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ON THE BOUNDEDNESS OF FRACTIONAL TYPE MARCINKIEWICZ INTEGRAL OPERATORS

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Abstract. We show that a broad family of fractional type Marcinkiewicz integral operators with the kernel belonging to $L^1(S^{n-1})$ is bounded from the Triebel-Lizorkin space $F^{\alpha}_{pq}(\mathbb{R}^n)$ to Lebesgue space $L^p(\mathbb{R}^n)$, which improves some known results significantly. This is done by exploiting a local but more general fractional version of Littlewood-Paley g-function.

1. Introduction

Let Ω be a homogeneous function of degree zero on $\mathbb{R}^n$ with $\Omega \in L^1(S^{n-1})$ and its integration on the unit sphere vanishes. Define the fractional type Marcinkiewicz integral operator $\mu_{\Omega,\alpha,\rho,q}$ by

$$\mu_{\Omega,\alpha,\rho,q} f(x) = \left(\int_0^{\infty} \left| \frac{1}{t^{\rho+\alpha}} \int_{|y|\leqslant t} f(x-y) \frac{\Omega(y)}{|y|^{n-\rho}} dy \right|^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

There are many known results concerning with the above operator. Among such achievements are the celebrated and earlier work of Marcinkiewicz [10] and Zygmund [16]. For generalizing Marcinkiewicz and Zygmund's work, in 1958, Stein [12] defined and studied the n-dimensional Marcinkiewicz integral $\mu_{\Omega,0,1,2}$ by the famous real variable techniques introduced by Calderón and Zygmund. Stein showed that the Marcinkiewicz integral $\mu_{\Omega,0,1,2}$ was of type weak $(1,1)$ and type (p,p) $(1 < p \leqslant 2)$ with $\Omega \in \mathrm{Lip}_{\alpha}$ $(0 < \alpha \leqslant 1)$. In 1960, in connection with the well known Marcinkiewicz integral, the L^p boundedness of the parametric Marcinkiewicz integral operator $\mu_{\Omega,0,\rho,2}$ was considered by Hörmander in [9]. From then on, the Marcinkiewicz integral and fractional type Marcinkiewicz integral of higher dimensions with more rough kernel were studied by many authors, see for example [14], [5], [6], [7], [1], [2] and the references therein.

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In 2002, Chen, Fan and Ying [4] proved that $\mu_{\Omega,\alpha,1,q}$ is bounded from the homogeneous Triebel-Lizorkin space $\dot{F}^{\alpha}_{pq}(\mathbb{R}^n)$ to Lebesgue space $L^p(\mathbb{R}^n)$ for suitable α and $\Omega\in L^r(S^{n-1})$ $(r>1)$.

Now we define a more general type fractional Marcinkiewicz integral as follows:

$$S_{\Gamma,h_\nu,\alpha,q,\rho}f(x)=\left(\int_0^\infty\left|\int_{|y|\leqslant t}\frac{h_\nu(t^\alpha|y|)}{t^\rho}f(x-y)\frac{\Gamma(y)}{|y|^{n-\rho}}dy\right|^q\frac{dt}{t}\right)^{\frac{1}{q}}. \tag{1.1}$$

If $-1<\nu<0$, $\beta\in\mathbb{R}$, $\alpha=-1-\beta/\nu$, $\rho=1$, $\delta=1+\nu$, $h_\nu(t)=t^\nu$ and $\Gamma(y)=\Omega(y)$, then the corresponding operator $S_{\Gamma,h_\nu,\alpha,q,\rho}$ reduces to the fractional type Marcinkiewicz operator $\mu_{\Omega,\beta,\delta,q}$. Recently, Al-Salman studied the operator $\mu_{\Omega,\beta,\delta,2}$ and obtain the following result.

THEOREM A. *Let $\alpha>-1$, $-1<\nu<0$ and $\nu(1+\alpha)>-1$. Suppose that h_ν satisfies*

$$|h_\nu(t)|\leqslant c_1t^\nu \quad \text{for } 0<t<1; \tag{1.2}$$

$$|h_\nu(t)-A_\nu t^\nu|\leqslant c_2t^{\beta_\nu} \quad \text{for } 0<t<1; \tag{1.3}$$

$$h_\nu(t)|=O(t^{-\varepsilon_\nu}) \quad \text{for } t\geqslant 1. \tag{1.4}$$

for some $0<\varepsilon_\nu<1$, $\beta_\nu>\max(0,-\nu)$, $c_1,c_2>0$ and real A_ν.

