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BOUNDEDNESS FOR RIESZ–TYPE POTENTIAL OPERATORS ON HERZ–MORREY SPACES WITH VARIABLE EXPONENT

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Abstract. In this paper, the Riesz-type potential operator of variable order $\beta(x)$ is shown to be bounded from the Herz-Morrey spaces $M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)$ with variable exponent $q_1(x)$ into the weighted space $M\dot{K}^{\alpha,\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n,\omega)$, where $\omega=(1+|x|)^{-\gamma(x)}$ with some $\gamma(x)>0$ and $1/q_1(x)-1/q_2(x)=\beta(x)/n$ when $q_1(x)$ is not necessarily constant at infinity. It is assumed that the exponent $q_1(x)$ satisfies the logarithmic continuity condition both locally and at infinity and $1<q_1(\infty)\leqslant q_1(x)\leqslant (q_1)_+<\infty\ (x\in\mathbb{R}^n)$.

1. Introduction

Last decade, we witness a strong rise of interest to the study of various mathematical problems in the so-called spaces with non-standard growth. This expression mainly relates to the generalized Lebesgue spaces $L^{p(\cdot)}(\Omega)$ $(\Omega\subset\mathbb{R}^n)$ with variable order $p(x)$ (the generalized Lebesgue spaces with variable exponent), and to the corresponding generalized Sobolev spaces $W^{m,p(\cdot)}$.

Function spaces with variable exponent are being watched with keen interest not in real analysis but also in partial differential equations and in applied mathematics because they are applicable to the modeling for electrorheological fluids, mechanics of the continuum medium and image restoration. In some problems of mechanics, there arise variational problems with Lagrangians more complicated than is usually assumed in variational calculus, for example, of the form $|\xi|^{\beta(x)}$ when the character of nonlinearity varies from point to point (Lagrangians of the plasticity theory, Lagrangians of the mechanics of the so-called rheological fluids and others).

The theory of function spaces with variable exponent has rapidly made progress in the past twenty years since some elementary properties were established by Kováčik-Rákosník [1]. One of the main problems on the theory is the boundedness of the Hardy-Littlewood maximal operator on variable Lebesgue spaces. By virtue of the fine works [2–13], some important conditions on variable exponent, for example, the log-Hölder

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conditions and the Muckenhoupt type condition, have been obtained (for more details see [5, 6] et al).

In 2012, Almeida and Drihem [14] discuss the boundedness of a wide class of sublinear operators on Herz spaces $K^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n)$ and $\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n)$ with variable exponent $\alpha(\cdot)$ and $q(\cdot)$. Meanwhile, they also established Hardy-Littlewood-Sobolev theorems for fractional integrals on variable Herz spaces. In 2013, Samko [15, 16] introduce a new Herz type function spaces with variable exponent, where all the three parameters are variable, and proved the boundedness of some sublinear operators. In [17], the boundedness of operators are established in variable exponent Morrey spaces (for more results see [18, 19, 20, 21] et al).

In this paper, the author will investigate mapping properties of the operator $I_{\beta(\cdot)}$ within the framework of the Herz-Morrey spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ with variable exponent $q(\cdot)$ but fixed $\alpha\in\mathbb{R}$ and $p\in(0,\infty)$, where the Riesz-type potential operator of variable order

$$I_{\beta(\cdot)}(f)(x)=\int_{\Omega}\frac{f(y)}{|x-y|^{n-\beta(x)}}\mathrm{d}y,\ x\in\Omega\subset\mathbb{R}^n,\ 0<\beta(x)<n.$$

2. Preliminaries

In this section, we define some function spaces with variable exponent, and give basic properties and useful lemmas. Throughout this paper we will use the following notation:

NOTATION

- Denote by $|S|$ the Lebesgue measure and by χ_S the characteristic function for a measurable set $S\subset\mathbb{R}^n$.
- f_S denotes the mean value of f on measurable set S, namely

$$f_S:=\frac{1}{|S|}\int_S f(x)\mathrm{d}x.$$

- $B(x,r)$ is the ball cenetered at x and of radius r; $B_0=B(0,1)$.
- C denotes a constant that is independent of the main parameters involved but whose value may differ from line to line.
- For any exponent $1<q(x)<\infty$, we denote by $q'(x)$ its conjugate exponent, namely, $1/q(x)+1/q'(x)=1$.
- For $A\sim D$, we mean that there is a constant $C>0$ such that$C^{-1}D\leqslant A\leqslant CD$.

