr{M}, \tag{3.8}$$

hold, then it follows that $K = L$.

Proof. Suppose that (3.7) holds. Take K for Q in (3.7), use Theorem 3.5 to get

$$\widetilde{W}_i(\mathscr{I}^C K) \leqslant \widetilde{W}_i(\mathscr{I}^C L), \tag{3.9}$$

with equality if and only if K and L are dilates. Take L for Q in (3.7), use Theorem 3.5 to get

$$\widetilde{W}_i(\mathscr{I}^C L) \leqslant \widetilde{W}_i(\mathscr{I}^C K).$$

Hence, there is equality in (3.9) and thus, there is a $\lambda > 0$ for which $K = \lambda L$. But equality in (3.9) implies that $\lambda = 1$.

The same argument shows that condition (3.8) implies that $K = L$. □

Taking $L = B$ in Theorem 3.5, we have that

COROLLAY 3.7. *Let K be a star body in $\mathbb{C}^n$. If $0 \leqslant i \leqslant 2n-2$ and $0 < j \leqslant 2n-2$, then*

$$\widetilde{W}_i(\mathscr{I}_j^C K)^{2n-2} \leqslant \left(\frac{\omega_{2n-2}}{\pi}\right)^{\frac{(2n-i)j}{2}} \omega_{2n}^j \widetilde{W}_i(\mathscr{I}^C K)^{2n-j-2},$$

with equality if and only if K is a ball.

Moreover, a generalization of Corollay 3.7 will be established.

THEOREM 3.8. *Let K be a star body in $\mathbb{C}^n$. If $0 \leqslant i < j \leqslant 2n-2$ and $0 \leqslant k \leqslant 2n-2$, then*

$$\widetilde{W}_k(\mathscr{I}_j^C K)^{2n-i-2} \leqslant \left(\sqrt{\frac{\omega_{2n-2}}{\pi}}\,\omega_{2n}^{\frac{1}{2n-k}}\right)^{(j-i)(2n-k)} \widetilde{W}_k(\mathscr{I}_i^C K)^{2n-j-2}, \tag{3.10}$$

with equality if and only if K is a ball.

Proof. From (3.3), it follows that the case $k = 2n-2$ of inequality (3.10) reduces to Lemma 2.2, and hence, it may be assumed that $k < 2n-2$.

Suppose that Q is a star body in $\mathbb{C}^n$, from Lemma 3.3,

$$\widetilde{W}_{k,2}(Q, \mathscr{I}_j^C K) = \widetilde{W}_{j,2}(K, \mathscr{I}_k^C Q). \tag{3.11}$$

By (2.6), we have

$$\widetilde{W}_{j,2}(K, \mathscr{I}_k^C Q)^{2n-i-2} \leqslant \widetilde{W}_{2n-2}(\mathscr{I}_k^C Q)^{j-i} \widetilde{W}_{i,2}(K, \mathscr{I}_k^C Q)^{2n-j-2}. \tag{3.12}$$

From (3.3) and Lemma 2.2, it follows that

$$\widetilde{W}_{2n-2}(\mathscr{I}_k^C Q) = \frac{\omega_{2n-2}}{\pi}\widetilde{W}_{k+2}(Q) \leqslant \frac{\omega_{2n-2}}{\pi}\omega_{2n}^{\frac{2}{2n-k}}\widetilde{W}_k(Q)^{\frac{2n-k-2}{2n-k}}, \tag{3.13}$$

with equality if and only if Q is a ball.

For the second term on the right of (3.12), note that by Lemma 3.3,

$$\widetilde{W}_{i,2}(K,\mathscr{I}_k^C Q) = \widetilde{W}_{k,2}(Q,\mathscr{I}_i^C K).$$

Apply Lemma 2.1 to the quantity on the right and get:

$$\widetilde{W}_{i,2}(K,\mathscr{I}_k^C Q) = \widetilde{W}_{k,2}(Q,\mathscr{I}_i^C K) \leqslant \widetilde{W}_k(Q)^{\frac{2n-k-2}{2n-k}}\widetilde{W}_k(\mathscr{I}_i^C K)^{\frac{2}{2n-k}}, \tag{3.14}$$

with equality if and only if Q and $\mathscr{I}_i K$ are dilates.

Now take $Q = \mathscr{I}_j^C K$, note that $\widetilde{W}_{k,2}(Q,Q) = \widetilde{W}_k(Q)$, and combine (3.11) with (3.12), (3.13) and (3.14) to obtain the desired inequality of Theorem 1.1.