In addition, $\Omega\in L\log L(S^{n-1})$ satisfies the cancellation condition. Then for $2/(2+\nu(1+\alpha))<p<2/(-\nu(1+\alpha))$, there exists a constant $C_p>0$ such that

$$\|S_{\Omega,h_\nu,\alpha,2,1}f\|_{L^p(\mathbb{R}^n)}\leqslant C_p\|f\|_{L^p_{-\nu(1+\alpha)}(\mathbb{R}^n)}, \tag{1.5}$$

where $L^p_{-\nu(1+\alpha)}(\mathbb{R}^n)$ is the Sobolev space.

In this paper, our first main aim is to show that we can weaken the assumption $\Omega\in L\log L(S^{n-1})$ in Theorem A to $\Omega\in L^1(S^{n-1})$, and Theorem A even holds for general q which is bigger than one and general ρ bigger than zero. This is done by using a recent result in [15] concerning with a local but more general fractional version of Littlewood-Paley g-function. Let I be an interval in $(0,+\infty)$. Let $\{\sigma_t : t\in I\}$ be a family of $L^1(\mathbb{R}^n)$ functions, $\alpha\in\mathbb{R}$, $q\in[1,\infty)$. Define the localized Littlewood-Paley g-function as follows:

$$g_{I,\alpha,q}(f)(x)=\left(\int_I|\sigma_t*f(x)|^q\frac{dt}{t^{1+q\alpha}}\right)^{\frac{1}{q}}.$$

If $I=(0,\infty)$, we simply denote $g_{I,\alpha,q}(f)=g_{\alpha,q}(f)$. If $\alpha=0$ and $q=2$, then $g_{0,2}$ was studied by [8] Duoandikoetxea and Rubio de Francia, Here and hereafter, for $1<p<\infty$ we denote $\max(p,p')$ by $\tilde{p}$, where p' is the conjugate exponent of p, i.e. $p'=p/(p-1)$.

The following theorem was obtained in [15].

THEOREM 1. ([15]) *Let* $1 < p,q < \infty$. *Assume* $\|\sup_{t\in I}|\sigma_t| * f\|_{L^\gamma(\mathbb{R}^n)} \leqslant C_p\|f\|_{L^\gamma(\mathbb{R}^n)}$ *for all* $f \in S(\mathbb{R}^n)$ *and all* $\gamma \in (1,\infty)$. *If one of the following conditions holds*

(i) $|\hat{\sigma}_t(\xi)| \leqslant C(|\xi|t)^{\beta_+}$ *for some* $\beta_+ > 0$, *all* $t \in I$ *and* $\alpha \in (0, 4\beta_+/\tilde{p}\tilde{q})$.

(ii) $|\hat{\sigma}_t(\xi)| \leqslant C(|\xi|t)^{-\beta_-}$ *for some* $\beta_- > 0$, *all* $t \in I$ *and* $\alpha \in (-4\beta_-/\tilde{p}\tilde{q}, 0)$.

Then

$$\|g_{I,\alpha,q}(f)\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{\dot{F}^\alpha_{pq}(\mathbb{R}^n)}, \tag{1.6}$$

where $\dot{F}^\alpha_{pq}(\mathbb{R}^n)$ *is the homogeneous Triebel-Lizorkin space.*

From this we deduce the following consequence.

COROLLARY 2. *Let* $1 < p,q < \infty$, $\alpha \in (0,4/\tilde{p}\tilde{q})$, I_1 *and* I_2 *be disjoint intervals with* $\mathbb{R}_+ = \overline{I_1}\bigcup\overline{I_2}$. *Assume*

$$\sigma_t(x) = \frac{\Gamma(x)h(t,x)t^\alpha}{|x|^n} \quad \text{with } \Gamma^*(x') = \sup_{r>0}|\Gamma(rx')| \in L^1(S^{n-1}),$$

and $|h(t,x)| \leqslant \tilde{h}(t,|x|)$ ($t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$) *for some* $\tilde{h}(t,r)$. *Assume the following three conditions:*

$$N_\alpha f = \sup_{t\in I_1}|\sigma_t * f| \text{ is a bounded operator on } L^\gamma(\mathbb{R}^n) \text{ for all } \gamma \in (1,\infty), \tag{1.7}$$

$$\int_0^\infty \left(\int_{I_2} |\tilde{h}(t,r)|^q \frac{dt}{t}\right)^{\frac{1}{q}} \frac{dr}{r} < \infty, \tag{1.8}$$

and for any $t \in I_1$,

$$\int_{\mathbb{R}^n} \frac{\Gamma(x)h(t,x)}{|x|^n}dx = 0, \qquad \int_0^\infty \tilde{h}(t,r)dr \leqslant Ct^{1-\alpha}. \tag{1.9}$$

