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NEW INEQUALITIES AND INFINITE PRODUCT FORMULAS FOR TRIGONOMETRIC AND THE LEMNISCATE FUNCTIONS

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Abstract. In this paper, we give a new approach to prove inequalities for the Schwab-Borchardt mean, the lemniscatic mean and the arithmetic geometric mean. Additionally, we apply these means to inequalities for trigonometric functions or the lemniscate functions by considering several functional inequalities. One of these applications includes infinite product formulas for the lemniscate function and the arithmetic geometric mean by considering several functional equations.

1. Introduction

In mathematical literature, the Schwab-Borchardt mean, the lemniscatic mean and the arithmetic geometric mean have attracted many researchers. For example, inequalities for these means were obtained by differentiation or a sequential method. These were applied to get trigonometric or lemniscatic inequalities [5, 6, 7, 8, 9]. The following trigonometric inequality

$$\frac{2+\cos x}{3} > \frac{\sin x}{x} > \cos^{1/3} x \tag{1}$$

for $0 < x \leqslant \pi/2$ were established by use of Schwab-Borchardt mean in [8]. The first inequality in the above is a well known inequality which was obtained by N. Cusa and C. Huygens. The second inequality in the above was obtained by D. D. Adamović and D. S. Mitrinović [5].

In this paper, we obtain a general approach to get these inequalities for means. We apply our approach to inequalities for trigonometric or the lemniscate functions by considering several functional inequalities. For example, we obtain

$$\frac{4+\mathrm{sl}' x}{5} > \frac{\mathrm{sl}\, x}{x} > (\mathrm{sl}' x)^{1/5} \tag{2}$$

for $0 < x \leqslant L/2$ where $\mathrm{sl}\, x$ is the lemniscatic sine and $L = 2.622057\cdots$ is the lemniscate constant. These inequalities are lemniscatic analogs of (1). Furthermore, by considering several functional equations, we get infinite product formulas for the lemniscate function and the arithmetic geometric mean as follows:

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$$\frac{\operatorname{sl} x}{x} = \sqrt{\frac{1+\operatorname{sl}' x}{2}} \times \sqrt{\frac{1+\sqrt{\frac{2}{1+\operatorname{sl}' x}}}{2}} \times \sqrt{\frac{1+\sqrt{\frac{2}{1+\sqrt{\frac{2}{1+\operatorname{sl}' x}}}}}{2}} \times \cdots \tag{3}$$

for $0 < x \leqslant L/2$. When $x = L/2$ in (3), we obtain

$$\frac{2}{L} = \sqrt{\frac{1}{2}} \times \sqrt{\frac{1+\sqrt{2}}{2}} \times \sqrt{\frac{1+\sqrt{\frac{2}{1+\sqrt{2}}}}{2}} \times \cdots$$

which was obtained in [3, 4] by different way (for more details, see section 3).

2. Theorems for the Schwab-Borchardt mean

The Schwab-Borchardt mean SB [1, 2, 8, 9] is defined as follows:
Given positive constants a_0, b_0, we set a_n, b_n inductively by

$$a_{n+1} = \frac{a_n + b_n}{2}, \, b_{n+1} = (a_{n+1} b_n)^{1/2}, \tag{4}$$

and define

$$SB(a_0, b_0) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n,$$

which is given as

$$SB(a_0,b_0) = \begin{cases} \dfrac{\sqrt{b_0^2 - a_0^2}}{\arccos(a_0/b_0)} & 0 \leqslant a_0 < b_0, \\ \dfrac{\sqrt{a_0^2 - b_0^2}}{\operatorname{arccosh}(a_0/b_0)} & b_0 < a_0. \end{cases}$$

It is known [9] that this mean satisfies

$$a_0 < a_1 < \cdots < a_n < \cdots < SB < \cdots < b_n < \cdots < b_1 < b_0 \quad (a_0 < b_0)$$

and

$$b_0 < b_1 < \cdots < b_n < \cdots < SB < \cdots < a_n < \cdots < a_1 < a_0 \quad (b_0 < a_0).$$

There exists an attempt to get bounds for SB. The following bounds were established in [9]. For arbitrary $a_0, b_0 (\neq a_0)$, we have

$$\frac{a_n + 2b_n}{3} > SB(a_0, b_0) > (a_n b_n^2)^{1/3}. \tag{5}$$

We then give the following lemma to show (5). Thorough out this paper, p, q denote real numbers.

