Mathematical
Inequalities
& Applications
Volume 18, Number 2 (2015), 453–469 doi:10.7153/mia-18-34

L^p BOUNDS FOR PARAMETRIC MARCINKIEWICZ INTEGRALS WITH MIXED HOMOGENEITY

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(*Communicated by J. Pečarić*)

Abstract. In this paper we consider the parametric Marcinkiewicz integrals with mixed homogeneity along certain compound surfaces. Under the rather weakened size conditions on the integral kernels both on the unit sphere and in the radial direction, the L^p boundedness for such operators are given. As applications, the corresponding results for parametric Marcinkiewicz integral operators related to area integrals and Littlewood-Paley g_λ^* functions are also obtained.

1. Introduction

Let $\mathbb{R}^n$, $n \geqslant 2$, be the n-dimensional Euclidean space and S^{n-1} denote the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure $d\sigma$. Let $\alpha_j \geqslant 1$ $(j = 1,\dots,n)$ be fixed real numbers. Define the function $F : \mathbb{R}^n \times (0,\infty) \longrightarrow \mathbb{R}$ by $F(x,\rho) = \sum_{j=1}^n x_j^2 \rho^{-2\alpha_j}$, $x = (x_1,\dots,x_n)$. It is clear that for each fixed $x \in \mathbb{R}^n$, the function $F(x,\rho)$ is a decreasing function in $\rho > 0$. We let $\rho(x)$ denote the unique solution of the equation $F(x,\rho) = 1$. Fabes and Rivière [14] showed that $(\mathbb{R}^n,\rho)$ is a metric space which is often called the mixed homogeneity space related to $\{\alpha_j\}_{j=1}^n$. For $\lambda > 0$, we let A_λ be the diagonal $n\times n$ matrix $A_\lambda = \mathrm{diag}\{\lambda^{\alpha_1},\dots,\lambda^{\alpha_n}\}$. Let $\mathbb{R}^+ := (0,\infty)$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, we denote $A_{\varphi(\rho(y))}y'$ by $A_\varphi(y)$ for $y \in \mathbb{R}^n$, where $y' = A_{\rho(y)^{-1}}y \in S^{n-1}$.

The change of variables related to the spaces $(\mathbb{R}^n,\rho)$ is given by the transformation

$$\begin{aligned}
x_1 &= \rho^{\alpha_1}\cos\theta_1\cdots\cos\theta_{n-2}\cos\theta_{n-1},\\
x_2 &= \rho^{\alpha_2}\cos\theta_1\cdots\cos\theta_{n-2}\sin\theta_{n-1},\\
&\cdots\cdots\cdots,\\
x_{n-1} &= \rho^{\alpha_{n-1}}\cos\theta_1\sin\theta_2,\\
x_n &= \rho^{\alpha_n}\sin\theta_1.
\end{aligned}$$

Thus $dx = \rho^{\alpha-1}J(x')d\rho d\sigma(x')$, where $\rho^{\alpha-1}J(x')$ is the Jacobian of the above transform and $\alpha = \sum_{j=1}^n \alpha_j$, $J(x') = \sum_{j=1}^n \alpha_j (x_j')^2$. Obviously, $J(x') \in \mathscr{C}^\infty(S^{n-1})$ and there exists $M > 0$ such that

$$1 \leqslant J(x') \leqslant M, \quad \forall\, x' \in S^{n-1}.$$

Mathematics subject classification (2010): 42B20, 42B25, 42B99.
Keywords and phrases: Marcinkiewicz integrals, rough kernel, mixed homogeneity, maximal operators, Littlewood-Paley theory.
Supported by the NNSF of China (11101339, 11371295).

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It is easy to see that

$$\rho(x) = |x|, \quad \text{if } \alpha_1 = \alpha_2 = \ldots = \alpha_n = 1.$$

Let Ω be integrable on S^{n-1} and satisfy

$$\int_{S^{n-1}} \Omega(u)J(u)d\sigma(u) = 0, \tag{1.1}$$

$$\Omega(A_s x) = \Omega(x), \ \forall\, s > 0 \text{ and } x \in \mathbb{R}^n. \tag{1.2}$$

Define the parabolic Marcinkiewicz integral operator $\mathscr{M}_{h,\Omega,\rho}$ by

$$\mathscr{M}_{h,\Omega,\rho}(f)(x) = \Big(\int_0^\infty \Big|\frac{1}{t^\rho}\int_{\rho(y)\leqslant t} \frac{\Omega(y)h(\rho(y))}{\rho(y)^{\alpha-\rho}} f(x-y)dy\Big|^2 \frac{dt}{t}\Big)^{1/2}, \ x \in \mathbb{R}^n, \tag{1.3}$$

where $\rho = \sigma + i\tau(\sigma, \tau \in \mathbb{R}$ with $\sigma > 0)$ and $h \in \Delta_1(\mathbb{R}^+)$. Here $\Delta_\gamma(\mathbb{R}^+)$ for $\gamma \geqslant 1$ denotes the set of all measurable functions h on $\mathbb{R}^+$ satisfying the condition

$$\|h\|_{\Delta_\gamma(\mathbb{R}^+)} = \sup_{R>0}\Big(R^{-1}\int_0^R |h(t)|^\gamma dt\Big)^{1/\gamma} < \infty.$$

It is easy to check that $L^\infty(\mathbb{R}^+) = \Delta_\infty(\mathbb{R}^+) \subsetneq \Delta_{\gamma_2}(\mathbb{R}^+) \subsetneq \Delta_{\gamma_1}(\mathbb{R}^+)$ for $0 < \gamma_1 < \gamma_2 < \infty$.

When $h(t) \equiv 1$, we denote $\mathscr{M}_{h,\Omega,\rho}$ by $\mathscr{M}_{\Omega,\rho}$. When $\alpha_1 = \ldots = \alpha_n = 1$, the operator $\mathscr{M}_{\Omega,\rho}$ reduces to the classical parametric Marcinkiewicz integral operator denoted by $\mu_{\Omega,\rho}$, which has been studied by many authors (see [1, 11, 18] et al.). When $\rho = 1$, $\mu_{\Omega,\rho}$ is the classical Marcinkiewicz integral operator, which was introduced by Stein [23] and investigated by many authors (see [2, 4, 12, 22, 26] for example). When $\alpha_j \geqslant 1$ $(j = 1, \ldots, n)$ and $\rho = 1$, Xue, Ding and Yabuta [27] first proved that $\mathscr{M}_{\Omega,\rho}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, provided that $\Omega \in L^q(S^{n-1})$ for fixed $q > 1$. Afterwards, Chen and Ding [5] (resp., [6]) extended the above result to the case $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$ (resp., $\Omega \in H^1(S^{n-1})$). Moreover, it follows from Wang, Chen and Yu's work [24] (also see [3]) that $\mathscr{M}_{\Omega,\rho}$ is of type (p, p) for $2\beta/(2\beta - 1) < p < 2\beta$ if $\Omega \in \mathscr{F}_\beta(S^{n-1})$ for some $\beta > 1$, where

$$\mathscr{F}_\beta(S^{n-1}) := \Big\{\Omega \in L^1(S^{n-1}) : \sup_{\xi \in S^{n-1}} \int_{S^{n-1}} |\Omega(y')|\Big(\log\frac{1}{|\xi \cdot y'|}\Big)^\beta d\sigma(y') < \infty\Big\}, \ \forall\, \beta > 0, \tag{1.4}$$

which was introduced by Grafakos and Stefanov [16] in the study of L^p bounds for singular integral operator with rough kernels. It follows from [16] that $\mathscr{F}_{\beta_1}(S^{n-1}) \subsetneq \mathscr{F}_{\beta_2}(S^{n-1})$ for $0 < \beta_2 < \beta_1$, and $\bigcup_{q>1} L^q(S^{n-1}) \subsetneq \mathscr{F}_\beta(S^{n-1})$ for any $\beta > 0$. Moreover,

$$\bigcap_{\beta>1} \mathscr{F}_\beta(S^{n-1}) \nsubseteq H^1(S^{n-1}) \nsubseteq \bigcup_{\beta>1} \mathscr{F}_\beta(S^{n-1})$$

and

$$\bigcap_{\beta>1} \mathscr{F}_\beta(S^{n-1}) \nsubseteq L\log^+ L(S^{n-1}).$$

Recently, we [19] improved the result of [24] to the case: $\Omega \in \mathscr{F}_\beta(S^{n-1})$ for some $\beta > 1/2$ with $1+1/(2\beta) < p < 1+2\beta$.