Then there exists a constant $C > 0$ *such that*

$$\|g_{\alpha,q}(f)\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{F^\alpha_{pq}(\mathbb{R}^n)}.$$

In the above inequality, $F^\alpha_{pq}(\mathbb{R}^n)$ is the Triebel-Lizorkin space, namely the collection of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that $\|f\|_{F^\alpha_{pq}(\mathbb{R}^n)} = \left\|\left(\sum_{j=0}^\infty 2^{j\alpha q}|(\varphi_j\hat{f})^\vee|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)} < \infty$, for $\alpha \in \mathbb{R}, 0 < p < \infty$, and $0 < q < \infty$. With usual modification if $q = \infty$.

Using Corollary 2, we obtain the following improvement and extension of Theorem A.

THEOREM 3. *Let* $1 < p,q < \infty$, $\rho > 0$, $\alpha > -1$, $\nu < 0$. *Assume* $|h_\nu(t)| \leqslant Ct^\nu$ $(0 < t < 1)$ *and* $|h_\nu(t)| \leqslant Ct^{-\varepsilon_\nu}$ $(t \geqslant 1)$ *for some* $\varepsilon_\nu > 0$. *If* $\rho + \nu > 0$, $\alpha\varepsilon_\nu + \rho > 0$, $-\nu(1+\alpha) < \frac{4}{\tilde{p}\tilde{q}}$, *and* $\Omega \in L^1(S^{n-1})$ *satisfies the cancellation condition* $\int_{S^{n-1}}\Omega(x')d\sigma(x') = 0$, *then we have*

$$\|S_{\Omega,h_\nu,\alpha,q,\rho}\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{F^{-\nu(1+\alpha)}_{pq}(\mathbb{R}^n)}. \tag{1.10}$$

REMARK 1. Note that if $q=2$, $\rho=1$, then $-\nu(1+\alpha)<4/\tilde{p}\tilde{q}$ coincides with $2/(2+\nu(1+\alpha))<p<2/(-\nu(1+\alpha))$, which appears in Theorem A.

Our second purpose of this paper is to study the truncated operators defined by

$$S^{(0)}_{\Gamma,h_\nu,\alpha,q,\rho}f(x)=\left(\int_0^1\left|\int_{|y|\leqslant u}f(x-y)\frac{\Gamma(y)}{|y|^{n-\rho}}\frac{h_\nu(u^\alpha|y|)}{u^\rho}dy\right|^q\frac{du}{u}\right)^{\frac{1}{q}} \tag{1.11}$$

and

$$S^{(\infty)}_{\Gamma,h_\nu,\alpha,q,\rho}f(x)=\left(\int_1^\infty\left|\int_{|y|\leqslant u}f(x-y)\frac{\Gamma(y)}{|y|^{n-\rho}}\frac{h_\nu(u^\alpha|y|)}{u^\rho}dy\right|^q\frac{du}{u}\right)^{\frac{1}{q}}. \tag{1.12}$$

We obtain the following result:

THEOREM 4. *Suppose that* $\Gamma\in L^1(S^{n-1})$ *satisfies* $\Gamma^*(x')=\sup_{r>0}|\Gamma(rx')|\in L^1(S^{n-1})$*, and that* h_ν *satisfies* (1.2) *and* (1.4)*. Let* $\alpha>-1$*. Then for* $1\leqslant q\leqslant\infty$ *we have*

(a) $\|S^{(\infty)}_{\Gamma,h_\nu,\alpha,q,\rho}f\|_{L^p(\mathbb{R}^n)}\leqslant C\|\Gamma^*\|_{L^1(S^{n-1})}\|f\|_{L^p(\mathbb{R}^n)}$, $\quad 1\leqslant p\leqslant\infty$ *whenever* $\nu>-\rho$.

