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A YOUNG–LIKE INEQUALITY WITH APPLICATIONS TO THE COMMUTATOR ESTIMATES

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Abstract. In the recent decade, Fourier analysis techniques based on the Littlewood-Paley decomposition have proved to be very efficient in the study of PDEs, since J.-M. Bony introduced the paradifferential calculus. Of those techniques, commutator estimates play the crucial role in dealing with bilinear estimates. In this paper, we develop a Young-like inequality which is the generalization of the classical Young's convolution inequality. The new inequality enables us to obtain various commutator estimates in the L^p-framework, which creates the basis to obtain different nonlinear a priori estimates in the analysis of PDEs.

1. Introduction

One of the most basic inequalities in Fourier analysis is the convolution inequality, which is also usually referred to as Young's inequality (see *e.g.*, [11]).

THEOREM 1.1. (W. H. Young, 1912) *Let* $1 \leqslant p,q,r \leqslant \infty$ *satisfy*

$$1+\frac{1}{q}=\frac{1}{p}+\frac{1}{r}.$$

Then for all f *in* $L^p(G)$ *and all* g *in* $L^r(G)$ *we have*

$$\|f*g\|_{L^q(G)} \leqslant \|g\|_{L^r(G)}\|f\|_{L^p(G)}, \tag{1.1}$$

where G *is the fixed locally compact group with a left invariant Haar measure* μ.

The inequality has its origin in the efforts of W. H. Young in 1912 to generalize Parseval's theorem for Fourier series to other L^p-classes, and they extend naturally in the context of analysis on locally compact groups. The most useful case reads as follows.

THEOREM 1.2. *Let* $1 \leqslant p \leqslant \infty$. *For all* f *in* $L^p(G)$ *and all* g *in* $L^1(G)$ *we have*

$$\|f*g\|_{L^p(G)} \leqslant \|g\|_{L^1(G)}\|f\|_{L^p(G)}. \tag{1.2}$$

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In above theorems, the convolution $f*g$ is defined as

$$(f*g)(x)=\int_G f(y)g(y^{-1}x)d\mu(y). \tag{1.3}$$

For instance, if $G=\mathbb{R}^d$ with the usual additive structure, then $y^{-1}=-y$ and the integral in (1.3) is written as

$$(f*g)(x)=\int_{\mathbb{R}^d} f(y)g(x-y)dy.$$

In this paper, we present the following Young-like inequality.

THEOREM 1.3. *Let* $1\leqslant p,p_1,p_2\leqslant\infty$ *and* $1/p=1/p_1+1/p_2$. *Then,*

$$\Big\|\int_{\mathbb{R}^d} f(y)F_1(x-\lambda_1 y)F_2(x-\lambda_2 y)dy\Big\|_{L^p}\leqslant \|f\|_{L^1}\|F_1\|_{L^{p_1}}\|F_2\|_{L^{p_2}} \tag{1.4}$$

for all $f\in L^1(\mathbb{R}^d), F_i\in L^{p_i}(\mathbb{R}^d)$ *and* $\lambda_i\in\mathbb{R}(i=1,2)$.

REMARK 1.1. In the case when $\lambda_1=\lambda_2=\lambda$, the integral inside L^p norm on the left-hand side of (1.4) can be rewritten as

$$\frac{1}{\lambda^d}f\Big(\frac{\cdot}{\lambda}\Big)*(F_1F_2),$$

and hence it can be seen that (1.4) is an immediate consequence of the classical Young's inequality as well as the Hölder inequality for $\|F_1F_2\|_{L^p}\leqslant\|F_1\|_{L^{p_1}}\|F_2\|_{L^{p_2}}$, where we also notice $\|\lambda^{-d}f(\cdot/\lambda)\|_{L^1}=\|f\|_{L^1}$. However, in (1.4) parameters λ_1,λ_2 can be *different* and *independent of each other*, and thus such process fails to prove (1.4). An important observation is that λ_1 and λ_2 only appear in front of the integral variable y. It turns out that (1.4) follows from the elementary calculations on the basis of the original proof of the classical Young's inequality, see Section 3 for details.

REMARK 1.2. Let us mention that the inequality (1.4) has not been published in the literature although it is very elementary. Additionally, we can also obtain the corresponding Young-like inequality in more general measure spaces endowed with a group structure, however, we only give (1.4) on Euclidean spaces for application purposes, and the general case is left to the interesting reader.