In this paper, we will focus on the general operator $\mathscr{M}_{h,\Omega,\rho}$ with $h \in \Delta_\gamma(\mathbb{R}^+)$. Due to the presence of h, the kernel of $\mathscr{M}_{h,\Omega,\rho}$ has the additional roughness in the radial direction, which has received a large amount of attention of many authors in the Euclidean setting, for example, see [10] for the case $\Omega \in L(\log^+ L)(S^{n-1})$, [8, 9, 11] for the case $\Omega \in H^1(S^{n-1})$, [1] for the case $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$ or certain block spaces on S^{n-1}. On the other hand, in the Euclidean setting, to extend the results in [16] to the singular integral operator with rough kernel both on the unit sphere and in the radial direction

$$T_{h,\Omega}f(x) := \text{p.v.} \int_{\mathbb{R}^n} \frac{h(|y|)\Omega(y')}{|y|^n} f(x-y)dy, \;\; x \in \mathbb{R}^n.$$

Fan and Sato [15] introduced the functions class $\tilde{\mathscr{F}}_\beta(S^{n-1})$ in more general form, which denotes the set of all functions $\Omega \in L^1(S^{n-1})$ with satisfying

$$\sup_{\xi' \in S^{n-1}} \iint_{S^{n-1}\times S^{n-1}} |\Omega(\theta)\Omega(w)| \Big(\log \frac{1}{|(\theta - w)\cdot \xi'|}\Big)^\beta d\sigma(\theta)d\sigma(w) < \infty, \quad \beta > 0, \tag{1.5}$$

and showed that $\mathscr{F}_\beta(S^1) \subset \tilde{\mathscr{F}}_\beta(S^1)$ (for $n > 2$, the relation between $\mathscr{F}_\beta(S^{n-1})$ and $\tilde{\mathscr{F}}_\beta(S^{n-1})$ remains to be open). Moreover, they proved that $T_{h,\Omega}$ is bounded on $L^p(\mathbb{R}^n)$ provided that $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$ and $\Omega \in \tilde{\mathscr{F}}_\beta(S^{n-1})$ for $\beta > \max\{2,\gamma'\}$ and $|1/p - 1/2| < \min\{1/\gamma', 1/2\} - 1/\beta$. Recently, we [21] extended the result of [15] to the singular integral operators with mixed homogeneity in more general form, which is listed as follows.

THEOREM A. ([21]) *Let P_N be a real polynomial on $\mathbb{R}$ with $P_N(0) = 0$ and $P_N(t) > 0$ for $t \neq 0$, where N is the degree of P_N, and let $\varphi \in \mathfrak{F}$. Here $\mathfrak{F}$ is the set of all functions ϕ which satisfy:*

(a) *$\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous increasing $\mathscr{C}^1$ function satisfying that ϕ' is monotonic;*

(b) *there exist constants C_ϕ and c_ϕ such that $t\phi'(t) \geqslant C_\phi \phi(t)$ and $\phi(2t) \leqslant c_\phi \phi(t)$ for all $t > 0$.*

Suppose that $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$ and $\Omega \in \tilde{\mathscr{F}}_\beta(S^{n-1})$ for some $\beta > \max\{2,\gamma'\}$ satisfying (1.1)–(1.2). Then the singular integral operators $T_{h,\Omega,P_N,\varphi}$ defined by

$$T_{h,\Omega,P_N,\varphi}(f)(x) := \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y)h(\rho(y))}{\rho(y)^\alpha} f(x - A_{P_N(\varphi)}(y))dy, \;\; x \in \mathbb{R}^n, \tag{1.6}$$

are bounded on $L^p(\mathbb{R}^n)$ for p with satisfying $|1/p - 1/2| < \min\{1/\gamma', 1/2\} - 1/\beta$. The bounds are independent of the coefficients of P_N, but depend on φ, N, γ, n and β.

REMARK 1. We remark that there are some model examples in the class $\mathfrak{F}$, such as t^α $(\alpha > 0)$, $t^\alpha(\ln(1+t))^\beta$ $(\alpha,\beta > 0)$, $t\ln\ln(e+t)$, real-valued polynomials P on $\mathbb{R}$ with positive coefficients and $P(0) = 0$ and so on. For $\varphi \in \mathfrak{F}$, there exists a constant $B_\varphi > 1$ such that $\varphi(2t) \geqslant B_\varphi \varphi(t)$ (see [3]).

Based on the above, we find it is natural to ask the following question.

QUESTION. *For the general case* $\alpha_j \geqslant 1\,(j=1,\ldots,n)$, *is* $\mathscr{M}_{h,\Omega,\rho}$ *bounded on* $L^p(\mathbb{R}^n)$ *under the same assumptions on* Ω *and* h *as in Theorem A?*

In this paper, we will give an affirmative answer to this question by treating a family of operators, which is broader than $\mathscr{M}_{h,\Omega,\rho}$. Precisely, let h, Ω, ρ be as in (1.3). For a suitable mapping $\Phi:\mathbb{R}^n\to\mathbb{R}^n$, we define the parabolic Marcinkiewicz integral operator $\mathscr{M}_{h,\Omega,\Phi,\rho}$ on $\mathbb{R}^n$ by

$$\mathscr{M}_{h,\Omega,\Phi,\rho}(f)(x) := \Big(\int_0^\infty |F_{h,\Omega,\Phi,\rho}(f)(x,t)|^2\frac{dt}{t}\Big)^{1/2},\ \ x\in\mathbb{R}^n, \tag{1.7}$$

where

$$F_{h,\Omega,\Phi,\rho}(f)(x,t) := \frac{1}{t^\rho}\int_{\rho(y)\leqslant t}\frac{\Omega(y)h(\rho(y))}{\rho(y)^{\alpha-\rho}}f(x-\Phi(y))dy.$$

Clearly, $\mathscr{M}_{h,\Omega,\rho}$ is the special case of $\mathscr{M}_{h,\Omega,\Phi,\rho}$ for $\Phi(y)=y$. Our main results can be stated as follows:

THEOREM 1. *Let* $\Phi(y)=(P_1(\varphi(\rho(y)))y_1',\ldots,P_n(\varphi(\rho(y)))y_n')$ *with* $P_j(t)$ *being real valued polynomials on* $\mathbb{R}$ *satisfying* $P_j(0)=0$ *and* $\varphi\in\mathfrak{F}$. *Suppose that* $h\in\Delta_\gamma(\mathbb{R}^+)$ *for some* $\gamma>1$ *and* $\Omega\in\tilde{\mathscr{F}}_\beta(S^{n-1})$ *for some* $\beta>\max\{2,\gamma'\}/2$ *satisfying* (1.1)–(1.2). *Then* $\mathscr{M}_{h,\Omega,\Phi,\rho}$ *defined as in* (1.7) *are bounded on* $L^p(\mathbb{R}^n)$ *for* p *with satisfying* $|1/p-1/2|<\min\{1/\gamma',1/2\}-\min\{1/\gamma'+1/2,1\}/(\beta+1)$. *The bounds are independent of the coefficients of* P_j *for all* $1\leqslant j\leqslant n$, *but depend on* $\max_{1\leqslant j\leqslant n}\deg(P_j)$, ρ *and* φ.

In particular, when $P_j(\varphi(\rho(y)))y_j' = P_N(\varphi(\rho(y)))^{\alpha_j}y_j'$ $(1\leqslant j\leqslant n)$, we have

THEOREM 2. *Let* $\Phi(y)=A_{P_N(\varphi)}(y)$ *with* $\varphi\in\mathfrak{F}$ *and* $P_N(t)=\sum_{i=1}^N a_it^i$ *with* $P_N(t)>0$ *for* $t\neq0$. *Suppose that* $h\in\Delta_\gamma(\mathbb{R}^+)$ *for some* $\gamma>1$ *and* $\Omega\in\tilde{\mathscr{F}}_\beta(S^{n-1})$ *for some* $\beta>\max\{2,\gamma'\}/2$ *satisfying* (1.1)–(1.2). *Then* $\mathscr{M}_{h,\Omega,\Phi,\rho}$ *defined as in* (1.7) *are bounded on* $L^p(\mathbb{R}^n)$ *for* p *with satisfying* $|1/p-1/2|<\min\{1/\gamma',1/2\}-\min\{1/\gamma'+1/2,1\}/(\beta+1)$. *The bounds are independent of the coefficients of* P_N, *but depend on* N, ρ *and* φ.