2.1. Function spaces with variable exponent

Let Ω be a measurable set in $\mathbb{R}^n$ with $|\Omega|>0$. We first define Lebesgue spaces with variable exponent.

DEFINITION 2.1. Let $q(\cdot):\Omega\to(1,\infty)$ be a measurable function.

(I) The variable Lebesgue spaces $L^{q(\cdot)}(\Omega)$ is defined by

$$L^{q(\cdot)}(\Omega)=\{f \text{ is measurable function} : F_q(f/\eta)<\infty \text{ for some constant } \eta>0\},$$

where $F_q(f):=\int_\Omega |f(x)|^{q(x)}\mathrm{d}x$.

(II) The space $L^{q(\cdot)}_{\mathrm{loc}}(\Omega)$ is defined by

$$L^{q(\cdot)}_{\mathrm{loc}}(\Omega)=\{f \text{ is measurable function} : f\in L^{q(\cdot)}(\Omega_0) \text{ for all compact subsets } \Omega_0\subset\Omega\}.$$

(III) The weighted Lebesgue space $L^{q(\cdot)}_{\omega}(\Omega)$ is defined by as the set of all measurable functions for which

$$\|f\|_{L^{q(\cdot)}_{\omega}(\Omega)}=\|\omega^{1/q(\cdot)}f\|_{L^{q(\cdot)}(\Omega)}<\infty.$$

The Lebesgue space $L^{q(\cdot)}(\Omega)$ is a Banach space when equipped with the norm

$$\|f\|_{L^{q(\cdot)}(\Omega)}=\inf\Big\{\eta>0 : F_q(f/\eta)=\int_\Omega\Big(\frac{|f(x)|}{\eta}\Big)^{q(x)}\mathrm{d}x\leqslant 1\Big\}. \tag{2.1}$$

Next we define some classes of variable exponent functions. Given a function $f\in L^1_{\mathrm{loc}}(\Omega)$, the Hardy-Littlewood maximal operator M is defined by

$$Mf(x)=\sup_{r>0} r^{-n}\int_{B(x,r)\cap\Omega}|f(y)|\mathrm{d}y \quad (x\in\Omega),$$

where $B(x,r)=\{y\in\mathbb{R}^n : |x-y|<r\}$.

DEFINITION 2.2. Given a measurable function $q(\cdot)$ defined on Ω, we write

$$q_-:=\operatorname*{ess\,inf}_{x\in\Omega} q(x),\quad q_+:=\operatorname*{ess\,sup}_{x\in\Omega} q(x).$$

(I) $\quad q'_-=\operatorname*{ess\,inf}_{x\in\Omega} q'(x)=\frac{q_+}{q_+-1},\quad q'_+=\operatorname*{ess\,sup}_{x\in\Omega} q'(x)=\frac{q_-}{q_--1}.$

(II) Denote by $\mathscr{P}_0(\Omega)$ the set of all measurable functions $q(\cdot):\Omega\to(0,\infty)$ such that

$$0<q_-\leqslant q(x)\leqslant q_+<\infty,\quad x\in\Omega.$$

(III) Denote by $\mathscr{P}(\Omega)$ the set of all measurable functions $q(\cdot):\Omega\to(1,\infty)$ such that

$$1<q_-\leqslant q(x)\leqslant q_+<\infty,\quad x\in\Omega.$$

(IV) The set $\mathscr{B}(\Omega)=\{q(\cdot)\in\mathscr{P}(\Omega)$: the maximal operator M is bounded on $L^{q(\cdot)}(\Omega)\}$. $L^{q(\cdot)}(\Omega)$.