The motivation of (1.4) mainly originates from many kinds of nonlinear estimates by using L^p $(1\leqslant p\leqslant\infty)$ norm rather than L^∞ norm only. For instance, we meet with commutator estimates when dealing with the bilinear terms in the PDEs. As shown by [1, 2, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 19, 20, 21] and references therein, also including ourselves [16, 17, 18], the commutator estimates play the crucial role in the nonlinear analysis of fluid-dynamical equations, such as the compressible or incompressible Navier-Stokes equations, viscoelastic fluid equations, liquid-gas two phase flow model, Euler equations and related models, *etc.*. There are a number of variations on the commutator estimates. To the best of our knowledge, in the most case, from Taylor formula of first order, the convolution inequality in Theorem 1.2 and Bernstein's inequality, the operator $[T_A,\Delta_j]B$ can proceed as

$$\|[T_A,\Delta_j]B\|_{L^p} \leqslant \sum_{|j-j'|\leqslant 4} C2^{-j}\|\nabla S_{j'-1}A\|_{L^\infty}\|\Delta_{j'}B\|_{L^p}.$$

For the definition of paraproduct, see Section 2.

A natural question follows immediately. Whether the above L^∞-L^p bound can be replaced with L^{p_1}-L^{p_2} $(1/p=1/p_1+1/p_2,\ 1\leqslant p_1,\ p_2\leqslant\infty)$ one or not. As a matter of fact, the Young-like inequality in Theorem 1.3 enable us to obtain the desired bound so that almost all commutator estimates will be further extended. To illuminate this, we generalize those commutator estimates in the recent book [3] (see, Lemma 2.100, P. 112). Precisely, our main results read as follows.

THEOREM 1.4. *Let* $\sigma\in\mathbb{R}$, $1\leqslant r\leqslant\infty$, $1\leqslant p\leqslant p_3\leqslant\infty$ *and* $1/p=1/p_1+1/p_2=1/p_3+1/p_4$. *Let* v *be a vector field over* $\mathbb{R}^d$. *Assume that*

$$\sigma > -d\min\left\{\frac{1}{p_3},\frac{1}{p'}\right\} \quad \text{or} \quad \sigma > -1-d\min\left\{\frac{1}{p_3},\frac{1}{p'}\right\} \ \text{if}\ \operatorname{div}v=0 \tag{1.5}$$

with $1/p+1/p'=1$. *Define* $\mathscr{R}_j=[v\cdot\nabla,\Delta_j]f$ *(or* $\mathscr{R}_j=\operatorname{div}[v,\Delta_j]f$ *if* $\operatorname{div}v=0$*). There exists a constant* C, *depending continuously on* p,p_1,p_3,σ *and* d, *such that*

$$\left\|\left(2^{j\sigma}\|\mathscr{R}_j\|_{L^p}\right)_j\right\|_{\ell^r} \leqslant C\left(\|\nabla v\|_{L^{p_1}}\|f\|_{B^\sigma_{p_2,r}}+\|\nabla v\|_{B^{d/p_3}_{p_3,\infty}}\|f\|_{B^\sigma_{p,r}}\right) \tag{1.6}$$

if $\sigma<1+\frac{d}{p_3}$. *Furthermore, assume that* $\sigma>0$ *(or* $\sigma>-1$, *if* $\operatorname{div}v=0$*), then*

$$\left\|\left(2^{j\sigma}\|\mathscr{R}_j\|_{L^p}\right)_j\right\|_{\ell^r} \leqslant C\left(\|\nabla v\|_{L^{p_1}}\|f\|_{B^\sigma_{p_2,r}}+\|\nabla f\|_{L^{p_4}}\|\nabla v\|_{B^{\sigma-1}_{p_3,r}}\right). \tag{1.7}$$

In the limit case that $\sigma=-\frac{d}{p_3}$ *or* $\sigma=-\frac{d}{p'}$ $(\sigma=-1-\frac{d}{p_3}$ *or* $\sigma=-1-\frac{d}{p'}$, *if* $\operatorname{div}v=0$*), we have*

$$\sup_{j\geqslant-1}\left(2^{j\sigma}\|\mathscr{R}_j\|_{L^p}\right)_j \leqslant C\left(\|\nabla v\|_{L^{p_1}}\|f\|_{B^\sigma_{p_2,\infty}}+\|\nabla v\|_{B^{d/p_3}_{p_3,1}}\|f\|_{B^\sigma_{p,\infty}}\right). \tag{1.8}$$