Furthermore, applying Theorems 1–2 with the fact that $\mathscr{F}_\beta(S^1)\subset\tilde{\mathscr{F}}_\beta(S^1)$, we obtain

THEOREM 3. *Let* $n=2$ *and* Φ, h *be as in Theorem* 1. *Suppose that* $\Omega\in\mathscr{F}_\beta(S^1)$ *for some* $\beta>\max\{2,\gamma'\}/2$ *satisfying* (1.1)–(1.2). *Then* $\mathscr{M}_{h,\Omega,\Phi,\rho}$ *defined as in* (1.7) *are bounded on* $L^p(\mathbb{R}^2)$ *for* p *with satisfying* $|1/p-1/2|<\min\{1/\gamma',1/2\}-\min\{1/\gamma'+1/2,1\}/(\beta+1)$. *The bounds are independent of the coefficients of* P_j *for* $j=1,2$, *but depend on* $\max_{1\leqslant j\leqslant 2}\deg(P_j)$, ρ *and* φ.

THEOREM 4. *Let $n=2$ and Φ, h be as in Theorem 2. Suppose that $\Omega \in \mathscr{F}_\beta(S^1)$ for some $\beta > \max\{2,\gamma'\}/2$ satisfying (1.1)–(1.2). Then $\mathscr{M}_{h,\Omega,\Phi,\rho}$ defined as in (1.7) are bounded on $L^p(\mathbb{R}^2)$ for p with satisfying $|1/p-1/2| < \min\{1/\gamma',1/2\} - \min\{1/\gamma'+1/2,1\}/(\beta+1)$. The bounds are independent of the coefficients of P_N, but depend on N, ρ and φ.*

REMARK 2. Compared with Theorem A, in our theorems, the range of β is relaxed to that $\beta > \max\{1, \gamma'/2\}$ and for the same β and γ, the range of p is larger than one in Theorem A. However, we don't know whether the ranges of β and p are sharp, which is interesting.

The rest of this paper is organized as follows. After recalling some preliminary notations and lemmas in Section 2, we will prove the main results in Section 3. Finally, we consider the L^p bounds of the corresponding parametric Marcinkiewicz integral operators related to area integrals and Littlewood-Paley g_λ^* functions in Section 4. We would like to remark that the main methods employed in this paper is a combination of ideas and arguments from [7, 25]. Throughout this paper, we let p' satisfy $1/p+1/p'=1$. The letter C, sometimes with additional parameters, will stand for positive constants, not necessarily the same one at each occurrence but is independent of the essential variables.

2. Preliminary lemmas

Let $\mathscr{N} = \max_{1\leqslant j\leqslant n} \deg(P_j)$. For $1 \leqslant l \leqslant n$, let $P_l(t) = \sum_{i=1}^{\mathscr{N}} a_{i,l} t^i$. For $1 \leqslant s \leqslant \mathscr{N}$ and $1 \leqslant l \leqslant n$, let $P_l^{(s)}(t) = \sum_{i=1}^{s} a_{i,l} t^i$ and $P^{(s)}(t) = (P_1^{(s)}(t), \dots, P_n^{(s)}(t))$. Set $P^{(0)}(t) = 0$ and

$$\Phi_s(y) = (P_1^{(s)}(\varphi(\rho(y)))y'_1, \dots, P_n^{(s)}(\varphi(\rho(y)))y'_n).$$

Then we can write

$$\Phi_s(y)\cdot\xi = \sum_{l=1}^{n} \xi_l y'_l P_l^{(s)}(\varphi(\rho(y))) = \sum_{l=1}^{n}\sum_{i=1}^{s} \xi_l y'_l a_{i,l} \varphi(\rho(y))^i = \sum_{i=1}^{s} (L_i\xi\cdot y')\varphi(\rho(y))^i, \tag{2.1}$$

where $L_i : \mathbb{R}^n \to \mathbb{R}^n$ is the linear transformation given by

$$L_i\xi = (a_{i,1}\xi_1, \dots, a_{i,n}\xi_n).$$

For each $k \in \mathbb{Z}$, $t \in \mathbb{R}^+$ and $1 \leqslant s \leqslant \mathscr{N}$, we define the signed measures $\{\sigma_{k,t;s}\}$ and $\{|\sigma_{k,t;s}|\}$ on $\mathbb{R}^n$ by

$$\widehat{\sigma_{k,t;s}}(\xi) = \frac{1}{(2^k t)^\rho} \int_{2^{k-1}t<\rho(y)\leqslant 2^k t} \exp(-2\pi i \Phi_s(y)\cdot\xi) \frac{\Omega(y)h(\rho(y))}{\rho(y)^{\alpha-\rho}} dy;$$

$$\widehat{|\sigma_{k,t;s}|}(\xi) = \frac{1}{(2^k t)^\rho} \int_{2^{k-1}t<\rho(y)\leqslant 2^k t} \exp(-2\pi i \Phi_s(y)\cdot\xi) \frac{|\Omega(y)h(\rho(y))|}{\rho(y)^{\alpha-\rho}} dy.$$

LEMMA 1. *([22]) Suppose $\Phi(t) = t^{\alpha_1} + \mu_2 t^{\alpha_2} + \cdots + \mu_n t^{\alpha_n}$ and $\varphi \in \mathfrak{F}$, where $\mu_2, \ldots, \mu_n$ are real parameters, and $\alpha_1, \ldots, \alpha_n$ are distinct positive (not necessarily integer) exponents. Then for any $r > 0$ and $\lambda \in \mathbb{R}\backslash\{0\}$,*

$$\Big| \int_{r/2}^{r} \exp(i\lambda\Phi(\varphi(t)))\frac{dt}{t}\Big| \leqslant C(\varphi)|\lambda\varphi(r)^{\alpha_1}|^{-\varepsilon},$$

where $\varepsilon = \min\{1/\alpha_1, 1/n\}$ and $C(\varphi)$ doesn't depend on $\mu_2, \ldots, \mu_n$.

LEMMA 2. *Let $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $1 < \gamma \leqslant \infty$ and $\tilde{\gamma} = \max\{2, \gamma'\}$. Suppose that $\Omega \in \tilde{\mathscr{F}}_\beta(S^{n-1})$ for some $\beta > 0$ and satisfies (1.1)–(1.2). Then for any $k \in \mathbb{Z}$, $\xi \in \mathbb{R}^n$, $t > 0$ and $1 \leqslant s \leqslant \mathscr{N}$, there exists a positive constant C depends on φ such that*

(i) $\big|\widehat{\sigma_{k,t;s}}(\xi) - \widehat{\sigma_{k,t;s-1}}(\xi)\big| \leqslant C|\varphi(2^k t)^s L_s \xi|$;

(ii) $|\widehat{\sigma_{k,t;s}}(\xi)| \leqslant C(\log|\varphi(2^k t)^s L_s \xi|)^{-\beta/\tilde{\gamma}}$, if $|\varphi(2^k t)^s L_s \xi| > 1$.

Proof. By (2.1) and a change of variable, we have

$$\begin{aligned}
&\big|\widehat{\sigma_{k,t;s}}(\xi) - \widehat{\sigma_{k,t;s-1}}(\xi)\big| \\
&\leqslant \frac{1}{(2^k t)^\rho}\int_{2^{k-1}t<\rho(y)\leqslant 2^k t} |\exp(-2\pi i\Phi_s(y)\cdot\xi) - \exp(-2\pi i\Phi_{s-1}(y)\cdot\xi)|\frac{|\Omega(y)h(\rho(y))|}{\rho(y)^{\alpha-\rho}}dy \\
&\leqslant C|\varphi(2^k t)^s L_s \xi| \textstyle\int_{2^{k-1}t}^{2^k t} |h(r)|\frac{dr}{r}\int_{S^{n-1}} |\Omega(\theta)| d\sigma(\theta) \\
&\leqslant C|\varphi(2^k t)^s L_s \xi|.
\end{aligned}$$

Then (i) holds. On the other hand, by a change of variable and Hölder's inequality, we have

$$\begin{aligned}
|\widehat{\sigma_{k,t;s}}(\xi)| &= \Big|\frac{1}{(2^k t)^\rho}\int_{2^{k-1}t}^{2^k t}\int_{S^{n-1}} \exp\Big(-2\pi i\sum_{\eta=1}^{s} L_\eta\xi\cdot\theta\varphi(r)^\eta\Big)\Omega(\theta)J(\theta)d\sigma(\theta)h(r)\frac{dr}{r^{1-\rho}}\Big| \\
&\leqslant C\int_{2^{k-1}t}^{2^k t}\Big|\int_{S^{n-1}} \exp\Big(-2\pi i\sum_{\eta=1}^{s} L_\eta\xi\cdot\theta\varphi(r)^\eta\Big)\Omega(\theta)J(\theta)d\sigma(\theta)\Big||h(r)|\frac{dr}{r} \\
&\leqslant C\Big(\int_{2^{k-1}t}^{2^k t}\Big|\int_{S^{n-1}} \exp\Big(-2\pi i\sum_{\eta=1}^{s} L_\eta\xi\cdot\theta\varphi(r)^\eta\Big)\Omega(\theta)J(\theta)d\sigma(\theta)\Big|^{\gamma'}\frac{dr}{r}\Big)^{1/\gamma'}.
\end{aligned} \tag{2.2}$$