LEMMA 2.1. *Let f be a real valued function which is continuous at 1. Suppose one of the following conditions holds.*

1. $t^p f(t) > f(2t^2-1)$ *for* $0 < t < 1$ *and* $a_0 < b_0$
2. $t^p f(t) > f(2t^2-1)$ *for* $1 < t$ *and* $b_0 < a_0$.

Then, with a real number q, the sequence

$$b_n^q f\left(\frac{a_n}{b_n}\right) \tag{6}$$

is strictly increasing as n increase. If $t^q f(t) < f\left(2t^2-1\right)$ holds for above condition, with a real number q, then the sequence (6) is strictly decreasing as n increases.

Proof. We note that for all n we have $0 < a_{n+1}/b_{n+1} < 1$ or $1 < a_{n+1}/b_{n+1}$. First, we assume that $t^q f(t) > f\left(2t^2-1\right)$ holds. Since $b_{n+1}/b_n = a_{n+1}/b_{n+1}$ and $a_n/b_n = 2(a_{n+1}/b_{n+1})^2 - 1$, we have

$$\begin{aligned}
&b_{n+1}^q f\left(\frac{a_{n+1}}{b_{n+1}}\right) - b_n^q f\left(\frac{a_n}{b_n}\right)\\
&= b_n^q \left\{\left(\frac{b_{n+1}}{b_n}\right)^q f\left(\frac{a_{n+1}}{b_{n+1}}\right) - f\left(\frac{a_n}{b_n}\right)\right\}\\
&= b_n^q \left\{\left(\frac{a_{n+1}}{b_{n+1}}\right)^q f\left(\frac{a_{n+1}}{b_{n+1}}\right) - f\left(\frac{2a_{n+1}^2}{b_{n+1}^2} - 1\right)\right\} > 0.
\end{aligned}$$

If $t^q f(t) < f\left(2t^2-1\right)$ holds, we get the second assertion in the same way as above. This completes the proof. □

We then give the following theorem which refines the inequalities (5).

THEOREM 2.2. *Let $\{a_n\}$, $\{b_n\}$ be sequences determined by (4). We then have*

$$\begin{aligned}
\frac{a_n + 2b_n}{3} &> \left\{\frac{(a_n+2b_n)^3 + a_n b_n^2}{28}\right\}^{1/3} >\\
&SB(a_0,b_0) > \left\{\frac{a_n b_n^2 (a_n + 2b_n)}{3}\right\}^{1/4} > (a_n b_n^2)^{1/3}.
\end{aligned}$$

Proof. We can show the first and the fourth inequalities by direct computations. We can show the second inequality by setting $f(t) = (t+2)^3 + t$ and $q = 3$ and the third inequality by setting $f(t) = t(t+2)$ and $q = 4$ in Lemma 2.1. □

As we saw, if $a_0 < b_0$, the Schwab-Borchardt mean SB is represented by $\arccos(a_0/b_0)$. We then get the following theorem for trigonometric inequalities.

THEOREM 2.3. *Let q be a real number and f be a real valued function which is continuous at* 1. *If $t^q f(t) > f\left(2t^2-1\right)$ holds for $0<t<1$, then we have*

$$f(1)\left(\frac{\sin x}{x}\right)^q > f(\cos x) \quad (0 < x \leqslant \pi/2). \tag{7}$$

The inequality in (7) is reversed if $t^q f(t) < f\left(2t^2-1\right)$ holds for $0<t<1$.

Proof. We put $a_0/b_0 = \cos x$ and if $t^q f(t) > f(2t^2-1)$, we have

$$\lim_{n\to\infty}\left\{\left(\frac{b_n}{b_0}\right)^q f\left(\frac{a_n}{b_n}\right)\right\} = f(1)\left(\frac{\sqrt{1-\frac{a_0^2}{b_0^2}}}{\arccos(a_0/b_0)}\right)^q > f\left(\frac{a_0}{b_0}\right)$$

from Lemma 2.1. This shows (7). If $t^q f(t) < f\left(2t^2-1\right)$ holds, we get the second assertion in the same way as above. □

We can get the inequality (1) in the same way as the proof of Theorem 2.2.
We here note that if the function

$$f(t) = \sqrt{\frac{1+t}{2}} \times \sqrt{\frac{1+\sqrt{\frac{1+t}{2}}}{2}} \times \sqrt{\frac{1+\sqrt{\frac{1+\sqrt{\frac{1+t}{2}}}{2}}}{2}} \times \cdots$$

converges, it satisfies $tf(t) = f(2t^2-1)$ and this suggests $f(1)(\sin x/x) = f(\cos x)$. Indeed, the following well known theorem which was established by Euler holds.