(V) The set $\mathscr{C}_0^{\log}(\Omega)$ consists of all locally log-Hölder continuous functions $q(\cdot):\Omega\to(0,\infty)$ satisfies the condition

$$|q(x)-q(y)|\leqslant\frac{-C}{\ln(|x-y|)},\quad |x-y|\leqslant 1/2,\ x,y\in\Omega. \tag{2.2}$$

(VI) The set $\mathscr{C}_\infty^{\log}(\Omega)$ consists of all log-Hölder decay continuous functions $q(\cdot):\Omega\to(0,\infty)$ at infinity satisfies the condition

$$|q(x)-q(\infty)|\leqslant\frac{C_\infty}{\ln(e+|x|)},\quad x\in\Omega, \tag{2.3}$$

where $q(\infty)=\lim_{|x|\to\infty}q(x)$.

(VII) Denote by $\mathscr{C}^{\log}(\Omega):=\mathscr{C}_0^{\log}(\Omega)\cap\mathscr{C}_\infty^{\log}(\Omega)$ the set of all globally log-Hölder continuous functions (i.e. locally log-Hölder continuous and satisfies the log-Hölder decay condition.) $q(\cdot):\Omega\to(0,\infty)$.

REMARK 1.

(i) The logarithmic condition (2.2) is usually called the locally log-Hölder continuity or the Dini-Lipschitz condition.

(ii) The $\mathscr{C}_\infty^{\log}(\Omega)$ condition is equivalent to the uniform continuity condition

$$|q(x)-q(y)|\leqslant\frac{C}{\ln(e+|x|)},\quad |y|\geqslant|x|,\ x,y\in\Omega. \tag{2.4}$$

The $\mathscr{C}_\infty^{\log}(\Omega)$ condition was originally defined in this form in [4].

Next we define the Herz-Morrey spaces with variable exponent. Let $B_k=B(0,2^k)$ $=\{x\in\mathbb{R}^n:|x|\leqslant 2^k\}$, $A_k=B_k\setminus B_{k-1}$ and $\chi_k=\chi_{A_k}$ for $k\in\mathbb{Z}$.

DEFINITION 2.3. Suppose that $\alpha\in\mathbb{R}$, $0\leqslant\lambda<\infty$, $0<p<\infty$, $q(\cdot)\in\mathscr{P}(\mathbb{R}^n)$. The Herz-Morrey space with variable exponent $M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ is definded by

$$M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)=\Big\{f\in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\}):\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)}<\infty\Big\},$$

where

$$\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)}=\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0}2^{k\alpha p}\|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{p}}.$$

Compare the variable Herz-Morrey space $M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ with the variable Herz space $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, where

$$\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \Big\{ f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\}) : \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \| f\chi_k \|^p_{L^{q(\cdot)}(\mathbb{R}^n)} < \infty \Big\},$$

Obviously, $M\dot{K}^{\alpha,0}_{p,q(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

we can see that our result below generalize the result in the setting of the Herz-Morrey space with variable exponent, which proved by Izuki in [22]. And when $\lambda = 0$, our main result is also valid.

In this paper, we denote by $q(x)$ bounded exponents on Ω. Thus, $q(x)$ is not allowed to tend to infinity. Similarly, when dealing with the conjugate space and considering singular and maximal operators, we have to exclude the tendency of $q(x)$ to 1. Therefore, in the sequel, we assume that variable exponent belongs to $\mathscr{P}(\Omega)$.

2.2. Recent results for Riesz-type potential $I_{\beta(\cdot)}$

In this part we recall some recent results for Riesz-type potential operator $I_{\beta(\cdot)}$. The order $\beta(x)$ of the potential is not assumed to be continuous. We assume that it is a measurable function on Ω satisfying the following assumptions

$$\left.\begin{aligned} \beta_0 := \operatorname*{ess\,inf}_{x\in\Omega} \beta(x) > 0 \\ \operatorname*{ess\,sup}_{x\in\Omega} p(x)\beta(x) < n \end{aligned}\right\}. \tag{2.5}$$

The boundedness of the Riesz-type potential operator $I_{\beta(\cdot)}$ from the space $L^{p(\cdot)}(\mathbb{R}^n)$ with the variable exponent $p(x)$ into the space $L^{q(\cdot)}(\mathbb{R}^n)$ with the limiting Sobolev exponent

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\beta(x)}{n} \tag{2.6}$$

was an open problem for a long time. It was solved in the case of bounded domains. First, in [23], in the case of bounded domains Ω, there was proved the following conditional result.