If $1 < \gamma \leqslant 2$, note that $\gamma' \geqslant 2$, we get from (2.2) that

$$\begin{aligned}
|\widehat{\sigma_{k,t;s}}(\xi)| &\leqslant C\Big(\int_{2^{k-1}t}^{2^k t}\Big|\int_{S^{n-1}} \exp\Big(-2\pi i\sum_{\eta=1}^{s} L_\eta\xi\cdot\theta\varphi(r)^\eta\Big)\Omega(\theta)J(\theta)d\sigma(\theta)\Big|^2 \\
&\qquad\times\Big|\int_{S^{n-1}} \exp\Big(-2\pi i\sum_{\eta=1}^{s} L_\eta\xi\cdot\theta\varphi(r)^\eta\Big)\Omega(\theta)J(\theta)d\sigma(\theta)\Big|^{\gamma'-2}\frac{dr}{r}\Big)^{1/\gamma'} \\
&\leqslant C\Big(\int_{2^{k-1}t}^{2^k t}\Big|\int_{S^{n-1}} \exp\Big(-2\pi i\sum_{\eta=1}^{s} L_\eta\xi\cdot\theta\varphi(r)^\eta\Big)\Omega(\theta)J(\theta)d\sigma(\theta)\Big|^2\frac{dr}{r}\Big)^{1/\gamma'}.
\end{aligned}$$

If $\gamma > 2$, then $\gamma' \in [1,2)$. By Hölder's inequality we have

$$|\widehat{\sigma_{k,t;s}}(\xi)| \leqslant C\Big(\int_{2^{k-1}t}^{2^k t}\Big|\int_{S^{n-1}} \exp\Big(-2\pi i\sum_{\eta=1}^{s} L_\eta\xi\cdot\theta\varphi(r)^\eta\Big)\Omega(\theta)J(\theta)d\sigma(\theta)\Big|^2\frac{dr}{r}\Big)^{1/2}.$$

Thus we have

$$|\widehat{\sigma_{k,t;s}}(\xi)| \leqslant C\Big(\int_{2^{k-1}t}^{2^k t}\Big|\int_{S^{n-1}}\exp\Big(-2\pi i\sum_{\eta=1}^{s}L_\eta\xi\cdot\theta\varphi(r)^\eta\Big)\Omega(\theta)J(\theta)d\sigma(\theta)\Big|^2\frac{dr}{r}\Big)^{1/\tilde{\gamma}}, \tag{2.3}$$

Let

$$I_{k,t,s}(\xi) := \int_{2^{k-1}t}^{2^k t}\Big|\int_{S^{n-1}}\exp\Big(-2\pi i\sum_{\eta=1}^{s}L_\eta\xi\cdot\theta\varphi(r)^\eta\Big)\Omega(\theta)J(\theta)d\sigma(\theta)\Big|^2\frac{dr}{r}.$$

We can write

$$\begin{aligned}
&|I_{k,t,s}(\xi)|\\
&=\Big|\int_{2^{k-1}t}^{2^k t}\iint_{(S^{n-1})^2}\exp\Big(-2\pi i\sum_{\eta=1}^{s}L_\eta\xi\cdot(\theta-w)\varphi(r)^\eta\Big)\Omega(\theta)\overline{\Omega(w)}J(\theta)\overline{J(w)}d\sigma(\theta)d\sigma(w)\tfrac{dr}{r}\Big|\\
&\leqslant\iint_{(S^{n-1})^2}|\Omega(\theta)\overline{\Omega(w)}|\Big|\int_{2^{k-1}t}^{2^k t}\exp\Big(-2\pi i\sum_{\eta=1}^{s}L_\eta\xi\cdot(\theta-w)\varphi(r)^\eta\Big)\frac{dr}{r}\Big|d\sigma(\theta)d\sigma(w).
\end{aligned} \tag{2.4}$$

Let

$$\tilde{I}_{k,t,s,\xi}(\theta,w) := \int_{2^{k-1}t}^{2^k t}\exp\Big(-2\pi i\sum_{\eta=1}^{s}L_\eta\xi\cdot(\theta-w)\varphi(r)^\eta\Big)\frac{dr}{r}.$$

Applying Lemma 1, we have

$$|\tilde{I}_{k,t,s,\xi}(\theta,w)| \leqslant C(\varphi)|\varphi(2^k t)^s L_s\xi\cdot(\theta-w)|^{-1/s}.$$

Combining the trivial inequality $|\tilde{I}_{k,t,s,\xi}(\theta,w)| \leqslant C$ with the fact that $t/(\log t)^\beta$ is increasing in (e^β,∞), we have

$$|\tilde{I}_{k,t,s,\xi}(\theta,w)| \leqslant C(\varphi)\frac{\big(\log 2e^{\beta s}|L_s'\xi\cdot(\theta-w)|^{-1}\big)^\beta}{\big(\log|\varphi(2^k t)^s L_s\xi|\big)^\beta},\ \text{ if } |\varphi(2^k t)^s L_s\xi|>1, \tag{2.5}$$

where $L_s'\xi = L_s\xi/|L_s\xi|$. This together (2.4) with the fact that $\Omega\in\tilde{\mathscr{F}}_\beta(S^{n-1})$ implies

$$|I_{k,t,s}(\xi)| \leqslant C(\varphi)(\log|\varphi(2^k t)^s L_s\xi|)^{-\beta},\ \text{ if } |\varphi(2^k t)^s L_s\xi|>1. \tag{2.6}$$

Together with (2.3) leads to (ii). This proves Lemma 2. □

LEMMA 3. *Let $\mathscr{P}:\mathbb{R}^+\to\mathbb{R}^n$ be a polynomial mapping, where $\mathscr{P}(t)=(P_1(t),\dots,P_n(t))$ with P_j being are real polynomial defined on $\mathbb{R}^+$. Suppose that $h\in\Delta_\gamma(\mathbb{R}^+)$ for some $\gamma>1$ and $\varphi\in\mathfrak{F}$, then the operator $\mathscr{M}_{\mathscr{P},\varphi,h}$ defined as*

$$\mathscr{M}_{\mathscr{P},\varphi,h}(f)(x) := \sup_{r>0}\Big|\int_r^{2r} f(x-\mathscr{P}(\varphi(t)))h(t)\frac{dt}{t}\Big|$$

satisfies

$$\|\mathscr{M}_{\mathscr{P},\varphi,h}(f)\|_{L^p(\mathbb{R}^n)} \leqslant C_p\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|f\|_{L^p(\mathbb{R}^n)},\quad \gamma'<p\leqslant\infty.$$

The constant C_p is independent of the coefficients of P_j for $1\leqslant j\leqslant n$, but depends on φ.

Proof. By Hölder's inequality we have

$$\Big|\int_r^{2r}|f(x-\mathscr{P}(\varphi(t)))h(t)\frac{dt}{t}\Big|\leqslant C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\Big(\int_r^{2r}|f(x-\mathscr{P}(\varphi(t)))|^{\gamma'}\frac{dt}{t}\Big)^{1/\gamma'}.$$

From this and invoking Lemma 2.2 in [20], Lemma 3 is obtained. □

LEMMA 4. *Let* $\Omega\in L^1(S^{n-1})$ *and* $h\in\Delta_\gamma(\mathbb{R}^+)$ *for some* $\gamma>1$. *Suppose that* $\varphi\in\mathfrak{F}$. *Then for* $|1/p-1/2|<\min\{1/2,1/\gamma'\}$ *and* $1\leqslant s\leqslant\mathscr{N}$, *we have*

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\sigma_{k,t;s}*g_k|^2dt\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)};\tag{2.7}$$

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\sigma_{k,t;s}*g_k|^2dt\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)},\quad\forall\,1\leqslant t\leqslant 2.\tag{2.8}$$

The constant C *is independent of the coefficients of* P_j *for* $1\leqslant j\leqslant n$, *but depends on* φ.