From the equality conditions of inequalities (3.13) and (3.14), we have that $\mathscr{I}_i^C K$ and $\mathscr{I}_j^C K$ must be centered balls. Thus there exist $\lambda,\mu > 0$, such that

$$\mathscr{I}_i^C K = \lambda B, \quad \text{and} \quad \mathscr{I}_j^C K = \mu B. \tag{3.15}$$

Let $r = \dfrac{\omega_{2n-2}}{\pi}$, if there is equality in (3.10), it follows that

$$\mu^{2n-i-2} = r^{\frac{j-i}{2}}\lambda^{2n-j-2},$$

equivalently,

$$\omega_{2n}^{j-i}\Big(\frac{\lambda^2\omega_{2n}}{r}\Big)^{2n-j-2} = \Big(\frac{\mu^2\omega_{2n}}{r}\Big)^{2n-i-2}.$$

Moreover, (3.3) and (3.15) imply

$$\widetilde{W}_{i+2}(K) = \frac{\lambda^2\omega_{2n}}{r} \quad \text{and} \quad \widetilde{W}_{j+2}(K) = \frac{\mu^2\omega_{2n}}{r}.$$

Hence, we have

$$\omega_{2n}^{j-i}\widetilde{W}_{i+2}(K)^{2n-j-2} = \widetilde{W}_{j+2}(K)^{2n-i-2},$$

which implies, by Lemma 2.2, that K is a ball. □

An important generalization of Theorem 1.2 will be established as follows:

THEOREM 3.9. *Let K and L be star bodies in $\mathbb{C}^n$. If $0 \leqslant i \leqslant 2n-2$ and $0 \leqslant j \leqslant 2n-4$, then*

$$\widetilde{W}_i(\mathscr{I}_j^C(K\widetilde{+}_2 L))^{\frac{4}{(2n-i)(2n-j-2)}} \leqslant \widetilde{W}_i(\mathscr{I}_j^C K)^{\frac{4}{(2n-i)(2n-j-2)}} + \widetilde{W}_i(\mathscr{I}_j^C L)^{\frac{4}{(2n-i)(2n-j-2)}}, \tag{3.16}$$

with equality if and only if K and L are dilates.

Proof. Suppose that M is a star body in $\mathbb{C}^n$. From Lemma 3.3, Lemma 2.3 and Lemma 2.1, it follows that

$$\begin{aligned}
&\widetilde{W}_{i,2}(M,\mathscr{I}_j^C(K\widetilde{+}_2L))^{\frac{2}{2n-j-2}}\\
&=\widetilde{W}_{j,2}(K\widetilde{+}_2L,\mathscr{I}_i^C M)^{\frac{2}{2n-j-2}}\\
&\leqslant \widetilde{W}_{j,2}(K,\mathscr{I}_i^C M)^{\frac{2}{2n-j-2}}+\widetilde{W}_{j,2}(L,\mathscr{I}_i^C M)^{\frac{2}{2n-j-2}}\\
&=\widetilde{W}_{i,2}(M,\mathscr{I}_j^C K)^{\frac{2}{2n-j-2}}+\widetilde{W}_{i,2}(M,\mathscr{I}_j^C L)^{\frac{2}{2n-j-2}}\\
&\leqslant \widetilde{W}_i(M)^{\frac{2(2n-i-2)}{(2n-i)(2n-j-2)}}\left[\widetilde{W}_i(\mathscr{I}_j^C K)^{\frac{4}{(2n-i)(2n-j-2)}}+\widetilde{W}_i(\mathscr{I}_j^C L)^{\frac{4}{(2n-i)(2n-j-2)}}\right].
\end{aligned}$$

Take $\mathscr{I}_j^C(K\widetilde{+}_2L)$ for M to obtain the desired inequality (3.16).

By the equality conditions of Lemma 2.1, equality in (3.16) holds if and only if $M,\mathscr{I}_j^C K$ and $\mathscr{I}_j^C L$ are dilates. If there is equality in (3.16), then there exist $\lambda_1,\lambda_2>0$, such that

$$\mathscr{I}_j^C(K)=\lambda_1\mathscr{I}_j(K\widetilde{+}_2L) \quad\text{and}\quad \mathscr{I}_j^C L=\lambda_2\mathscr{I}_j(K\widetilde{+}_2L). \tag{3.17}$$

From equality in (3.16), it follows that

$$\lambda_1^{\frac{4}{2n-j-2}}+\lambda_2^{\frac{4}{2n-j-2}}=1.$$

On the other hand, (3.3) and (3.17) imply

$$\lambda_1^2=\frac{\widetilde{W}_{j+2}(K)}{\widetilde{W}_{j+2}(K\widetilde{+}_2L)} \quad\text{and}\quad \lambda_2^2=\frac{\widetilde{W}_{j+2}(L)}{\widetilde{W}_{j+2}(K\widetilde{+}_2L)}.$$