THEOREM A. *Let Ω be a bounded open set in $\mathbb{R}^n$, $p(\cdot) \in \mathscr{C}^{\log}(\Omega) \cap \mathscr{P}(\Omega)$ and $\beta(x)$ satisfy assumptions (2.5). Define $q(x)$ by (2.6). If the maximal operator is bounded in the space $L^{p(\cdot)}(\Omega)$, then the Sobolev theorem*

$$\|I_{\beta(\cdot)}(f)\|_{L^{q(\cdot)}(\Omega)} \leqslant C \|f\|_{L^{p(\cdot)}(\Omega)}$$

is valid.

After Diening [8] proved the boundedness of the maximal operator over bounded domains, the validity of the Sobolev theorem for bounded domains became an unconditional statement.

In 2008, in the case of bounded sets, Almeida, Hasanov and Samko [17] proved the boundedness of the maximal operator in variable exponent Morrey spaces, and in 2009, Hästö [24] used his new ”local-to-global” approach to extend the result of [17] about the maximal operator to the whole space $\mathbb{R}^n$.

In 2010, in the case of bounded sets, Guliyev, Hasanov and Samko [25] considered the boundedness of the Riesz-type potential operator $I_{\beta(\cdot)}$ on the generalized variable exponent Morrey type spaces.

For the whole space $\mathbb{R}^n$, the Sobolev theorem was proved by Diening [26], under the condition that the exponent $p(x)$ is constant outside some ball of large radius.

THEOREM B. *Let $\Omega=\mathbb{R}^n$, $0<\beta=const<n$, and let $p(\cdot)\in\mathscr{C}_0^{\log}(\mathbb{R}^n)\cap\mathscr{P}(\mathbb{R}^n)$ be constant outside some large ball $B(0,r)$. Define $q(x)$ by (2.6). If $\operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x)<n/\beta$, then*

$$\|I_\beta(f)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Another version of the Sobolev theorem for the space $\mathbb{R}^n$ was proved in [27] for the exponents $p(x)$ not necessarily constant in a neigbourhood of infinity, but with some “extra” power weight fixed to infinity and under the assumption that $p(x)$ takes its minimal value at infinity.

THEOREM C. *Let $\Omega=\mathbb{R}^n$, $\beta(x)$ meet conditions (2.5) and*

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(\infty)\beta(x)<n. \tag{2.7}$$

Suppose that $p(\cdot)\in\mathscr{C}^{\log}(\mathbb{R}^n)\cap\mathscr{P}(\mathbb{R}^n)$ and

$$1<p(\infty)\leqslant p(x)\leqslant p_+<\infty. \tag{2.8}$$

Then the following weighted Sobolev-type estimate is valid for the operator $I_{\beta(\cdot)}$:

$$\left\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(f)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $q(x)$ is defined by (2.6), and

$$\gamma(x)=C_\infty\beta(x)\left(1-\frac{\beta(x)}{n}\right)\leqslant\frac{n}{4}C_\infty, \tag{2.9}$$

C_∞ being the Dini-Lipschitz constant from (2.3) which $q(\cdot)$ is replaced by $p(\cdot)$.

In 2013, in the case of unbounded sets, Guliyev and Samko [28] considered the boundedness of the Riesz-type potential operator $I_{\beta(\cdot)}$ on the generalized variable exponent Morrey type spaces.

REMARK 2. Under the assumptions of Theorem 2.2, similar conclusion of Theorem 2.2 is also valid to the fractional maximal operator (see [27])

$$M_{\beta(\cdot)}(f)(x)=\sup_{r>0}\frac{1}{|B(x,r)|^{n-\beta(x)}}\int_{B(x,r)}|f(y)|\mathrm{d}y.$$

2.3. Auxiliary propositions and lemmas

In this part we state some auxiliary propositions and lemmas which will be needed for proving our main theorems. And we only describe partial results we need.

PROPOSITION 2.1. Let $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$.