Proof. To prove this lemma, we use a similar argument as in the proof of [13, Theorem 7.5]. Since $\Delta_\gamma(\mathbb{R}^+)\subset\Delta_2(\mathbb{R}^+)$ for $\gamma\geqslant 2$, we only prove this lemma for the case $1<\gamma\leqslant 2$ and $|1/p-1/2|<1/\gamma'$. By the duality, it suffices to prove (2.7) for $2<p<2\gamma/(2-\gamma)$. Let $q=(p/2)'$ and $\{g_k\}_{k\in\mathbb{Z}}\in L^p(\mathbb{R}^n,\ell^2)$. Then there exists a nonnegative function $f\in L^q(\mathbb{R}^n)$ with unit norm such that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\sigma_{k,t;s}*g_k|^2dt\Big)^{1/2}\Big\|^2_{L^p(\mathbb{R}^n)}=\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\int_1^2|\sigma_{k,t;s}*g_k(x)|^2dt\,f(x)dx.\tag{2.9}$$

By a change of variable and Hölder's inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_1^2|\sigma_{k,t;s}*g_k(x)|^2dt\,f(x)dx\\ &\leqslant\int_{\mathbb{R}^n}\int_1^2\Big(\int_{2^{k-1}t}^{2^kt}\int_{S^{n-1}}|g_k(x-\Phi_s(A_ry'))||\Omega(y')|d\sigma(y')|h(r)|\frac{dr}{r}\Big)^2dt\,f(x)dx\\ &\leqslant C\|\Omega\|_{L^1(S^{n-1})}\int_{\mathbb{R}^n}\int_1^2\Big(\int_{2^{k-1}t}^{2^kt}\Big(\int_{S^{n-1}}|g_k(x-\Phi_s(A_ry'))|^2|\Omega(y')|d\sigma(y')\Big)^{1/2}|h(r)|\frac{dr}{r}\Big)^2dt\,f(x)dx\\ &\leqslant C\|h\|^{\gamma}_{\Delta_\gamma(\mathbb{R}^+)}\int_{\mathbb{R}^n}\int_1^2\int_{2^{k-1}t}^{2^kt}\int_{S^{n-1}}|g_k(x-\Phi_s(A_ry'))|^2|\Omega(y')|d\sigma(y')|h(r)|^{2-\gamma}\frac{dr}{r}dt\,f(x)dx\\ &\leqslant C\|h\|^{\gamma}_{\Delta_\gamma(\mathbb{R}^+)}\int_{\mathbb{R}^n}\tilde{M}(f)(x)|g_k(x)|^2dx,\end{aligned}\tag{2.10}$$

where

$$\tilde{M}(f)(x)=\int_1^2\int_{2^{k-1}t}^{2^kt}\int_{S^{n-1}}f(x+\Phi_s(A_ry'))|h(r)|^{2-\gamma}|\Omega(y')|d\sigma(y')\frac{dr}{r}dt.$$

Note that $|h(\cdot)|^{2-\gamma}\in\Delta_{\gamma/(2-\gamma)}(\mathbb{R}^+)$ and $q>(\gamma/(2-\gamma))'$. Using Lemma 3, Minkowski's inequality and the similar arguments as in getting [13, Theorem 7.4], we have

$$\|\tilde{M}(f)\|_{L^q(\mathbb{R}^n)}\leqslant C_q\|\Omega\|_{L^1(S^{n-1})}\|h\|^{2-\gamma}_{\Delta_\gamma(\mathbb{R}^+)}\|f\|_{L^q(\mathbb{R}^n)}.\tag{2.11}$$

Combining (2.9)–(2.11) with Hölder's inequality implies

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\sigma_{k,t;s}*g_k|^2dt\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}\leqslant C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}$$

for $2<p<2\gamma/(2-\gamma)$. This proves (2.7). It remains to prove (2.8). By the similar argument as in getting (2.7), it suffices to prove (2.8) for $2<p<2\gamma/(2-\gamma)$. Fixed $t\in[1,2]$, let $q=(p/2)'$ and $\{g_k\}_{k\in\mathbb{Z}}\in L^p(\mathbb{R}^n,\ell^2)$. Then there exists a nonnegative function $f\in L^q(\mathbb{R}^n)$ with unit norm such that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\sigma_{k,t;s}*g_k|^2\Big)^{1/2}\Big\|^2_{L^p(\mathbb{R}^n)}=\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}|\sigma_{k,t;s}*g_k(x)|^2f(x)dx. \tag{2.12}$$

By a change of variable and Hölder's inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}|\sigma_{k,t;s}*g_k(x)|^2f(x)dx\\
&\leqslant\int_{\mathbb{R}^n}\Big|\int_{2^{k-1}t}^{2^kt}\int_{S^{n-1}}|g_k(x-\Phi_s(A_ry'))||\Omega(y')|d\sigma(y')|h(r)|\frac{dr}{r}\Big|^2f(x)dx\\
&\leqslant C\|\Omega\|_{L^1(S^{n-1})}\int_{\mathbb{R}^n}\Big|\int_{2^{k-1}t}^{2^kt}\Big(\int_{S^{n-1}}|g_k(x-\Phi_s(A_ry'))|^2|\Omega(y')|d\sigma(y')\Big)^{1/2}|h(r)|\frac{dr}{r}\Big|^2f(x)dx\\
&\leqslant C\|h\|^{\gamma}_{\Delta_\gamma(\mathbb{R}^+)}\int_{\mathbb{R}^n}\int_{2^{k-1}t}^{2^kt}\int_{S^{n-1}}|g_k(x-\Phi_s(A_ry'))|^2|\Omega(y')|d\sigma(y')|h(r)|^{2-\gamma}\frac{dr}{r}f(x)dx\\
&\leqslant C\|h\|^{\gamma}_{\Delta_\gamma(\mathbb{R}^+)}\int_{\mathbb{R}^n}\tilde{G}(f)(x)|g_k(x)|^2dx,
\end{aligned}$$

where

$$\tilde{G}(f)(x)=\int_{2^{k-1}t}^{2^kt}\int_{S^{n-1}}f(x+\Phi_s(A_ry'))|h(r)|^{2-\gamma}|\Omega(y')|d\sigma(y')\frac{dr}{r}.$$

Thus

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\sigma_{k,t;s}*g_k|^2\Big)^{1/2}\Big\|^2_{L^p(\mathbb{R}^n)}\leqslant C\|h\|^{\gamma}_{\Delta_\gamma(\mathbb{R}^+)}\int_{\mathbb{R}^n}\tilde{G}(f)(x)\sum_{k\in\mathbb{Z}}|g_k(x)|^2dx. \tag{2.13}$$

By the similar argument as in getting (2.11), we have

$$\|\tilde{G}(f)\|_{L^q(\mathbb{R}^n)}\leqslant C_q\|\Omega\|_{L^1(S^{n-1})}\|h\|^{2-\gamma}_{\Delta_\gamma(\mathbb{R}^+)}\|f\|_{L^q(\mathbb{R}^n)}. \tag{2.14}$$

Combining (2.13) with Hölder's inequality implies

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\sigma_{k,t;s}*g_k|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}\leqslant C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)},$$

for $2<p<2\gamma/(2-\gamma)$, where C is independent of t. This proves (2.8). Lemma 4 is proved. □

Let $\{\lambda_k\}_{k\in\mathbb{Z}}$ be a collection of $\mathscr{C}^\infty(0,\infty)$ functions with satisfying the following conditions:

$$\mathrm{supp}(\lambda_k)\subset[\varphi(2^{k+1})^{-1},\varphi(2^{k-1})^{-1}];$$
$$0\leqslant\lambda_k\leqslant 1;\ \sum_{k\in\mathbb{Z}}\lambda_k^2(t)=1;\ |d\lambda_k(t)/dt|\leqslant C/t,$$

where C is independent of t and k. For each $k\in\mathbb{Z}$, we define the multiplier operators S_k in $\mathbb{R}^n$ by

$$\widehat{S_kf}(\xi)=\lambda_k(|L_s\xi|)\hat{f}(\xi). \tag{2.15}$$

By the arguments similar to those used in [26, Proposition 3.1], one can easily get the following lemma. The details are omitted here.