REMARK 1.3. Theorem 1.4 is exactly consistent with Lemma 2.100 in [3], if one takes the limiting case $p_1=\infty$. Therefore, the current commutator estimates can be regarded as a generalization of Lemma 2.100. The key to the proof is the Young-like inequality in Theorem 1.3, which can be also applied to establish a number of various commutator estimates. Below are two useful cases in the study of partial differential equations:

(i) Assume that $f=v$, then

$$\left\|\left(2^{j\sigma}\|\mathscr{R}_j\|_{L^p}\right)_j\right\|_{\ell^r} \leqslant C\|\nabla v\|_{L^{p_1}}\|f\|_{B^\sigma_{p_2,r}}, \tag{1.9}$$

if $\sigma>0$ (or $\sigma>-1$, if $\operatorname{div}v=0$);

(ii) When $p_1=\infty$, assume that $\sigma>1+d/p_3$, or $\sigma=1+d/p_3$ and $r=1$. We note that $B^{\sigma-1}_{p,r}\hookrightarrow L^{p_4}$, so the inequality (1.7) ensures that

$$\left\|\left(2^{j\sigma}\|\mathscr{R}_j\|_{L^p}\right)_j\right\|_{\ell^r}\leqslant C\|\nabla v\|_{B^{\sigma-1}_{p_3,r}}\|f\|_{B^{\sigma}_{p,r}}. \tag{1.10}$$

2. Preliminary

For convenience of reader, we try to make the context self-contained. This section is devoted to the Littlewood-Paley decomposition theory briefly. The interested reader is also referred to [3] for more details.

Let (φ,χ) be a couple of smooth functions valued in $[0,1]$ such that φ is supported in the shell $\mathscr{C}(0,\frac{3}{4},\frac{8}{3})=\{\xi\in\mathbb{R}^d|\frac{3}{4}\leqslant|\xi|\leqslant\frac{8}{3}\}$, χ is supported in the ball $\mathscr{B}(0,\frac{4}{3})=\{\xi\in\mathbb{R}^d||\xi|\leqslant\frac{4}{3}\}$ and

$$\chi(\xi)+\sum_{j=0}^{\infty}\varphi(2^{-j}\xi)=1,\quad j\in\mathbb{Z},\ \ \xi\in\mathbb{R}^d.$$

Let $\mathscr{S}'$ be the dual space of the Schwartz class $\mathscr{S}$. For $f\in\mathscr{S}'$, the nonhomogeneous dyadic blocks are defined as follows

$$\Delta_{-1}f:=\chi(D)f=\tilde{h}*f\quad\text{with}\ \ \tilde{h}=\mathscr{F}^{-1}\chi,$$

$$\Delta_j f:=\varphi(2^{-j}D)f=2^{jd}\int h(2^jy)f(x-y)dy\quad\text{with}\ \ h=\mathscr{F}^{-1}\varphi,\ \ \text{if}\ \ j\geqslant 0.$$

Here $*$, $\mathscr{F}^{-1}$ represent the convolution operator and the inverse Fourier transform, respectively. Furthermore, we define by S_jf the low frequency cut-off

$$S_jf:=\sum_{i\leqslant j-1}\Delta_i f.$$

It is clear that the above spectral truncation operators are almost orthogonal in L^2.

PROPOSITION 2.1. *For any $f,g\in\mathscr{S}'$, the following properties hold:*

$$\Delta_i\Delta_j f\equiv 0\quad\text{if}\quad|i-j|\geqslant 2,$$

$$\Delta_j(S_{j-1}f\Delta_i g)\equiv 0\quad\text{if}\quad|i-j|\geqslant 5.$$

Additionally, for any $f\in\mathscr{S}'$, yields

$$f=\sum_{j\geqslant -1}\Delta_j f,$$

which leads to the definition of inhomogeneous Besov spaces.