(I) If $q(\cdot) \in \mathscr{C}^{\log}(\mathbb{R}^n)$, then we have $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$.

(II) $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ if and only if $q'(\cdot) \in \mathscr{B}(\mathbb{R}^n)$.

The first part in Proposition 2.1 is independently due to Cruz-Uribe, Fiorenza and Neugebauer [4] and to Nekvinda [12] respectively. The second of Proposition 2.1 belongs to Diening (see Theorem 8.1 in [7] or Theorem 1.2 in [3]).

REMARK 3. Since

$$|q'(x) - q'(y)| \leqslant \frac{|q(x) - q(y)|}{(q_- - 1)^2},$$

it follows at once that if $q(\cdot) \in \mathscr{C}^{\log}(\mathbb{R}^n)$, then so does $q'(\cdot)$—i.e., if the condition hold, then M is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$ and $L^{q'(\cdot)}(\mathbb{R}^n)$. Furthermore, Diening has proved general results on Musielak-Orlicz spaces.

The next lemma is known as the generalized Hölder's inequality on Lebesgue spaces with variable exponent, and the proof can be found in [1, 5, 6].

LEMMA 2.1. (generalized Hölder's inequality) *Suppose that $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, then for any $f \in L^{q(\cdot)}(\mathbb{R}^n)$ and any $g \in L^{q'(\cdot)}(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n} |f(x)g(x)| \mathrm{d}x \leqslant C_q \|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{q'(\cdot)}(\mathbb{R}^n)}, \tag{2.10}$$

where $C_q = 1 + 1/q_- - 1/q_+$.

The following lemma can be found in [22]. Here we only state the parts what we need.

LEMMA 2.2.

(I) *Let $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Then there exist positive constants $\delta \in (0,1)$ and $C > 0$ such that*

$$\frac{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \leqslant C \left(\frac{|S|}{|B|} \right)^{\delta}$$

for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$.

(II) *Let* $q(\cdot)\in\mathscr{B}(\mathbb{R}^n)$. *Then there exists a positive constant* $C>0$ *such that*

$$C^{-1}\leqslant\frac{1}{|B|}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\leqslant C \tag{2.11}$$

for all balls B *in* $\mathbb{R}^n$.

REMARK 4.
(i) If $q_1(\cdot),\ q_2(\cdot)\in\mathscr{C}^{\log}(\mathbb{R}^n)\cap\mathscr{P}(\mathbb{R}^n)$, then we see that $q_1'(\cdot),\ q_2(\cdot)\in\mathscr{B}(\mathbb{R}^n)$. Hence we can take positive constants $0<\delta_1<1/(q_1')_+$, $0<\delta_2<1/(q_2)_+$ such that

$$\frac{\|\chi_S\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}\leqslant C\left(\frac{|S|}{|B|}\right)^{\delta_1},\quad \frac{\|\chi_S\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}\leqslant C\left(\frac{|S|}{|B|}\right)^{\delta_2} \tag{2.12}$$

hold for all balls B in $\mathbb{R}^n$ and all measurable subsets $S\subset B$ (see [22, 29]).

(ii) On the other hand, Kopaliani [10] has proved the conclusion: If the exponent $q(\cdot)\in\mathscr{P}(\mathbb{R}^n)$ equals to a constant outside some large ball, then $q(\cdot)\in\mathscr{B}(\mathbb{R}^n)$ if and only if $q(\cdot)$ satisfies the Muckenhoupt type condition

$$\sup_{Q:\text{cube}}\frac{1}{|Q|}\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{q'(\cdot)}(\mathbb{R}^n)}<\infty.$$

3. Main result and its proof

Our main result can be stated as follows.