THEOREM 2.4. *If $0 < x \leqslant \pi/2$, we have*

$$\begin{aligned}\frac{\sin x}{x} &= \sqrt{\frac{1+\cos x}{2}} \times \sqrt{\frac{1+\sqrt{\frac{1+\cos x}{2}}}{2}} \times \sqrt{\frac{1+\sqrt{\frac{1+\sqrt{\frac{1+\cos x}{2}}}{2}}}{2}} \times \cdots \\ &= \cos\frac{x}{2} \times \cos\frac{x}{4} \times \cos\frac{x}{8} \times \cdots.\end{aligned}$$

Proof. We put $c_n = a_n/b_n$ in the Schwab-Borchardt mean. First, we note that $a_{n+1}/b_{n+1} = b_{n+1}/b_n$ holds. We then have

$$\begin{aligned}\frac{b_n}{b_0} &= \frac{c_1 \times c_2 \times \cdots \times c_n}{\frac{b_0}{b_n} \times \frac{a_1}{b_1} \times \frac{a_2}{b_2} \times \cdots \times \frac{a_n}{b_n}} \\ &= \frac{c_1 \times c_2 \times \cdots \times c_n}{\frac{b_0}{b_n} \times \frac{b_1}{b_0} \times \frac{b_2}{b_1} \times \cdots \times \frac{b_n}{b_{n-1}}} \\ &= c_1 \times c_2 \times \cdots \times c_n.\end{aligned}$$

We note that $c_{n+1}^2 = (1+c_n)/2$ holds. By setting $n \to \infty$ and $c_0 = \cos x$, we get

$$\begin{aligned}\frac{\sqrt{1-c_0^2}}{\arccos c_0} &= \lim_{n\to\infty} \frac{b_n}{b_0} \\ &= c_1 \times c_2 \times \cdots \times c_n \times \cdots \\ &= \sqrt{\frac{1+c_0}{2}} \times \sqrt{\frac{1+\sqrt{\frac{1+c_0}{2}}}{2}} \times \sqrt{\frac{1+\sqrt{\frac{1+\sqrt{\frac{1+c_0}{2}}}{2}}}{2}} \times \cdots\end{aligned}$$

which is equivalent to

$$\begin{aligned}\frac{\sin x}{x} &= \sqrt{\frac{1+\cos x}{2}} \times \sqrt{\frac{1+\sqrt{\frac{1+\cos x}{2}}}{2}} \times \sqrt{\frac{1+\sqrt{\frac{1+\sqrt{\frac{1+\cos x}{2}}}{2}}}{2}} \times \cdots \\ &= \cos\frac{x}{2} \times \cos\frac{x}{4} \times \cos\frac{x}{8} \times \cdots,\end{aligned}$$

where we used half-angle formula for $\cos x$. This completes the proof. □

3. Theorems for the lemniscatic mean

We establish similar statements for the lemniscate functions as of [1]. The arclemniscate sine is defined by

$$\operatorname{arcsl} y = \int_0^y (1-s^4)^{-1/2} ds \quad (y^2 \leqslant 1). \tag{8}$$

Similarly, the hyperbolic arclemniscate sine is defined by

$$\operatorname{arcslh} y = \int_0^y (1+s^4)^{-1/2} ds.$$

The lemniscatic constant is given as

$$\frac{L}{2} = \int_0^1 (1-s^4)^{-1/2} ds = 1.311028\cdots.$$

Then, the lemniscatic sine $\operatorname{sl} x$ $(0 < x \leqslant L/2)$ is defined as of the inverse function of $\operatorname{arcsl} y$. The following lemma for $\operatorname{sl} x$ is valid.