DEFINITION 2.1. For $s \in \mathbb{R}$ and $1 \leqslant p, r \leqslant \infty$, the inhomogeneous Besov spaces $B^s_{p,r}$ is defined by

$$B^s_{p,r} = \{f \in S' : \|f\|_{B^s_{p,r}} < \infty\},$$

where

$$\|f\|_{B^s_{p,r}} = \begin{cases} \left(\sum_{j=-1}^{\infty}(2^{js}\|\Delta_j f\|_{L^p})^r\right)^{1/r}, & r < \infty, \\ \sup_{j \geqslant -1} 2^{js}\|\Delta_j f\|_{L^p}, & r = \infty. \end{cases}$$

To show the proof of commutator estimates, we recall the Bony's decomposition.

DEFINITION 2.2. Let f, g be two temperate distributions. The product $f \cdot g$ has the Bony's decomposition formally:

$$f \cdot g = T_f g + T_g f + R(f, g),$$

where $T_f g$ is paraproduct of g by f,

$$T_f g = \sum_{j' \leqslant j-2} \Delta_{j'} f \Delta_j g = \sum_j S_{j-1} f \Delta_j g$$

and the remainder $R(f, g)$ is denoted by

$$R(f, g) = \sum_j \Delta_j f \tilde{\Delta}_j g \quad \text{with } \tilde{\Delta}_j := \Delta_{j-1} + \Delta_j + \Delta_{j+1}.$$

The bilinear paraproduct and remainder operators enjoy continuity properties in most useful functional spaces. Here, we present the continuity property for the remainder only. The reader is referred to [3] for more results on the subject.

PROPOSITION 2.2. *There is a constant C such that the following inequalities hold. Let $(s_1, s_2) \in \mathbb{R}^2$ and $1 \leqslant p_1, p_2, r_1, r_2 \leqslant \infty$. Assume that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leqslant 1, \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \leqslant 1.$$

If $s_1 + s_2 > 0$, then we have, for any (f, g) in $B^{s_1}_{p_1,r_1} \times B^{s_2}_{p_2,r_2}$,

$$\|R(f, g)\|_{B^{s_1+s_2}_{p,r}} \leqslant \frac{C^{|s_1+s_2|+1}}{s_1 + s_2}\|f\|_{B^{s_1}_{p_1,r_1}}\|g\|_{B^{s_2}_{p_2,r_2}}.$$

If $r = 1$ and $s_1 + s_2 = 0$, then we have, for any (f, g) in $B^{s_1}_{p_1,r_1} \times B^{s_2}_{p_2,r_2}$,

$$\|R(f, g)\|_{B^0_{p,\infty}} \leqslant C^{|s_1+s_2|+1}\|f\|_{B^{s_1}_{p_1,r_1}}\|g\|_{B^{s_2}_{p_2,r_2}}.$$

On the other hand, in the study of non-stationary partial differential equations, Chemin-Lerner's spaces (a class of mixed space-time Besov spaces) are useful, which was first introduced by J.-Y. Chemin and N. Lerner in [4]. We present the definition as follows.

DEFINITION 2.3. For $T>0$, $s\in\mathbb{R}$, $1\leqslant r,\rho\leqslant\infty$, set (with the usual convention if $r=\infty$)

$$\|f\|_{\widetilde{L}^{\rho}_T(B^s_{p,r})}:=\Big(\sum_{j\geqslant -1}(2^{js}\|\Delta_j f\|_{L^{\rho}_T(L^p)})^r\Big)^{\frac{1}{r}}.$$

Then we define the space $\widetilde{L}^{\rho}_T(B^s_{p,r})$ as the completion of $\mathscr{S}$ over $(0,T)\times\mathbb{R}^N$ by the above norm.

Finally, we state the classical Bernstein's inequality to end this section.

PROPOSITION 2.3. *Let $k\in\mathbb{N}$ and $0<R_1<R_2$. There exists a constant C, depending only on R_1, R_2 and d, such that for all $1\leqslant a\leqslant b\leqslant\infty$ and $f\in L^a$, we have*

$$\mathrm{Supp}\,\mathscr{F}f\subset\mathscr{B}(0,R_1\lambda)\Rightarrow\sup_{|\alpha|=k}\|\partial^{\alpha}f\|_{L^b}\leqslant C^{k+1}\lambda^{k+N(\frac{1}{a}-\frac{1}{b})}\|f\|_{L^a};$$

$$\mathrm{Supp}\,\mathscr{F}f\subset\mathscr{C}(0,R_1\lambda,R_2\lambda)\Rightarrow C^{-k-1}\lambda^k\|f\|_{L^a}\leqslant\sup_{|\alpha|=k}\|\partial^{\alpha}f\|_{L^a}\leqslant C^{k+1}\lambda^k\|f\|_{L^a},$$

where $\mathscr{F}f$ represents the Fourier transform on f.