Hence, we have

$$\widetilde{W}_{j+2}(K\widetilde{+}_2L)^{\frac{2}{2n-j-2}}=\widetilde{W}_{j+2}(K)^{\frac{2}{2n-j-2}}+\widetilde{W}_{j+2}(L)^{\frac{2}{2n-j-2}},$$

which implies, by Lemma 2.3, that K and L are dilates. □

REMARK 2. Taking $j=0$ in Theorem 3.9, it becomes

$$\widetilde{W}_i(\mathscr{I}^C(K\widetilde{+}_2L))^{\frac{2}{(2n-i)(n-1)}}\leqslant \widetilde{W}_i(\mathscr{I}^C K)^{\frac{2}{(2n-i)(n-1)}}+\widetilde{W}_i(\mathscr{I}^C L)^{\frac{2}{(2n-i)(n-1)}},$$

with equality if and only if K and L are dilates.

REFERENCES

[1] J. ABARDIA AND A. BERNIG, *Projection bodies in complex vector spaces*, Adv. Math., **211** (2011), 830–846.
[2] S. CAMPI, *Convex intersection bodies in three and four dimensions*, Mathematika, **46** (1999), 15-27.

[3] R. J. GARDNER, *On the Busemann-Petty problem concerning central sections of centrally symmetric convex bodies*, Bull. Amer. Math. Soc., **30** (1994), 222–226.

[4] R. J. GARDNER, *Intersection bodies and the Busemann-Petty problem*, Trans. Amer. Math. Soc., **342** (1994), 435–445.

[5] R. J. GARDNER, *A positive answer to the Busemann-Petty problem in three dimensions*, Ann. Math., **140** (1994), 435–447.

[6] R. J. GARDNER, *Geometric Tomography*, second ed., Cambridge University Press, New York, 2006.

[7] R. J. GARDNER, A. KOLDOBSKY AND T. SCHLUMPRECHT, *An analytic solution to the Busemann-Petty problem on sections of convex bodies*, Ann. Math., **149** (1999), 691–703.

[8] P. GOODEY AND W. WEIL, *Intersection bodies and ellipsoids*, Mathematika, **42** (1995), 295–304.

[9] E. GRINBERG AND G. ZHANG, *Convolutions, transforms, and convex bodies*, Proc. Lond. Math. Soc., **78** (1999), 77–115.

[10] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, second ed., Cambridge University Press, Cambridge, 1988.

[11] A. KOLDOBSKY, *Intersection bodies in* $\mathbb{R}^4$, Adv. Math., **136** (1998), 1–14.

[12] A. KOLDOBSKY, *Intersection bodies, positive definite distributions, and the Busemann-Petty problem*, Amer. J. Math., **120** (1998), 827–840.

[13] A. KOLDOBSKY, *A functional analytic approach to intersection bodies*, Geom. Funct. Anal., **10** (2000), 1507–1526.

[14] A. KOLDOBSKY, *Fourier Analysis in Convex Geometry*, American Mathematical Society Press, Providence, 2005.

[15] A. KOLDOBSKY, G. PAOURIS AND M. ZYMONOPOULOU, *Complex Intersection Bodies*, J. London Math. Soc., **88** (2013), 538–562.

[16] M. LUDWIG, *Intersection bodies and valuations*, Amer. J. Math., **128** (2006), no. 6, 1409–1428.

[17] E. LUTWAK, *Dual mixed volumes*, Pac. J. Math., **58** (1975), 531–538.

[18] E. LUTWAK, *On some affine isoperimetric inequalities*, J. Differential Geom., **23** (1986), 1–13.

[19] E. LUTWAK, *Volume of mixed bodies*, Trans. Amer. Math. Soc., **294** (1986), 487–500.

[20] E. LUTWAK, *Intersection bodies and dual mixed volumes*, Adv. Math., **71** (1988), 232–261.

[21] E. LUTWAK, *Inequalities for mixed projecton bodies*, Trans. Amer. Math. Soc., **339** (1993), 901–916.

[22] M. MOSZYŃSKA, *Quotient star bodies, intersection bodies, and star duality*, J. Math. Anal. Appl., **232** (1999), 45–60.

[23] F. E. SCHUSTER, *Volume inequalities and additive maps of convex bodies*, Mathematika, **53** (2006), no. 2, 211–234.

[24] G. ZHANG, *Intersection bodies and the Busemann-Petty inequalities in* $\mathbb{R}^4$, Ann. Math., **140** (1994), 331–346.

[25] G. ZHANG, *A positive solution to the Busemann-Petty problem in* $\mathbb{R}^4$, Ann. Math., **149** (1999), 535–543.

[26] C. J. ZHAO, *On intersection and mixed intersection bodies*, Geom. Dedicata, **144** (2009), 109–122.

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