LEMMA 5. *Let S_k be as in (2.15) and $\{g_{j,k,t}\}$ be arbitrary functions in $L^p(\mathbb{R}^n)$. Then*

(i) *for each fixed $1<p<2$ and $1<q<p$,*

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\int_1^2\Big|\sum_{k\in\mathbb{Z}}S_{j+k}g_{j,k,t}\Big|^2dt\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}^q\leqslant C\sum_{k\in\mathbb{Z}}\Big\|\Big(\sum_{j\in\mathbb{Z}}\int_1^2|g_{j,k,t}|^2dt\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}^q; \tag{2.16}$$

(ii) *for each fixed $2<p<\infty$ and $1<q<p'$,*

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\int_1^2\Big|\sum_{k\in\mathbb{Z}}S_{j+k}g_{j,k,t}\Big|^2dt\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}^q\leqslant C\sum_{k\in\mathbb{Z}}\Big(\int_1^2\Big\|\Big(\sum_{j\in\mathbb{Z}}|g_{j,k,t}|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}^2dt\Big)^{q/2}. \tag{2.17}$$

In order to prove our results, we need the following lemma.

LEMMA 6. ([13]) *Let r and d be two positive integers and $\{v_1,\dots,v_d\}\subset\mathbb{R}^r$ be a collection of vectors which spans $\mathbb{R}^r$. Then there exists a subcollection $\{u_1,\dots,u_r\}\subset\{v_1,\dots,v_d\}$ and constants $\{k_{js}\}_{1\leqslant j\leqslant d,1\leqslant s\leqslant r}$ such that*

$$v_j=k_{j1}u_1+\cdots+k_{jr}u_r$$

for $j=1,\dots,d$.

3. Proofs of main results

This section is devoted to the proofs of main results. We need only to prove Theorem 1 in section 1.

Proof. Now we begin to prove Theorem 1. This proof is based on the ideas in [7] and some techniques from [25]. Let $\varphi\in\mathfrak{F}$ and B_φ be as in Remark 1. For simplicity, we denote $\tilde{\gamma}=\max\{2,\gamma'\}$. For $s\in\{1,\dots,\mathscr{N}\}$, let $r_s=\mathrm{rank}(L_s)$. By Lemma 6, there are two nonsingular linear transformations $H_s:\mathbb{R}^{r_s}\to\mathbb{R}^{r_s}$ and $G_s:\mathbb{R}^n\to\mathbb{R}^n$ such that

$$|H_s\pi_{r_s}^nG_s\xi|\leqslant|L_s\xi|\leqslant C_n|H_s\pi_{r_s}^nG_s\xi|,\ \text{ for }\xi\in\mathbb{R}^n, \tag{3.1}$$

where $C_n > 1$ and $\pi^n_{r_s}$ is a projection operator from $\mathbb{R}^n$ to $\mathbb{R}^{r_s}$. For a function $\phi \in \mathscr{C}_0^\infty(\mathbb{R})$ such that $\phi \equiv 1$ for $|t| \leqslant 1/2$ and $\phi \equiv 0$ for $|t| > 1$. Let $\psi(t) = \phi(t^2)$. For $k \in \mathbb{Z}, t \in \mathbb{R}^+, \xi \in \mathbb{R}^n$ and $1 \leqslant s \leqslant \mathscr{N}$, we define the signed measures $\{\mu_{k,t;s}\}$ by

$$\widehat{\mu_{k,t;s}}(\xi) = \widehat{\sigma_{k,t;s}}(\xi) \prod_{l=s+1}^{\mathscr{N}} \psi(|\varphi(2^k t)^l H_l \pi^n_{r_l} G_l \xi|) - \widehat{\sigma_{k,t;s-1}}(\xi) \prod_{l=s}^{\mathscr{N}} \psi(|\varphi(2^k t)^l H_l \pi^n_{r_l} G_l \xi|). \tag{3.2}$$

Here we use the convention $\prod_{j\in\emptyset} a_j = 1$. It is easy to see that

$$\sigma_{k,t;\mathscr{N}} = \sum_{s=1}^{\mathscr{N}} \mu_{k,t;s}. \tag{3.3}$$

It follows from Lemma 2, (3.1) and the trivial estimate $|\widehat{\sigma_{k,t;s}}(\xi)| \leqslant C$ that for $1 \leqslant s \leqslant \mathscr{N}$,

$$|\widehat{\mu_{k,t;s}}(\xi)| \leqslant C(\varphi) \min\{1, \varphi(2^k t)^s |L_s \xi|\}, \tag{3.4}$$

$$|\widehat{\mu_{k,t;s}}(\xi)| \leqslant C(\varphi)(\log|\varphi(2^k t)^s L_s \xi|)^{-\beta/\tilde{\gamma}}, \text{ if } |\varphi(2^k t)^s L_s \xi| > 1. \tag{3.5}$$

By the definition of $\sigma_{k,t;s}$, we have

$$F_{h,\Omega,\Phi,\rho}(f)(x,t) = \sum_{k=-\infty}^{0} 2^{k\rho} \sigma_{k,t;\mathscr{N}} * f(x). \tag{3.6}$$

Then by (3.3) and (3.6) we can write

$$\begin{aligned} \mathscr{M}_{h,\Omega,\Phi,\rho}(f)(x) &= \Big(\int_0^\infty \Big| \sum_{k=-\infty}^{0} 2^{k\rho} \sigma_{k,t;\mathscr{N}} * f(x) \Big|^2 \tfrac{dt}{t} \Big)^{1/2} \\ &\leqslant \sum_{k=-\infty}^{0} 2^{k\sigma} \Big(\int_0^\infty |\sigma_{k,t;\mathscr{N}} * f(x)|^2 \tfrac{dt}{t} \Big)^{1/2} \\ &= \frac{1}{1-2^{-\sigma}} \Big(\sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} |\sigma_{0,t;\mathscr{N}} * f(x)|^2 \frac{dt}{t} \Big)^{1/2} \\ &= \frac{1}{1-2^{-\sigma}} \Big(\int_1^2 \sum_{k\in\mathbb{Z}} |\sigma_{k,t;\mathscr{N}} * f(x)|^2 \frac{dt}{t} \Big)^{1/2} \\ &\leqslant \frac{1}{1-2^{-\sigma}} \sum_{s=1}^{\mathscr{N}} \Big(\int_1^2 \sum_{k\in\mathbb{Z}} |\mu_{k,t;s} * f(x)|^2 dt \Big)^{1/2} \\ &:= \frac{1}{1-2^{-\sigma}} \sum_{s=1}^{\mathscr{N}} \mathscr{M}_{h,\Omega,\Phi,\rho,s}(f)(x). \end{aligned} \tag{3.7}$$

It suffices to prove that

$$\|\mathscr{M}_{h,\Omega,\Phi,\rho,s}\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)} \tag{3.8}$$

for $2\tilde{\gamma}(\beta+1)/((\tilde{\gamma}+2)\beta) < p < 2\tilde{\gamma}(\beta+1)/((\tilde{\gamma}-2)\beta + 2\tilde{\gamma})$ and $s \in \{1, \dots, \mathscr{N}\}$. By the definition of S_k, we can write

$$\mathscr{M}_{h,\Omega,\Phi,\rho,s}(f)(x) = \Big(\sum_{k\in\mathbb{Z}} \int_1^2 \Big| \mu_{k,t;s} * \Big(\sum_{i\in\mathbb{Z}} S_{i+k} S_{i+k} f \Big)(x) \Big|^2 dt \Big)^{1/2}. \tag{3.9}$$

In what follows, we will estimate the $\|\mathscr{M}_{h,\Omega,\Phi,\rho,s}(f)\|_{L^p(\mathbb{R}^n)}$ in two cases: $p>2$ and $p<2$. The idea is taken from [25], which was originated from [17]. We first establish a rough L^p estimate by the Littlewood-Paley theory, and then give a delicate L^2-boundedness by Plancherel's theorem and the Fourier transform estimates. Finally, the desired estimates will be obtained by the interpolation.

Case 1. $2\tilde{\gamma}(\beta+1)/((\tilde{\gamma}+2)\beta)<p<2$: Applying (2.16), we know that for $1<q<p$

$$\|\mathscr{M}_{h,\Omega,\Phi,\rho,s}(f)\|_{L^p(\mathbb{R}^n)}^q \leqslant C\sum_{i\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\mu_{k,t;s}*S_{i+k}f|^2dt\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}^q. \tag{3.10}$$

For fixed $i\in\mathbb{Z}$, let

$$\mathscr{I}_{i,s}f(x):=\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\mu_{k,t;s}*S_{i+k}f(x)|^2dt\Big)^{1/2}.$$

Next, we prove the following inequality

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\mu_{k,t;s}*g_k|^2dt\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \tag{3.11}$$

for arbitrary functions $\{g_k\}_{k\in\mathbb{Z}}$ in $L^p(\mathbb{R}^n,\ell^2)$.