(b) $\|S^{(0)}_{\Gamma,h_\nu,\alpha,q,\rho}f\|_{L^p(\mathbb{R}^n)}\leqslant C\|\Gamma^*\|_{L^1(S^{n-1})}\|f\|_{L^p(\mathbb{R}^n)}$, $\quad 1\leqslant p\leqslant\infty$ *whenever* $\nu>0$.

(c) $\|S_{\Gamma,h_\nu,\alpha,q,\rho}f\|_{L^p(\mathbb{R}^n)}\leqslant C\|\Gamma^*\|_{L^1(S^{n-1})}\|f\|_{L^p(\mathbb{R}^n)}$, $\quad 1\leqslant p\leqslant\infty$ *whenever* $\nu>0$.

REMARK 2. The case $\rho=1$, $q=2$ and $1<p<\infty$ was treated in Theorem 1.3 in [1].

2. Proofs of Corollary 2 and Theorems 3 and 4

Proof of Corollary 2. It is easily seen by Minkowski's inequality that for $1\leqslant p,q\leqslant\infty$, $\left(\int_{I_2}|\sigma_t(x)|^q\frac{dt}{t^{1+q\alpha}}\right)^{\frac{1}{q}}\in L^1(\mathbb{R}^n)$ would imply $\|g_{I_2,\alpha,q}(f)\|_{L^p(\mathbb{R}^n)}\leqslant C\|f\|_{L^p(\mathbb{R}^n)}$. By our choice of $\sigma_t(x)$, this suffices to demand $\int_0^\infty\left(\int_{I_2}|\tilde{h}(t,r)|^q\frac{dt}{t}\right)^{\frac{1}{q}}\frac{dr}{r}<\infty$.

We apply Theorem 1 to estimate $\|g_{I_1,\alpha,q}(f)\|_{\dot{F}^\alpha_{pq}(\mathbb{R}^n)}$. By (1.9),

$$\begin{aligned}|\hat{\sigma}_t(\xi)|&=t^\alpha\left|\int_{\mathbb{R}^n}(e^{-2\pi ix\cdot\xi}-1)\frac{\Gamma(x)h(t,x)}{|x|^n}dx\right|\leqslant 2\pi t^\alpha\int_{\mathbb{R}^n}|x\cdot\xi|\frac{|\Gamma(x)h(t,x)|}{|x|^n}dx\\&\leqslant 2\pi t^\alpha\|\Gamma^*\|_{L^1(S^{n-1})}\int_0^\infty\tilde{h}(t,r)dr|\xi|\leqslant Ct|\xi|.\end{aligned}$$

Hence by Theorem 1, we have $\|g_{I_1,\alpha,q}(f)\|_{L^p(\mathbb{R}^n)}\leqslant C\|f\|_{\dot{F}^\alpha_{pq}(\mathbb{R}^n)}$. This completes the proof. □

Now, we are in the position to prove Theorem 3.

Proof of Theorem 3. Set $h(t,x) = \chi_{[0,1]}(\frac{|x|}{t})(\frac{|x|}{t})^{\rho} h_{\nu}(t^{\alpha}|x|)$, $\Gamma(x) = \Omega(x/|x|)$, and $\sigma_t(x) = \Omega(x/|x|)h(x,t)t^{-(1+\alpha)\nu}/|x|^n$. Then $g_{-(1+\alpha)\nu,q}$ reduces to $S_{\Omega,h_\nu,\alpha,q,\rho}$. Define $\tilde{h}(t,r) = \chi_{[0,1]}\left(\frac{r}{t}\right)\left(\frac{r}{t}\right)^{\rho}|h_{\nu}(t^{\alpha}r)|$. Hence by Corollary 2 with $I_1 = (0,1)$ and $I_2 = (1,\infty)$, it suffices to show the following three claims:

$$\int_0^t \left(\frac{r}{t}\right)^{\rho} |h_{\nu}(t^{\alpha}r)|dr \leqslant Ct^{1+\nu(1+\alpha)} \text{ for any } t \in (0,1); \tag{2.1}$$

$$N_{-(1+\alpha)\nu}f = \sup_{t\in(0,1)} t^{-(1+\alpha)\nu}\left|\frac{\Omega(\cdot)h(t,\cdot)}{|\cdot|^n} * f\right| \text{ is a bounded operator on } L^p(\mathbb{R}^n); \tag{2.2}$$

$$\int_0^{\infty}\Big(\int_{\max\{1,r\}}^{\infty} |h_{\nu}(t^{\alpha}r)|^q \frac{dt}{t^{1+\rho q}}\Big)^{\frac{1}{q}} \frac{dr}{r^{1-\rho}} < \infty. \tag{2.3}$$