3. Proof of Young-like inequality

This section is devoted to the

Proof of Theorem 1.3. By Hölder inequality for $1/p'+1/p=1$,

$$\begin{aligned}&\Big|\int_{\mathbb{R}^d}f(y)F_1(x-\lambda_1y)F_2(x-\lambda_2y)dy\Big|\\&\leqslant\int_{\mathbb{R}^d}|f(y)|^{\frac{1}{p'}+\frac{1}{p}}|F_1(x-\lambda_1y)||F_2(x-\lambda_2y)|dy\\&\leqslant\|f\|_{L^1}^{\frac{1}{p'}}\Big(\int_{\mathbb{R}^d}|f(y)||F_1(x-\lambda_1y)|^p|F_2(x-\lambda_2y)|^pdy\Big)^{\frac{1}{p}}.\end{aligned}\tag{3.1}$$

Then, taking further the L^p-norm on both sides,

$$\begin{aligned}&\Big\|\int_{\mathbb{R}^d}f(y)F_1(x-\lambda_1y)F_2(x-\lambda_2y)dy\Big\|_{L^p}\\&\leqslant\|f\|_{L^1}^{\frac{1}{p'}}\Big(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|f(y)||F_1(x-\lambda_1y)|^p|F_2(x-\lambda_2y)|^pdydx\Big)^{\frac{1}{p}}\\&=\|f\|_{L^1}^{\frac{1}{p'}}\Big(\int_{\mathbb{R}^d}|f(y)|dy\int_{\mathbb{R}^d}|F_1(x-\lambda_1y)|^p|F_2(x-\lambda_2y)|^pdx\Big)^{\frac{1}{p}}\\&\leqslant\|f\|_{L^1}^{\frac{1}{p'}}\Big(\int_{\mathbb{R}^d}|f(y)|dy\|F_1(\cdot-\lambda_1y)\|_{L^{p_1}}^p\|F_2(\cdot-\lambda_2y)\|_{L^{p_2}}^p\Big)^{\frac{1}{p}}\\&=\|f\|_{L^1}^{\frac{1}{p'}}\Big(\int_{\mathbb{R}^d}|f(y)|dy\|F_1\|_{L^{p_1}}^p\|F_2\|_{L^{p_2}}^p\Big)^{\frac{1}{p}}\\&=\|f\|_{L^1}^{\frac{1}{p'}}\|f\|_{L^1}^{\frac{1}{p}}\|F_1\|_{L^{p_1}}\|F_2\|_{L^{p_2}}\\&=\|f\|_{L^1}\|F_1\|_{L^{p_1}}\|F_2\|_{L^{p_2}},\end{aligned}\tag{3.2}$$

where Fubini's theorem was used in the third line, Hölder inequality for $1=\frac{1}{p_1/p}+\frac{1}{p_2/p}$ was used in the fourth line, and the substitutions $x-\lambda_1 y\to x$ and $x-\lambda_2 y\to x$ were used in the fifth line. Thus (1.4) is proved. $\square$

From the above proof, it is also straightforward to establish the following more general Young-like inequality.

COROLLARY 3.1. *Let $1\leqslant p\leqslant\infty$ and $1/p=\sum_{i=1}^n 1/p_i$. Then,*

$$\Big\|\int_{\mathbb{R}^d} f(y)\prod_{i=1}^n F_i(x-\lambda_i y)dy\Big\|_{L^p}\leqslant\|f\|_{L^1}\prod_{i=1}^n\|F_i\|_{L^{p_i}} \tag{3.3}$$

for all $f\in L^1(\mathbb{R}^d)$, $F_i\in L^{p_i}(\mathbb{R}^d)$ and $\lambda_i\in\mathbb{R}$ $(i=1,2,\cdots,n)$.