The main idea of the proof of (3.11) is taken from [13]. For $1\leqslant l\leqslant\mathscr{N}$, let Ψ be a radial function in $\mathbb{R}^{r_l}$ defined by $\hat{\Psi}(\xi^0)=\psi(|\xi^0|)$, where $\xi^0\in\mathbb{R}^{r_l}$, ψ is as in (3.2). Define J_l and $X_{k,t;l}$ by

$$J_lf(x):=f(G_l^\tau(H_l^\tau\otimes id_{\mathbb{R}^{n-r_l}})x)$$

and

$$X_{k,t;l}f(x):=J_l^{-1}((\Psi_{k,t;l}\otimes\delta_{\mathbb{R}^{n-r_l}})*J_lf)(x),$$

where $\Psi_{k,t;l}=\varphi(2^kt)^{-lr_l}\Psi(\varphi(2^kt)^{-l}x^0)$, $x^0\in\mathbb{R}^{r_l}$, H_l^τ (resp., G_l^τ) is the transpose of H_l (resp., G_l), H_l and G_l are as in (3.2). Let X_l be given by

$$X_lf(x):=\sup_{t\in[1,2]}\sup_{k\in\mathbb{Z}}|X_{k,t;l}f(x)|.$$

Then for $s\leqslant l\leqslant\mathscr{N}$, we have

$$X_lf(x)\leqslant C_l[J_l^{-1}\circ(M_{(l)}\otimes id_{\mathbb{R}^{n-r_l}})\circ J_l]f(x),$$

where $M_{(l)}$ denotes the Hardy-Littlewood maximal operator on $\mathbb{R}^{r_l}$, $x=(x^0,x^1)\in$

$\mathbb{R}^{r_l} \times \mathbb{R}^{n-r_l}$. Hence we have

$$\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2 |X_l(g_k)(\cdot)|^2 dt\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}^p \\
&\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2 |J_l^{-1}\circ(M_{(l)}\otimes id_{\mathbb{R}^{n-r_l}})\circ J_l(g_k)(\cdot)|^2 dt\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}^p \\
&\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}|J_l^{-1}\circ(M_{(l)}\otimes id_{\mathbb{R}^{n-r_l}})\circ J_l(g_k)(\cdot)|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}^p \\
&\leqslant C_l|J_l|\int_{\mathbb{R}^{n-r_l}}\int_{\mathbb{R}^{r_l}}\Big(\sum_{k\in\mathbb{Z}}|M_{(l)}[J_l(g_k)(\cdot,x^1)](x^0)|^2\Big)^{p/2}dx^0dx^1 \\
&\leqslant C_l|J_l|\int_{\mathbb{R}^{n-r_l}}\int_{\mathbb{R}^{r_l}}\Big(\sum_{k\in\mathbb{Z}}|J_l(g_k)(x^0,x^1)|^2\Big)^{p/2}dx^0dx^1 \\
&\leqslant C_l\Big\|\Big(\sum_{k\in\mathbb{Z}}|(g_k)(\cdot)|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}^p.
\end{aligned} \tag{3.12}$$

Also, by (3.2) and the definitions of $X_{k,t;l}$, we have

$$\mu_{k,t;s} * g_k(x) = \sigma_{k,t;s} * X_{k,t;s+1}\circ\cdots\circ X_{k,t;\mathscr{N}}\, g_k(x) - \sigma_{k,t;s-1} * X_{k,t;s}\circ\cdots\circ X_{k,t;\mathscr{N}}\, g_k(x). \tag{3.13}$$

Thus by (2.7), (3.13) and using the estimate (3.12) repeatedly, we can obtain (3.11).

Invoking (3.11) and the Littlewood-Paley theory, we get

$$\|\mathscr{I}_{i,s}f\|_{L^p(\mathbb{R}^n)} \leqslant C_p\Big\|\Big(\sum_{k\in\mathbb{Z}}|S_{i+k}f|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \leqslant C(p,\varphi)\|f\|_{L^p(\mathbb{R}^n)},\ \ |1/p-1/2|<1/\tilde{\gamma}. \tag{3.14}$$

On the other hand, by Plancherel's theorem, we know that

$$\begin{aligned}
\|\mathscr{I}_{i,s}f\|_{L^2(\mathbb{R}^n)}^2 &= \sum_{k\in\mathbb{Z}}\int_1^2\int_{\mathbb{R}^n}|\widehat{\mu_{k,t;s}}(\xi)|^2\lambda_{i+k}^2(|L_s\xi|)|\hat{f}(\xi)|^2 d\xi dt \\
&\leqslant \sum_{k\in\mathbb{Z}}\int_{\Lambda_{i+k}}|\hat{f}(\xi)|^2\int_1^2|\widehat{\mu_{k,t;s}}(\xi)|^2 dt d\xi,
\end{aligned}$$

where $\Lambda_{i+k} = \{\xi\in\mathbb{R}^n:\ \varphi(2^{i+k+1})^{-s}\leqslant|L_s\xi|\leqslant\varphi(2^{i+k-1})^{-s}\}$. Using (3.4)–(3.5), we have

$$\|\mathscr{I}_{i,s}(f)\|_{L^2(\mathbb{R}^n)} \leqslant C(s,\rho,\varphi)B_i\|f\|_{L^2(\mathbb{R}^n)}, \tag{3.15}$$

where

$$B_i := \begin{cases} B_\varphi^{-is}, & i>-2, \\ |i|^{-\beta/\tilde{\gamma}}, & i\leqslant -2. \end{cases} \tag{3.16}$$

By interpolation between (3.14) and (3.15), there exists a constant $\varepsilon_p\in((\tilde{\gamma}+2)/(2(\beta+1)),1)$ such that

$$\|\mathscr{I}_{i,s}(f)\|_{L^p(\mathbb{R}^n)} \leqslant C(s,\rho,\varphi)^{1-\varepsilon_p}B_i^{\varepsilon_p}\|f\|_{L^p(\mathbb{R}^n)},\ \ \text{for } 2\tilde{\gamma}(\beta+1)/((\tilde{\gamma}+2)\beta)<p<2.$$

Thus for fixed $2\tilde{\gamma}(\beta+1)/((\tilde{\gamma}+2)\beta) < p < 2$, we can choose $1 < q < p$ such that $q\varepsilon_p\beta/\tilde{\gamma} > 1$. Then

$$\sum_{i\in\mathbb{Z}} \|\mathscr{I}_{i,s}(f)\|_{L^p(\mathbb{R}^n)}^q \leqslant C(s,\rho,\varphi)\Big(\sum_{i>-2} B_\varphi^{-is\varepsilon_p q} + \sum_{i\leqslant -2} |i|^{-q\varepsilon_p\beta/\tilde{\gamma}}\Big)\|f\|_{L^p(\mathbb{R}^n)}^q$$
$$\leqslant C(s,\rho,\varphi)\|f\|_{L^p(\mathbb{R}^n)}^q$$

for $2\tilde{\gamma}(\beta+1)/((\tilde{\gamma}+2)\beta) < p < 2$, which together with (3.10) implies

$$\|\mathscr{M}_{h,\Omega,\Phi,\rho,s}(f)\|_{L^p(\mathbb{R}^n)} \leqslant C(s,\rho,\varphi)\|f\|_{L^p(\mathbb{R}^n)}, \text{ for } 2\tilde{\gamma}(\beta+1)/((\tilde{\gamma}+2)\beta) < p < 2. \tag{3.17}$$

Case 2. $2 < p < 2\tilde{\gamma}(\beta+1)/((\tilde{\gamma}-2)\beta+2\tilde{\gamma})$: It follows from (2.17) that

$$\|\mathscr{M}_{h,\Omega,\Phi,\rho,s}(f)\|_{L^p(\mathbb{R}^n)}^q \leqslant C\sum_{i\in\mathbb{Z}}\Big(\int_1^2 \Big\|\Big(\sum_{k\in\mathbb{Z}}|\mu_{k,t;s}*S_{i+k}f|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}^2 dt\Big)^{q/2} \tag{3.18}$$

for $2 < p < \infty$ and $1 < q < p'$. Let

$$\mathscr{J}_{i,t,s}f(x) := \Big(\sum_{k\in\mathbb{Z}}|\mu_{k,t;s}*S_{i+k}f(x)|^2\Big)^{1/2}.$$

By the similar arguments in getting (3.11) and (2.8), we have

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\mu_{k,t;s}*g_k|^2 dt\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \tag{3.19}$$

for arbitrary functions $\{g_k\}_{k\in\mathbb{Z}}$ in $L^p(\mathbb{R}^n,\ell^2)$. This inequality together with the Littlewood-Paley theory, we have for $i\in\mathbb{Z}$ and $t\in[1,2]$,

$$\|\mathscr{J}_{i,t,s}f\|_{L^p(\mathbb{R}^n)} \leqslant C(p,\varphi)\Big\|\Big(\sum_{k\in\mathbb{Z}}|S_{i+k}f|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)},\ |1/p-1/2| < 1/\tilde{\gamma}. \tag{3.20}$$