We first show (2.1). As $\alpha > -1$, we have $t^{\alpha}r < t^{1+\alpha} \leqslant t^0 = 1$ for $0 < r < t < 1$. Thus

$$\int_0^t (\frac{r}{t})^{\rho} h_{\nu}(t^{\alpha}r)dr \leqslant \frac{C}{t^{\rho-\alpha\nu}}\int_0^t r^{\rho+\nu}dr \leqslant \frac{t^{1+(1+\alpha)\nu}}{\rho+\nu+1}$$

if $\rho+\nu > -1$.

For (2.2), as $t^{\alpha}r < 1$,

$$\begin{aligned} N_{-(1+\alpha)\nu}f(x) &= \sup_{t\in(0,1)} t^{-(1+\alpha)\nu}\left|\frac{\Omega(\cdot)h(t,\cdot)}{|\cdot|^n} * f(x)\right| \\ &\leqslant C \sup_{t\in(0,1)} t^{-\rho-\nu}\int_{|y|\leqslant t}\frac{|\Omega(y')|}{|y|^{n-\rho-\nu}}|f(x-y)|dy. \end{aligned}$$

As $\Omega \in L^1(S^{n-1})$ and $\rho+\nu > 0$, by using the L^p boundedness of the Hardy-Littlewood maximal function, $N_{-(1+\alpha)\nu}f$ is indeed a bounded operator on $L^p(\mathbb{R}^n)$.

We next show (2.3). We rewrite the left side of (2.3) as follows.

$$\begin{aligned} &\int_0^{\infty}(\int_{\max\{1,r\}}^{\infty} |h_{\nu}(t^{\alpha}r)|^q \frac{dt}{t^{1+\rho q}})^{\frac{1}{q}} \frac{dr}{r^{1-\rho}} \\ &= \int_0^1 (\int_1^{\infty} |h_{\nu}(t^{\alpha}r)|^q \frac{dt}{t^{1+\rho q}})^{\frac{1}{q}} \frac{dr}{r^{1-\rho}} + \int_1^{\infty} (\int_r^{\infty} |h_{\nu}(t^{\alpha}r)|^q \frac{dt}{t^{1+\rho q}})^{\frac{1}{q}} \frac{dr}{r^{1-\rho}} \\ &=: I + II. \end{aligned}$$

For the term II, as $r > 1$ and $\alpha > -1$, it follows $t^{\alpha}r \geqslant 1$ for $t > r$. We obtain

$$\int_r^{\infty} |h_{\nu}(t^{\alpha}r)|^q \frac{dt}{t^{q\rho+1}} \leqslant C\int_r^{\infty} \frac{1}{t^{q\alpha\varepsilon_\nu}r^{q\varepsilon_\nu}}\frac{dt}{t^{q\rho+1}} = \frac{C}{q\rho+q\alpha\varepsilon_\nu}\frac{1}{r^{q(1+\alpha)\varepsilon_\nu+q\rho}} \tag{2.4}$$

because of $\alpha\varepsilon_\nu+\rho > 0$. So, it can be seen that $II < \infty$ as $(1+\alpha)\varepsilon_\nu > 0$.

For the term I, if $\alpha > 0$, we have $r^{-1/\alpha} \geqslant 1$ as $0 < r < 1$, and hence

$$\begin{aligned}
&\int_1^\infty |h_\nu(t^\alpha r)|^q \frac{dt}{t^{q\rho+1}} \\
&\leqslant C\int_1^{r^{-1/\alpha}} t^{q\alpha\nu} r^{q\nu} \frac{dt}{t^{q\rho+1}} + C\int_{r^{-1/\alpha}}^\infty \frac{1}{t^{q\alpha\varepsilon_\nu} r^{q\varepsilon_\nu}} \frac{dt}{t^{q\rho+1}} \\
&= \frac{Cr^{q\nu}}{q\alpha\nu - q\rho}(r^{-q\nu+q\rho/\alpha} - 1) + \frac{C}{r^{q\varepsilon_\nu}(q\alpha\varepsilon + q\rho)} r^{q\varepsilon_\nu + q\rho/\alpha} \\
&\leqslant \frac{C}{q\rho - q\alpha\nu} r^{q\nu} + \frac{C}{q\rho + q\alpha\varepsilon_\nu} r^{q\rho/\alpha},
\end{aligned} \tag{2.5}$$

because of $\alpha\varepsilon_\nu + \rho > 0$, $\alpha, \rho > 0$ and $\nu < 0$. Hence, we have $I < \infty$.