LEMMA 3.1.

$$\operatorname{sl}' x = \sqrt{1-\operatorname{sl}^4 x}.$$

We can show this lemma by setting $y = \operatorname{sl} x$ and by differentiating both sides in (8). We then obtain a lemma for the lemniscatic mean [2, 6, 7] which is defined as follows: Given real a_0, b_0 satisfying $a_0 > 0$, $b_0 \geqslant 0$, we set a_n, b_n inductively by

$$a_{n+1} = \frac{a_n + b_n}{2}, b_{n+1} = (a_n a_{n+1})^{1/2}, \tag{9}$$

and define

$$LM(a_0, b_0) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n,$$

which is given as

$$[LM(a_0,b_0)]^{-1/2} = \begin{cases} (a_0^2 - b_0^2)^{-1/4}\text{arcsl}\left(1 - \frac{b_0^2}{a_0^2}\right)^{1/4} & 0 \leqslant b_0 < a_0 \\ (b_0^2 - a_0^2)^{-1/4}\text{arcslh}\left(\frac{b_0^2}{a_0^2} - 1\right)^{1/4} & a_0 < b_0. \end{cases}$$

The following bounds were established in [6]. For arbitrary $a_0, b_0 (\neq a_0)$, we have

$$\frac{3a_n + 2b_n}{5} > LM(a_0, b_0) > (a_n^3 b_n^2)^{1/5}. \tag{10}$$

We then show the following lemma.

LEMMA 3.2. *Let f be a real valued function on $(0,\infty)$ which is continuous at 1. If $t^{2p} f(t^{-1}) > f(2t^2 - 1)$ holds for $t \neq 1$ with a real number p, then the sequence*

$$a_n^p f\left(\frac{b_n}{a_n}\right) \tag{11}$$

is strictly increasing as n increases. If $t^{2p} f(t^{-1}) < f(2t^2 - 1)$ holds for $t \neq 1$, with a real number p, then the sequence (11) is strictly decreasing as n increases.

Proof. We note that for all n we have $a_{n+1}/b_{n+1} \neq 1$ because $a_0/b_0 \neq 1$. First, we assume that $t^{2p} f(t^{-1}) > f(2t^2 - 1)$ holds. Since $b_{n+1}/a_{n+1} = (a_n/a_{n+1})^{1/2}$ and $a_{n+1}/a_n = (1 + b_n/a_n)/2$, we have

$$\begin{aligned} &a_{n+1}^p f\left(\frac{b_{n+1}}{a_{n+1}}\right) - a_n^p f\left(\frac{b_n}{a_n}\right) \\ &= a_n^p \left\{ \left(\frac{a_{n+1}}{a_n}\right)^p f\left(\frac{b_{n+1}}{a_{n+1}}\right) - f\left(\frac{b_n}{a_n}\right) \right\} \\ &= a_n^p \left\{ \frac{a_{n+1}^{2p}}{b_{n+1}^{2p}} f\left(\frac{b_{n+1}}{a_{n+1}}\right) - f\left(\frac{2a_{n+1}^2}{b_{n+1}^2} - 1\right) \right\} > 0. \end{aligned}$$

If $t^{2p} f(t^{-1}) < f(2t^2 - 1)$ holds, we get the second assertion in the same way as above. This completes the proof. □

By setting $p = 1$, $f(t) = t^{2/5}$ or $p = 1$, $f(t) = 2t + 3$, we have (10). Furthermore, we get the following theorem which refines the left side of (10).

THEOREM 3.3. *Let* $\{a_n\}$, $\{b_n\}$ *be the sequence determined by* (9). *We then have*

$$\frac{3a_n+2b_n}{5} > \left\{\frac{(3a_n+2b_n)^5+a_n^3b_n^2}{3126}\right\}^{1/5} > LM(a_0,b_0).$$

Proof. We can show the first inequality by a simple computation. We can show the second inequality by setting $f(t)=(2t+3)^5+t^2$ and $p=5$ in Lemma 3.2. □

We should note that inequalities for the lemniscatic mean and functions are established in [6, 7]. We here obtain the following general result to get lemniscatic inequalities.

THEOREM 3.4. *Let* p *be a real number and* f *be a real valued function on* $(0,\infty)$ *which is continuous at* 1. *If* $t^{2p}f(t^{-1}) > f(2t^2-1)$ *holds for* $t\neq 1$, *then we have*

$$f(1)\left(\frac{\mathrm{sl}\,x}{x}\right)^{2p} > f(\mathrm{sl}'x) \quad (0<x\leqslant L/2). \tag{12}$$

The inequality in (12) *is reversed if* $t^{2p}f(t^{-1}) < f(2t^2-1)$ *holds for* $t\neq 1$.