On the other hand, by the same argument as in getting (3.15), we have

$$\|\mathscr{J}_{i,t,s}f\|_{L^2(\mathbb{R}^n)} \leqslant C(s,\rho,\varphi)B_i\|f\|_{L^2(\mathbb{R}^n)}, \tag{3.21}$$

where B_i is as in (3.16). By interpolating between (3.20) and (3.21), for fixed $2 < p < 2\tilde{\gamma}(\beta+1)/((\tilde{\gamma}-2)\beta+2\tilde{\gamma})$, we can choose $q\in(1,p')$ and $\gamma_p\in((\tilde{\gamma}+2)/(2(\beta+1)),1)$ such that $q\gamma_p\beta/\tilde{\gamma} > 1$ and

$$\|\mathscr{J}_{i,t,s}f\|_{L^p(\mathbb{R}^n)} \leqslant C(s,\rho,\varphi)^{1-\gamma_p}B_i^{\gamma_p}\|f\|_{L^p(\mathbb{R}^n)}, \text{ for } 2 < p < 2\tilde{\gamma}(\beta+1)/((\tilde{\gamma}-2)\beta+2\tilde{\gamma}).$$

Combining this with (3.18) yields that

$$\|\mathscr{M}_{h,\Omega,\Phi,\rho,s}(f)\|_{L^p(\mathbb{R}^n)}^q \leqslant C(s,\rho,\varphi)\Big(\sum_{i>-2} B_\varphi^{-is\gamma_p q} + \sum_{i\leqslant -2}|i|^{-q\gamma_p\beta/\tilde{\gamma}}\Big)\|f\|_{L^p(\mathbb{R}^n)}^q$$
$$\leqslant C(s,\rho,\varphi)\|f\|_{L^p(\mathbb{R}^n)}^q$$

for $2 < p < 2\tilde{\gamma}(\beta+1)/((\tilde{\gamma}-2)\beta+2\tilde{\gamma})$. This together with (3.17) implies (3.8) and completes the proof of Theorem 1. $\square$

4. Additional results

As applications of our main results, we shall obtain the L^p bounds for the corresponding parametric Marcinkiewicz integral operators $\mathscr{M}^*_{h,\Omega,\Phi,\lambda,\rho}$ and $\mathscr{M}_{h,\Omega,\Phi,S,\rho}$ related to the Littlewood-Paley g^*_λ-function and the area integral S, respectively. In what follows, we set $\tilde{\gamma} = \max\{2,\gamma'\}$. Let $F_{h,\Omega,\Phi,\rho}(f)$ be as in (1.7), we define the operators $\mathscr{M}^*_{h,\Omega,\Phi,\lambda,\rho}$ and $\mathscr{M}_{h,\Omega,\Phi,S,\rho}$ by

$$\mathscr{M}^*_{h,\Omega,\Phi,\lambda,\rho}(f)(x) := \left(\iint_{\mathbb{R}^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}|F_{h,\Omega,\Phi,\rho}(f)(y,t)|^2\frac{dydt}{t^{n+1}}\right)^{1/2},$$

where $\lambda > 0$ and $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$;

$$\mathscr{M}_{h,\Omega,\Phi,S,\rho}(f)(x) := \left(\iint_{\Gamma(x)}|F_{h,\Omega,\Phi,\rho}(f)(y,t)|^2\frac{dydt}{t^{n+1}}\right)^{1/2},$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : \ |x-y| < t\}$.

THEOREM 5. *Let Φ, h, Ω be as in Theorem 1. Then for $2 \leqslant p < 2\tilde{\gamma}(\beta+1)/((\tilde{\gamma}-2)\beta+2\tilde{\gamma})$, there exists constants $C(\rho,\varphi)$ which are independent of the coefficients of P_j for $1 \leqslant j \leqslant n$ such that*

$$\|\mathscr{M}^*_{h,\Omega,\Phi,\lambda,\rho}(f)\|_{L^p(\mathbb{R}^n)} \leqslant C(\rho,\varphi)\|f\|_{L^p(\mathbb{R}^n)}; \tag{4.1}$$

$$\|\mathscr{M}_{h,\Omega,\Phi,S,\rho}(f)\|_{L^p(\mathbb{R}^n)} \leqslant C(\rho,\varphi)\|f\|_{L^p(\mathbb{R}^n)}. \tag{4.2}$$

The proof of Theorem 5 is based on the following lemma.

LEMMA 7. *Let $\lambda > 1$. Then there exists a constant $C(\lambda,n)$ such that for any nonnegative locally integrable function g on $\mathbb{R}^n$,*

$$\int_{\mathbb{R}^n}(\mathscr{M}^*_{h,\Omega,\Phi,\lambda,\rho}(f)(x))^2 g(x)dx \leqslant C(\lambda,n)\int_{\mathbb{R}^n}(\mathscr{M}_{h,\Omega,\Phi,\rho}(f)(x))^2 M(g)(x)dx,$$

where M is the usual Hardy-Littlewood maximal operator on $\mathbb{R}^n$.

Proof. By the definition of $\mathscr{M}^*_{h,\Omega,\Phi,\lambda,\rho}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}(\mathscr{M}^*_{h,\Omega,\Phi,\lambda,\rho}(f)(x))^2 g(x)dx \\
&= \int_{\mathbb{R}^n}\iint_{\mathbb{R}^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}|F_{h,\Omega,\Phi,\rho}(f)(y,t)|^2\frac{dydt}{t^{n+1}}g(x)dx \\
&\leqslant \int_{\mathbb{R}^n}\int_0^\infty |F_{h,\Omega,\Phi,\rho}(f)(y,t)|^2\left(\sup_{t>0}\frac{1}{t^n}\int_{\mathbb{R}^n}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}g(x)dx\right)\frac{dt}{t}dy \\
&\leqslant C(\lambda,n)\int_{\mathbb{R}^n}(\mathscr{M}_{h,\Omega,\Phi,\rho}(f)(y))^2 M(g)(y)dy
\end{aligned}$$

for $\lambda > 1$. This proves Lemma 7. $\square$

Proof. Now we prove Theorem 5. First we prove (4.1). For $2 \leqslant p < 2\tilde{\gamma}(\beta+1)/((\tilde{\gamma}-2)\beta+2\tilde{\gamma})$, by the duality we have

$$\|\mathscr{M}^*_{h,\Omega,\Phi,\lambda,\rho}(f)\|^2_{L^p(\mathbb{R}^n)} = \sup_{\|g\|_{L^q(\mathbb{R}^n)}\leqslant 1} \int_{\mathbb{R}^n} (\mathscr{M}^*_{h,\Omega,\Phi,\lambda,\rho}(f)(x))^2 g(x)dx,$$

where $q=(p/2)'$ and the supremum is taken over all g satisfying $\|g\|_{L^q(\mathbb{R}^n)} \leqslant 1$. By the L^p bounds of M, Hölder's inequality, Lemma 7 and Theorem 1, we get

$$\begin{aligned}\|\mathscr{M}^*_{h,\Omega,\Phi,\lambda,\rho}(f)\|^2_{L^p(\mathbb{R}^n)} &\leqslant C(\lambda,n) \sup_{\|g\|_{L^q(\mathbb{R}^n)}\leqslant 1} \int_{\mathbb{R}^n} (\mathscr{M}^\rho_{h,\Omega,\Phi}(f)(x))^2 M(g)(x)dx \\ &\leqslant C(\lambda,n)\|\mathscr{M}_{h,\Omega,\Phi,\rho}(f)\|^2_{L^p(\mathbb{R}^n)} \\ &\leqslant C(\lambda,n,\rho,\varphi)\|f\|^2_{L^p(\mathbb{R}^n)},\ \ 2\leqslant p<2\tilde{\gamma}(\beta+1)/((\tilde{\gamma}-2)\beta+2\tilde{\gamma}).\end{aligned}$$

Thus (4.1) holds. On the other hand, it is easy to check that

$$\mathscr{M}_{h,\Omega,\Phi,S,\rho}(f)(x) \leqslant 2^{n\lambda/2}\mathscr{M}^*_{h,\Omega,\Phi,\lambda,\rho}(f)(x),$$

which combining with (4.1) implies (4.2). Theorem 5 is proved. □

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(Received January 6, 2014)

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