If $-1 < \alpha \leqslant 0$, it follows $t^\alpha r \leqslant 1$ and we have

$$\int_1^\infty |h_\nu(t^\alpha r)|^q \frac{dt}{t^{q\rho+1}} \leqslant C\int_1^\infty t^{q\alpha\nu} r^{q\nu} \frac{dt}{t^{q\rho+1}} \leqslant \frac{C}{q\rho - q\alpha\nu} r^{q\nu} \tag{2.6}$$

provided $\rho - \alpha\nu > 0$. Since $\rho + \nu > 0$ and $0 > \alpha > -1$, we get $\rho - \alpha\nu > 0$. So, we have $I < \infty$ when $\rho + \nu > 0$.

In conclusion, (2.3) holds if $\rho + \nu > 0$ and $\alpha\varepsilon_\nu + \rho > 0$ and the theorem is proved. □

REMARK 3. Under $\rho = 1$ and $p = q = 2$, Corollary 1.5 [1] coincides with the above corollary. Theorem 1.6 in [1] deals with the case of general p but demanding additionally $\varepsilon_\nu < 1$ and $\Omega \in L(\log L)(S^{n-1})$.

REMARK 4. If we set $h_\nu(x) = |x|^\nu$ for $\nu < 0$, then the operator $S_{\Omega,h_\nu,\alpha,q,\rho}$ becomes

$$\mu_{\Omega,\beta,\tilde\rho,q} f(x) = \left(\int_0^\infty \left|\frac{1}{t^{\tilde\rho+\beta}} \int_{|y|\leqslant t} f(x-y)\frac{\Omega(y)}{|y|^{n-\tilde\rho}}dy\right|^q \frac{dt}{t}\right)^{\frac{1}{q}}, \tag{2.7}$$

with $\tilde\rho = \rho + \nu$, $\beta = -(1+\alpha)\nu$. By Theorem 3, we get the boundedness of the fractional Marcinkiewicz integral operator for $0 < \alpha < 4/\tilde\rho\tilde q$ and $\tilde\rho > 0$:

$$\|\mu_{\Omega,\beta,\tilde\rho,q} f\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{F^\alpha_{pq}(\mathbb{R}^n)}. \tag{2.8}$$

This was proved recently by the authors in [15] and improves Corollary 1.7 and Corollary 1.8 in [1].

Proof of Theorem 4. In the case $1 \leqslant q < \infty$, by Minkowski's inequality we have

$$S^{(0)}_{\Gamma,h_\nu,\alpha,q,\rho} f(x) \leqslant \int_{\mathbb{R}^n} |f(x-y)| \frac{|\Gamma(y)|}{|y|^{n-\rho}} \left(\int_0^1 \chi_{\{|y|\leqslant u\}} |h_\nu(u^\alpha|y|)|^q \frac{du}{u^{1+q\rho}}\right)^{\frac{1}{q}} dy \tag{2.9}$$

and

$$S^{(\infty)}_{\Gamma,h_\nu,\alpha,q,\rho} f(x) \leqslant \int_{\mathbb{R}^n} |f(x-y)| \frac{|\Gamma(y)|}{|y|^{n-\rho}} \left(\int_1^\infty \chi_{\{|y|\leqslant u\}} |h_\nu(u^\alpha|y|)|^q \frac{du}{u^{1+q\rho}}\right)^{\frac{1}{q}} dy. \tag{2.10}$$

We have similar estimates in the case $q=\infty$.

(1) We treat first $S^{(0)}_{\Gamma,h_\nu,\alpha,q,\rho}$. Let $1\leqslant q<\infty$. If $|y|\leqslant 1$, $|y|\leqslant u<1$ and $\alpha>-1$ $(\rho\neq\alpha\nu)$, then we have $1>u^{1+\alpha}\geqslant u^{\alpha}|y|$. Thus we have by the assumption $|h_\nu(t)|\leqslant Ct^{\nu}$ $(0<t<1)$