Proof. If $t^{2p}f(t^{-1}) > f(2t^2-1)$, we have

$$\lim_{n\to\infty}\left\{\left(\frac{a_n}{a_0}\right)^p f\left(\frac{b_n}{a_n}\right)\right\} = f(1)\left(1-\frac{b_0^2}{a_0^2}\right)^{p/2}\mathrm{arcsl}^{-2p}\left(1-\frac{b_0^2}{a_0^2}\right)^{1/4} > f\left(\frac{b_0}{a_0}\right)$$

from Lemma 3.2. By setting $1-b_0^2/a_0^2=\mathrm{sl}^4 x$, we get the assertion because $\mathrm{sl}'x = \sqrt{1-\mathrm{sl}^4 x}$. If $t^{2p}f(t^{-1}) < f(2t^2-1)$ holds, we get the second assertion in the same way as above. □

Direct computations give the following corollary which is a lemniscatic analog of the inequality (1).

COROLLARY 3.5. *If* $0<x\leqslant L/2$, *we have*

$$\frac{4+\mathrm{sl}'x}{5} > \frac{\mathrm{sl}\,x}{x} > (\mathrm{sl}'x)^{1/5}.$$

The second inequality is shown in [7] by different representation.

We note that if the function

$$f(t)=\frac{1+t}{2}\times\frac{1+\sqrt{\frac{2}{1+t}}}{2}\times\frac{1+\sqrt{\frac{2}{1+\sqrt{\frac{2}{1+t}}}}}{2}\times\cdots$$

converges, it satisfies $t^2 f(t^{-1}) = f(2t^2-1)$ and this implies $f(1)(\mathrm{sl}\,x/x)^2 = f(\mathrm{sl}'x)$. Indeed, we obtain the following theorem which is a lemniscatic analog of Theorem 2.4.

THEOREM 3.6. *If $0 < x \leqslant L/2$, we have*

$$\frac{\mathrm{sl}\, x}{x} = \sqrt{\frac{1+\mathrm{sl}' x}{2}} \times \sqrt{\frac{1+\sqrt{\frac{2}{1+\mathrm{sl}' x}}}{2}} \times \sqrt{\frac{1+\sqrt{\frac{2}{1+\sqrt{\frac{2}{1+\mathrm{sl}' x}}}}}{2}} \times \cdots.$$

Proof. We put $a_n/b_n = c_n$ in the lemniscatic mean. First, we note that $a_{n+1}/b_{n+1} = b_{n+1}/a_n$. We then have

$$\begin{aligned}\frac{a_{n+1}}{a_0} &= \frac{c_1 \times c_2 \times \cdots \times c_{n+1}}{\frac{a_0}{a_{n+1}} \times \frac{a_1}{b_1} \times \frac{a_2}{b_2} \times \cdots \times \frac{a_{n+1}}{b_{n+1}}} \\ &= \frac{c_1 \times c_2 \times \cdots \times c_{n+1}}{\frac{a_0}{a_{n+1}} \times \frac{b_1}{a_0} \times \frac{b_2}{a_1} \times \cdots \times \frac{b_{n+1}}{a_n}} \\ &= c_1^2 \times \cdots \times c_{n+1}^2.\end{aligned}$$

We note that $c_{n+1}^2 = (1+1/c_n)/2$ holds. By setting $n \to \infty$ and $1/c_0 = \mathrm{sl}' x = \sqrt{1-\mathrm{sl}^4 x}$, we have

$$\begin{aligned}\frac{(1-1/c_0^2)^{1/2}}{\mathrm{arcsl}^2(1-1/c_0^2)^{1/4}} &= \lim_{n\to\infty} \frac{a_{n+1}}{a_0} \\ &= c_1^2 \times \cdots \times c_n^2 \times \cdots \\ &= \frac{1+1/c_0}{2} \times \frac{1+\sqrt{\frac{2}{1+1/c_0}}}{2} \times \frac{1+\sqrt{\frac{2}{1+\sqrt{\frac{2}{1+1/c_0}}}}}{2} \times \cdots,\end{aligned}$$

which is equivalent to

$$\frac{\mathrm{sl}^2 x}{x^2} = \frac{1+\mathrm{sl}' x}{2} \times \frac{1+\sqrt{\frac{2}{1+\mathrm{sl}' x}}}{2} \times \frac{1+\sqrt{\frac{2}{1+\sqrt{\frac{2}{1+\mathrm{sl}' x}}}}}{2} \times \cdots.$$

This completes the proof. □

In [3, 4], Levin established

$$\frac{2}{L} = \sqrt{\frac{1}{2}} \times \sqrt{\frac{1+\sqrt{2}}{2}} \times \sqrt{\frac{1+\sqrt{\frac{2}{1+\sqrt{2}}}}{2}} \times \cdots. \tag{13}$$

By setting $x = L/2$ in Thorem 3.6, we get the product (13) as $\mathrm{sl}\,(L/2) = 1$ and $\mathrm{sl}'\,(L/2)$ =0.