THEOREM 3.1. *Suppose that* $\Omega=\mathbb{R}^n$, $q_1(\cdot)\in\mathscr{C}^{\log}(\mathbb{R}^n)\cap\mathscr{P}(\mathbb{R}^n)$, *and* $\beta(x)$ *meet conditions (2.5) and (2.7) which* $q_1(\cdot)$ *instead of* $p(\cdot)$. *Define the variable exponent* $q_2(\cdot)$ *by*

$$\frac{1}{q_2(x)}=\frac{1}{q_1(x)}-\frac{\beta(x)}{n}.$$

Let $q_1(\cdot)$, $q_2'(\cdot)$ *satisfies condition (2.8), and* $0<p_1\leqslant p_2<\infty$, $\lambda>0$, $\lambda-n\delta_2<\alpha<\lambda+n\delta_1$, *where* $\delta_1\in(0,1/(q_1')_+)$ *and* $\delta_2\in(0,1/(q_2)_+)$ *are the constants appearing in (2.12). Then*

$$\left\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(f)\right\|_{M\dot{K}^{\alpha,\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n)}\leqslant C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)},$$

where $\gamma(x)$ *is defined as in (2.9), and the Dini-Lipschitz constant is* $\max(C_\infty,2C_\infty/(p_--1)^2)$ *when condition (2.3)'s* $q(\cdot)$ *is replaced by* $q_1(\cdot)$.

COROLLARY 3.1. *Under the assumptions of Theorem 3.1, in the Herz-Morrey space with variable exponent, the estimate of Sobolev exponent*

$$\frac{1}{q_2(x)}=\frac{1}{q_1(x)}-\frac{\beta(x)}{n}$$

is also valid for the fractional maximal operator

$$M_{\beta(\cdot)}(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{n-\beta(x)}} \int_{B(x,r)} |f(y)| \mathrm{d}y,$$

that is

$$\left\| (1+|x|)^{-\gamma(x)} M_{\beta(\cdot)}(f) \right\|_{M\dot{K}^{\alpha,\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n)} \leqslant C \|f\|_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}.$$

Proof of the theorem 3.1. For any $f \in M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$, if we denote $f_j := f \cdot \chi_j = f \cdot \chi_{A_j}$ for each $j \in \mathbb{Z}$, then we can write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x) \cdot \chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

Because of $0 < p_1/p_2 \leqslant 1$, we apply inequality

$$\left(\sum_{i=-\infty}^{\infty} |a_i| \right)^{p_1/p_2} \leqslant \sum_{i=-\infty}^{\infty} |a_i|^{p_1/p_2}, \tag{3.1}$$

and obtain

$$\begin{aligned}
&\left\| (1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f) \right\|_{M\dot{K}^{\alpha,\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n)} \\
&= \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p_2} \left\| (1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f) \cdot \chi_k \right\|^{p_2}_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1/p_2} \\
&\leqslant C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p_1} \left\| (1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f) \cdot \chi_k \right\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right) \\
&\leqslant C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p_1} \left(\sum_{j=-\infty}^{k-2} \left\| (1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f_j) \cdot \chi_k \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \right) \\
&\quad + C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p_1} \left(\sum_{j=k-1}^{k+1} \left\| (1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f_j) \cdot \chi_k \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \right) \\
&\quad + C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p_1} \left(\sum_{j=k+2}^{\infty} \left\| (1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f_j) \cdot \chi_k \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \right) \\
&=: C(E_1 + E_2 + E_3).
\end{aligned}$$

First we estimate E_2. Using the Theorem 2.2, we have

$$\begin{aligned}
E_2 &= \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p_1} \left(\sum_{j=k-1}^{k+1} \left\| (1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f_j) \cdot \chi_k \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \right) \\
&\leqslant C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p_1} \left(\sum_{j=k-1}^{k+1} \|f_j \cdot \chi_k\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \right)
\end{aligned}$$

$$\leqslant C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p_1}\|f\cdot\chi_k\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}\right)$$
$$\leqslant C\|f\|_{M\dot{K}_{p_1,q_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}^{p_1}.$$

For E_1. Note that when $x\in A_k, j\leqslant k-2$, and $y\in A_j$, then $|x-y|\backsim|x|, 2|y|\leqslant|x|$. Therefore, using the generalized Hölder's inequality, we have

$$\begin{aligned}|I_{\beta(\cdot)}(f_j)(x)\cdot\chi_k(x)| &\leqslant \int_{A_j}\frac{|f(y)|}{|x-y|^{n-\beta(x)}}\mathrm{d}y\cdot\chi_k(x)\\ &\leqslant C2^{-kn}|x|^{\beta(x)}\int_{A_j}|f_j(y)|\mathrm{d}y\cdot\chi_k(x)\\ &\leqslant C2^{-kn}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\cdot|x|^{\beta(x)}\chi_k(x).\end{aligned}\tag{3.2}$$