$$\begin{aligned}\int_0^1 \chi_{\{|y|\leqslant u\}}|h_\nu(u^\alpha|y|)|^q\frac{du}{u^{1+q\rho}} &\leqslant C\int_{|y|}^1 u^{q\alpha\nu}|y|^{q\nu}\frac{du}{u^{1+q\rho}}\\ &=\frac{C|y|^{q\nu}}{q\alpha\nu-q\rho}(1-|y|^{q\alpha\nu-q\rho})\leqslant\frac{C}{|q\rho-q\alpha\nu|}(|y|^{q\nu}+|y|^{q(1+\alpha)\nu-q\rho})\end{aligned}\tag{2.11}$$

for $|y|\leqslant 1$. If $|y|>1$, then we see that the first term in (2.11) is equal to 0. Set

$$G(x)=\begin{cases}\frac{\Gamma^*(y/|y|)}{|y|^{n-\rho}}(|y|^{\nu}+|y|^{(1+\alpha)\nu-\rho}), & |y|\leqslant 1\\ 0, & |y|>1.\end{cases}\tag{2.12}$$

In the case $\rho=\alpha\nu$, the left side of (2.11) is bounded by $C\log 1/|y|$. So, we set $G(x)=\Gamma^*(y/|y|)\log 1/|y|$ $(|y|\leqslant 1)$ $=0$ $(|y|>1)$. Thus, in any case, if $\nu>0$, $\alpha>-1$ and $\Gamma^*(y')=\sup_{r>0}|\Gamma(ry')|\in L^1(S^{n-1})$, we see that $G\in L^1(\mathbb{R}^n)$, $\|G\|_{L^1(\mathbb{R}^n)}\leqslant C\|\Gamma^*\|_{L^1(S^{n-1})}$ and $S^{(0)}_{\Gamma,h_\nu,\alpha,q,\rho}f(x)\leqslant C|f|*G(x)$. Therefore we have

$$\|S^{(0)}_{\Gamma,h_\nu,\alpha,q,\rho}f\|_{L^p(\mathbb{R}^n)}\leqslant C\|\Gamma^*\|_{L^1(S^{n-1})}\|f\|_{L^p(\mathbb{R}^n)},\quad 1\leqslant p\leqslant\infty.\tag{2.13}$$

In the case $q=\infty$, we obtain similarly

$$\sup_{\{|y|\leqslant u<1\}}\frac{|h_\nu(u^\alpha|y|)|}{u^\rho}\leqslant C(|y|^{\nu}+|y|^{(1+\alpha)\nu-\rho}),$$

and so we have the same estimate (2.13) in this case, too. This shows (b).

(2) Next we treat $S^{(\infty)}_{\Gamma,h_\nu,\alpha,q,\rho}$. Let $1\leqslant q<\infty$.

(i) If $|y|\leqslant 1$ and $\alpha>0$ $(\rho\neq\alpha\nu)$, it follows $|y|^{-1/\alpha}\geqslant 1$ and we have

$$\begin{aligned}&\int_1^\infty \chi_{\{|y|\leqslant u\}}|h_\nu(u^\alpha|y|)|^q\frac{du}{u^{1+q\rho}}\\ &\quad\leqslant C\int_1^{|y|^{-1/\alpha}}u^{q\alpha\nu}|y|^{q\nu}\frac{du}{u^{1+q\rho}}+C\int_{|y|^{-1/\alpha}}^\infty\frac{1}{u^{q\alpha\varepsilon_\nu}|y|^{q\varepsilon_\nu}}\frac{du}{u^{1+q\rho}}\\ &\quad=\frac{C|y|^{q\nu}}{q\alpha\nu-q\rho}(|y|^{-q\nu+q\rho/\alpha}-1)+\frac{C}{|y|^{q\varepsilon_\nu}(q\alpha\varepsilon_\nu+q\rho)}|y|^{q\varepsilon_\nu+q\rho/\alpha}\\ &\quad\leqslant\frac{C}{|q\rho-q\alpha\nu|}|y|^{q\nu}+C\Big(\frac{1}{|q\rho-q\alpha\nu|}+\frac{1}{q\rho+q\alpha\varepsilon_\nu}\Big)|y|^{q\rho/\alpha}.\end{aligned}\tag{2.14}$$

When $\rho=\alpha\nu$, we can bound the left side of (2.14) by $C\log 1/|y|$.