4. Theorems for the arithmetic-geometric mean

Similar statements hold for the arithmetic geometric mean $M(a_0,b_0)$ which is defined as follows [1, 2]: Given positive a_0,b_0 satisfying $a_0>b_0$, we set a_n,b_n inductively by

$$a_{n+1}=\frac{a_n+b_n}{2},\, b_{n+1}=(a_nb_n)^{1/2}, \tag{14}$$

and define

$$M(a_0,b_0)=\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n,$$

which is given as

$$[M(a_0,b_0)]^{-1}=\frac{2}{\pi}\int_0^{\pi/2}(a_0^2\cos^2\theta+b_0^2\sin^2\theta)^{-1/2}d\theta.$$

This mean satisfies

$$b_0<b_1<\cdots<b_n<\cdots<M(a_0,b_0)<\cdots<a_n<\cdots<a_1<a_0.$$

We get the following lemma for the arithmetic geometric mean.

LEMMA 4.1. *Let f be a real valued function on $(0,\infty)$ which is continuous at 1. If $t^q\left\{(1+t^2)/2t^2\right\}^p f\left((t+t^{-1})/2\right)>f\left(t^2\right)$ holds for $1<t$ with some real p,q, then the sequence*

$$a_n^pb_n^qf\left(\frac{a_n}{b_n}\right) \tag{15}$$

is strictly increasing as n increases. If $t^q\left\{(1+t^2)/2t^2\right\}^p f\left((t+t^{-1})/2\right)<f\left(t^2\right)$ holds for $1<t$, with some real p,q, then the sequence (15) is strictly decreasing as n increase.

Proof. We note that $a_n/b_n>1$. First, we assume that $t^q\left((1+t^2)/2t^2\right)^p f\left((t+t^{-1})/2\right)>f\left(t^2\right)$ holds. Since $a_{n+1}/a_n=(1+b_n/a_n)/2$, $b_{n+1}/b_n=(a_n/b_n)^{1/2}$ and $a_{n+1}/b_{n+1}=\left\{(a_n/b_n)^{1/2}+(b_n/a_n)^{1/2}\right\}/2$, we have

$$\begin{aligned}
&a_{n+1}^pb_{n+1}^qf\left(\frac{a_{n+1}}{b_{n+1}}\right)-a_n^pb_n^qf\left(\frac{a_n}{b_n}\right)\\
&=a_n^pb_n^q\left\{\left(\frac{a_{n+1}}{a_n}\right)^p\left(\frac{b_{n+1}}{b_n}\right)^qf\left(\frac{a_{n+1}}{b_{n+1}}\right)-f\left(\frac{a_n}{b_n}\right)\right\}\\
&=a_n^pb_n^q\left\{\left(\frac{a_n}{b_n}\right)^{q/2}\left(\frac{1+\frac{b_n}{a_n}}{2}\right)^pf\left(\frac{\left(\frac{a_n}{b_n}\right)^{1/2}+\left(\frac{b_n}{a_n}\right)^{1/2}}{2}\right)-f\left(\frac{a_n}{b_n}\right)\right\}>0.
\end{aligned}$$

If $t^q\left\{(1+t^2)/2t^2\right\}^p f\left((t+t^{-1})/2\right)<f\left(t^2\right)$ holds, we get the second assertion in the same way as above. This completes the proof. □

We then get the following theorem.

THEOREM 4.2. *Let* $\{a_n\}$, $\{b_n\}$ *be sequences determined by* (14). *We then have*

$$\frac{a_n+b_n}{2} > \frac{a_n+(a_nb_n)^{1/2}+b_n}{3} > M(a_0,b_0) > \left\{\frac{a_nb_n(a_n+b_n)}{2}\right\}^{1/3} > (a_nb_n)^{1/2}.$$

Proof. The first and fourth inequalities are shown by direct computations. By setting $p=0$, $q=1$ and set $f(t)=t+t^{1/2}+1$ in Lemma 4.1, we have

$$tf\big((t+t^{-1})/2\big)-f(t^2) = \frac{(t^2+1)^{1/2}}{2}\left\{(2t)^{1/2}-(t^2+1)^{1/2}\right\} < 0$$

as $2t-(t^2+1)<0$. This shows the second inequality.