Notice that

$$\begin{aligned}I_{\beta(\cdot)}(\chi_{B_k})(x)\geqslant I_{\beta(\cdot)}(\chi_{B_k})(x)\cdot\chi_{B_k}(x) &= \int_{B_k}\frac{1}{|x-y|^{n-\beta(x)}}\mathrm{d}y\cdot\chi_{B_k}(x)\\ &\geqslant C|x|^{\beta(x)}\cdot\chi_{B_k}(x)\\ &\geqslant C|x|^{\beta(x)}\cdot\chi_k(x).\end{aligned}\tag{3.3}$$

Using Theorem 2.2, Lemma 2.2, (2.12), (3.2) and (3.3), we have

$$\begin{aligned}&\left\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(f_j)\cdot\chi_k(\cdot)\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\ &\leqslant C2^{-kn}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\left\|(1+|x|)^{-\gamma(x)}|\cdot|^{\beta(x)}\cdot\chi_k(\cdot)\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\ &\leqslant C2^{-kn}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\left\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(\chi_{B_k})\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\ &\leqslant C2^{-kn}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\\ &\leqslant C2^{-kn}\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\\ &\leqslant C\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}\\ &\leqslant C2^{(j-k)n\delta_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.\end{aligned}\tag{3.4}$$

On the other hand, note the following fact

$$\begin{aligned}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} &= 2^{-j\alpha}\Big(2^{j\alpha p_1}\|f_j\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{1/p_1}\\ &\leqslant 2^{-j\alpha}\Big(\sum_{i=-\infty}^{j}2^{i\alpha p_1}\|f_i\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{1/p_1}\\ &= 2^{j(\lambda-\alpha)}\Big(2^{-j\lambda}\Big(\sum_{i=-\infty}^{j}2^{i\alpha p_1}\|f_i\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{1/p_1}\Big)\\ &\leqslant C2^{j(\lambda-\alpha)}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{3.5}$$

Thus, combining (3.4) and (3.5), and using $\alpha<\lambda+n\delta_1$, it follows that

$$\begin{aligned}E_1 &= \sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda p_1}\Big(\sum_{k=-\infty}^{k_0}2^{k\alpha p_1}\Big(\sum_{j=-\infty}^{k-2}\Big\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(f_j)\cdot\chi_k\Big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\Big)^{p_1}\Big)\\ &\leqslant C\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda p_1}\Big(\sum_{k=-\infty}^{k_0}2^{k\alpha p_1}\Big(\sum_{j=-\infty}^{k-2}2^{(j-k)n\delta_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{p_1}\Big)\\ &\leqslant C\|f\|^{p_1}_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda p_1}\Big(\sum_{k=-\infty}^{k_0}2^{k\alpha p_1}\Big(\sum_{j=-\infty}^{k-2}2^{(j-k)n\delta_1}2^{j(\lambda-\alpha)}\Big)^{p_1}\Big)\\ &\leqslant C\|f\|^{p_1}_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda p_1}\Big(\sum_{k=-\infty}^{k_0}2^{k\lambda p_1}\Big(\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_1+\lambda-\alpha)}\Big)^{p_1}\Big)\\ &\leqslant C\|f\|^{p_1}_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda p_1}\Big(\sum_{k=-\infty}^{k_0}2^{k\lambda p_1}\Big)\\ &\leqslant C\|f\|^{p_1}_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Now, let us turn to estimate for E_3. Note that when $x\in A_k, j\geqslant k+2$, and $y\in A_j$, then $|x-y|\backsim|y|, 2|x|\leqslant|y|$. Therefore, using the generalized Hölder's inequality, we have

$$\begin{aligned}&\Big|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(f_j)(x)\cdot\chi_k(x)\Big|\leqslant(1+|x|)^{-\gamma(x)}\int_{A_j}\frac{|f(y)|}{|x-y|^{n-\beta(x)}}\mathrm{d}y\cdot\chi_k(x)\\ &\leqslant C\int_{A_j}|f(y)|(1+|x|)^{-\gamma(x)}|y|^{\beta(x)-n}\mathrm{d}y\cdot\chi_k(x)\\ &\leqslant C2^{-jn}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big\|(1+|x|)^{-\gamma(x)}|\cdot|^{\beta(x)}\chi_j(\cdot)\Big\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\cdot\chi_k(x).\end{aligned} \tag{3.6}$$