(ii) If $|y|\leqslant 1$, $u>1$ and $-1<\alpha\leqslant 0$, it follows $u^{\alpha}|y|\leqslant 1$ and we have

$$\begin{aligned}\int_1^\infty \chi_{\{|y|\leqslant u\}}|h_\nu(u^\alpha|y|)|^q\frac{du}{u^{1+q\rho}}&\leqslant C\int_1^\infty u^{q\alpha\nu}|y|^{q\nu}\frac{du}{u^{1+q\rho}}\\ &\leqslant\frac{C}{q\rho-q\alpha\nu}|y|^{q\nu},\end{aligned}\tag{2.15}$$

provided $\rho-\alpha\nu>0$.

(iii) If $|y|>1$ and $\alpha>-1$, we obtain by $|h_\nu(t)|\leqslant C_\infty t^{-\varepsilon_\nu}$ $(t\geqslant 1)$

$$\begin{aligned}\int_1^\infty \chi_{\{|y|\leqslant u\}}|h_\nu(u^\alpha|y|)|^q\frac{du}{u^{1+q\rho}}&=\int_{|y|}^\infty |h_\nu(u^\alpha|y|)|^q\frac{du}{u^{1+q\rho}}\\ \leqslant C\int_{|y|}^\infty \frac{1}{u^{q\alpha\varepsilon_\nu}|y|^{q\varepsilon_\nu}}\frac{du}{u^{1+q\rho}}&=\frac{C}{q\rho+q\alpha\varepsilon_\nu}\frac{1}{|y|^{q(1+\alpha)\varepsilon_\nu+q\rho}}.\end{aligned}\tag{2.16}$$

Now we set in the case $\alpha>0$ $(\rho\neq\alpha\nu)$

$$G(x)=\begin{cases}\frac{\Gamma^*(y/|y|)}{|y|^{n-\rho}}(|y|^\nu+|y|^{\rho/\alpha}), & |y|\leqslant 1\\ \frac{\Gamma^*(y/|y|)}{|y|^{n-\rho}}|y|^{-(1+\alpha)\varepsilon_\nu-\rho}, & |y|>1,\end{cases}\tag{2.17}$$

in the case $\alpha>0$ $(\rho=\alpha\nu)$ we modify the above as in the case (1), and in the case $-1<\alpha\leqslant 0$ and $\rho-\alpha\nu>0$

$$G(x)=\begin{cases}\frac{\Gamma^*(y/|y|)}{|y|^{n-\rho}}|y|^\nu, & |y|\leqslant 1\\ \frac{\Gamma^*(y/|y|)}{|y|^{n-\rho}}|y|^{-(1+\alpha)\varepsilon_\nu-\rho}, & |y|>1.\end{cases}\tag{2.18}$$

Hence, if $\alpha>-1$ and $\nu>-\rho$ (in the case $-1<\alpha<0$, $\nu>-\rho$ implies $\nu>-\rho/(-\alpha)$ i.e. $\rho-\alpha\nu>0$), and $\Gamma^*(y')=\sup_{r>0}|\Gamma(ry')|\in L^1(S^{n-1})$, then we see that $G\in L^1(\mathbb{R}^n)$, $\|G\|_{L^1(\mathbb{R}^n)}\leqslant C\|\Gamma^*\|_{L^1(S^{n-1})}$ and $S^{(\infty)}_{\Gamma,h_\nu,\alpha,q,\rho}f(x)\leqslant C|f|*G(x)$. Thus

$$\|S^{(\infty)}_{\Gamma,h_\nu,\alpha,q,\rho}f\|_{L^p(\mathbb{R}^n)}\leqslant C\|\Gamma^*\|_{L^1(S^{n-1})}\|f\|_{L^p(\mathbb{R}^n)},\quad 1\leqslant p\leqslant\infty.\tag{2.19}$$

In the case $q=\infty$, we obtain similarly that

$$\sup_{\{u>1\}}\frac{\chi_{|y|\leqslant u}|h_\nu(u^\alpha|y|)|}{u^\rho}\leqslant\begin{cases}C(|y|^\nu+|y|^{\rho/\alpha}), & \text{if } \alpha>0 \text{ and } |y|<1,\\ C|y|^\nu & \text{if } 0>\alpha>-1,\ |y|<1 \text{ and } \rho-\alpha\nu>0,\\ C|y|^{-\rho-\varepsilon_\nu(1+\alpha)\nu} & \text{if } >\alpha>-1 \text{ and } |y|\geqslant 1.\end{cases}$$

Therefore, we have the same estimate (2.19) in this case, too.

Thus, we finished the proof for (a). The assertion (c) follows from (a) and (b). □

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