Similarly, we put $p=0$, $q=1$ and set $f(t)=\left\{t(t+1)/2\right\}^{1/3}$ in Lemma 4.1. We then have

$$tf\big((t+t^{-1})/2\big)-f(t^2) = \frac{\{t(t^2+1)\}^{1/3}}{2}\left\{(t+1)^{2/3}-(4t)^{1/3}\right\} > 0$$

as $(t+1)^2-4t>0$. This shows the third inequality and Theorem 4.2. □

We note that if the function

$$f(t) = \frac{t^{1/2}+t^{-1/2}}{2} \times \frac{\left(\frac{t^{1/2}+t^{-1/2}}{2}\right)^{1/2}+\left(\frac{t^{1/2}+t^{-1/2}}{2}\right)^{-1/2}}{2} \times \cdots$$

converges, it satisfies $t\big((1+t^2)/2t^2\big)f\big((t+t^{-1})/2\big)=f\left(t^2\right)$ and this suggests us that $f(a_0/b_0)=a_1b_1f(a_1/b_1)/(a_0b_0)=\cdots=f(1)[M(a_0,b_0)]^2/(a_0b_0)$. In fact, we get the following theorem.

THEOREM 4.3. *If* $t>0$, *we have*

$$[M(t^{1/2},t^{-1/2})]^2 = \frac{t^{1/2}+t^{-1/2}}{2} \times \frac{\left(\frac{t^{1/2}+t^{-1/2}}{2}\right)^{1/2}+\left(\frac{t^{1/2}+t^{-1/2}}{2}\right)^{-1/2}}{2} \times \cdots.$$

Proof. We put $c_n = a_n/b_n$ in the arithmetic geometric mean. Since $a_n/b_n = a_n^2/b_{n+1}^2$, we have

$$\begin{aligned}\frac{b_{n+1}^2}{a_0b_0} &= \frac{\frac{b_{n+1}^2}{a_0b_0}\times\frac{a_0}{b_0}\times\frac{a_1}{b_1}\times\cdots\times\frac{a_n}{b_n}}{c_0\times c_1\times\cdots\times c_n} \\ &= \frac{\frac{b_{n+1}^2}{a_0b_0}\left(\frac{a_0}{b_1}\times\frac{a_1}{b_2}\times\cdots\times\frac{a_n}{b_{n+1}}\right)^2}{c_0\times c_1\times\cdots\times c_n} \\ &= c_1\times c_2\times\cdots\times c_n \\ &= \frac{c_0^{1/2}+c_0^{-1/2}}{2}\times\frac{\left(\frac{c_0^{1/2}+c_0^{-1/2}}{2}\right)^{1/2}+\left(\frac{c_0^{1/2}+c_0^{-1/2}}{2}\right)^{-1/2}}{2}\times\cdots.\end{aligned}$$

We note that $c_{n+1} = (c_n^{1/2}+c_n^{-1/2})/2$ holds. By setting $n\to\infty$ and $c_0 = t$, we obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{b_{n+1}^2}{a_0b_0} &= \frac{[M(a_0,b_0)]^2}{a_0b_0} \\ &= \left\{\frac{2}{\pi}\int_0^{\pi/2}\left(\frac{a_0}{b_0}\cos^2\theta+\frac{b_0}{a_0}\sin^2\theta\right)^{-1/2}d\theta\right\}^{-2} \\ &= [M(t^{1/2},t^{-1/2})]^2 \\ &= \frac{t^{1/2}+t^{-1/2}}{2}\times\frac{\left(\frac{t^{1/2}+t^{-1/2}}{2}\right)^{1/2}+\left(\frac{t^{1/2}+t^{-1/2}}{2}\right)^{-1/2}}{2}\times\cdots\end{aligned}$$

for $1 < t$. By setting $t\to 1/t$, we can show this equation on $0 < t < 1$ as this equation is invariant with respect to the substitution. It is easy to see that this equation holds for $t = 1$. This completes the proof. □

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