Similar to (3.3), we have

$$\begin{aligned} I_{\beta(\cdot)}(\chi_{B_j})(x) \geqslant I_{\beta(\cdot)}(\chi_{B_j})(x)\cdot\chi_{B_j}(x) &= \int_{B_j}\frac{1}{|x-y|^{n-\beta(x)}}\mathrm{d}y\cdot\chi_{B_j}(x) \\ &\geqslant C|x|^{\beta(x)}\cdot\chi_{B_j}(x) \\ &\geqslant C|x|^{\beta(x)}\cdot\chi_j(x). \end{aligned} \tag{3.7}$$

Using Theorem 2.2, Lemma 2.2, (2.12), (3.6) and (3.7), we obtain

$$\begin{aligned} &\left\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(f_j)\cdot\chi_k\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C2^{-jn}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\left\|(1+|x|)^{-\gamma(x)}|\cdot|^{\beta(x)}\chi_j(\cdot)\right\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C2^{-jn}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\left\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(\chi_{B_j})\right\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C2^{-jn}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C2^{-jn}\|\chi_{B_j}\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}} \\ &\leqslant C2^{(k-j)n\delta_2}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{3.8}$$

Therefore, combining (3.5) and (3.8), and using $\alpha > \lambda - n\delta_2$, it follows that

$$\begin{aligned} E_3 &= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{k_0}2^{k\alpha p_1}\left(\sum_{j=k+2}^{\infty}\left\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(f_j)\cdot\chi_k\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{p_1}\right) \\ &\leqslant C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{k_0}2^{k\alpha p_1}\left(\sum_{j=k+2}^{\infty}2^{(k-j)n\delta_2}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{p_1}\right) \\ &\leqslant C\|f\|^{p_1}_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{k_0}2^{k\alpha p_1}\left(\sum_{j=k+2}^{\infty}2^{(k-j)n\delta_2}2^{j(\lambda-\alpha)}\right)^{p_1}\right) \\ &\leqslant C\|f\|^{p_1}_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{k_0}2^{k\lambda p_1}\left(\sum_{j=k+2}^{\infty}2^{(k-j)(\alpha+n\delta_2-\lambda)}\right)^{p_1}\right) \\ &\leqslant C\|f\|^{p_1}_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{k_0}2^{k\lambda p_1}\right) \\ &\leqslant C\|f\|^{p_1}_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Combining the estimates for E_1, E_2 and E_3 yields

$$\left\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(f)\right\|_{M\dot{K}^{\alpha,\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n)} \leqslant C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}$$

and then completes the proof of Theorem 3.1.

Proof of the corollary 3.1. The statement of the corollary follows from the pointwise estimate

$$M_{\beta(\cdot)}(f)(x) \leqslant CI_{\beta(\cdot)}(|f|)(x), \tag{3.9}$$

where C does not depend on f and x. To prove (3.9), we observe that for any $x \in \mathbb{R}^n$, there exists an $r = r_x$ such that

$$M_{\beta(\cdot)}(f)(x) \leqslant \frac{2}{|B(x,r_x)|^{n-\beta(x)}} \int_{B(x,r_x)} |f(y)|\mathrm{d}y.$$

and on the other hand

$$I_{\beta(\cdot)}(|f|)(x) \geqslant \int_{B(x,r_x)} \frac{|f(y)|}{|x-y|^{n-\beta(x)}}\mathrm{d}y \geqslant \frac{C}{|B(x,r_x)|^{n-\beta(x)}} \int_{B(x,r_x)} |f(y)|\mathrm{d}y. \quad \square$$

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