4. Proof of commutator estimates

Proof of Theorem 1.4. In order to show that the gradient part of v is involved in the estimate, we need to split v into low and high frequencies: $v=\Delta_{-1}v+\tilde{v}$. Obviously, there exists a constant $C>0$ such that

$$\|\Delta_{-1}\nabla v\|_{L^a}\leqslant C\|\nabla v\|_{L^a},\quad \|\nabla\tilde{v}\|_{L^a}\leqslant C\|\nabla v\|_{L^a},\quad a\in[1,\infty]. \tag{4.1}$$

Further, as $\tilde{v}$ is spectrally supported away from the origin, that is, there exists a radius $0<R<\frac{3}{4}$ such that Supp $\mathscr{F}\tilde{v}\bigcap B(0,R)=\emptyset$, Proposition 2.3 ensures that

$$\|\Delta_j\nabla\tilde{v}\|_{L^a}\approx 2^j\|\Delta_j\tilde{v}\|_{L^a},\quad a\in[1,\infty],\quad j\geqslant -1. \tag{4.2}$$

From Definition 2.1, we can decompose

$$\mathscr{R}_j:=\sum_{i=1}^8\mathscr{R}_j^i$$

with

$$\begin{aligned}
&\mathscr{R}_j^1=[T_{\tilde{v}^k},\Delta_j]\partial_k f, && \mathscr{R}_j^2=T_{\partial_k\Delta_j f}\tilde{v}^k\\
&\mathscr{R}_j^3=-\Delta_j T_{\partial_k f}\tilde{v}^k, && \mathscr{R}_j^4=\partial_k R(\tilde{v}^k,\Delta_j f),\\
&\mathscr{R}_j^5=-R(\mathrm{div}\tilde{v},\Delta_j f), && \mathscr{R}_j^6=-\partial_k\Delta_j R(\tilde{v}^k,f),\\
&\mathscr{R}_j^7=\Delta_j R(\mathrm{div}\tilde{v},f), && \mathscr{R}_j^8=[\Delta_{-1}v^k,\Delta_j]\partial_k f,
\end{aligned}$$

where the summation convention over repeated indices has been used.

In what follows, the constant $C>0$ depends continuously on σ,p and d, and we denote by $(c_j)_{j\geqslant -1}$ a sequence such that $\|(c_j)\|_{\ell^r}\leqslant 1$.

Note that Proposition 2.1, with the aid of the Taylor's formula of first order, we get

$$
\begin{aligned}
\mathscr{R}_j^1 &= T_{\tilde{v}^k}\partial_k\Delta_j f - \Delta_j(T_{\tilde{v}^k}\partial_k f)\\
&= \sum_{j'\in\mathbb{Z}}\Big\{S_{j'-1}\tilde{v}^k\Delta_j(\partial_k\Delta_{j'}f) - \Delta_j(S_{j'-1}\tilde{v}^k\Delta_{j'}\partial_k f)\Big\}\\
&= \sum_{|j-j'|\leqslant 4} 2^{jn}\int_{\mathbb{R}^d}\varphi_0(2^j(x-y))\Big\{S_{j'-1}\tilde{v}^k(x) - S_{j'-1}\tilde{v}^k(y)\Big\}\partial_k\Delta_{j'}f(y)dy\\
&= \sum_{|j-j'|\leqslant 4} 2^{jn}\int_{\mathbb{R}^d}\varphi_0(2^j(x-y))\int_0^1((x-y)\cdot\nabla)S_{j'-1}\tilde{v}^k(x+\tau(y-x))d\tau\partial_k\Delta_{j'}f(y)dy\\
&= \sum_{|j-j'|\leqslant 4} 2^{-j}\int_{\mathbb{R}^d}\varphi_0(z)\int_0^1(z\cdot\nabla)S_{j'-1}\tilde{v}^k(x-\tau 2^{-j}z)d\tau\partial_k\Delta_{j'}f(x-2^{-j}z)dz, \qquad (4.3)
\end{aligned}
$$

where $\varphi_0 := h$ without loss of generality.

Then, by Young-like inequality given in Theorem 1.3 and Bernstein's inequality, we arrive at

$$
\begin{aligned}
&\|\mathscr{R}_j^1\|_{L^p}\\
&\leqslant \sum_{|j-j'|\leqslant 4} 2^{-j}\int_0^1\Big\|\int_{\mathbb{R}^d}|z\varphi_0(z)||\nabla S_{j'-1}\tilde{v}^k(x-\tau 2^{-j}z)||\partial_k\Delta_{j'}f(x-2^{-j}z)|dz\Big\|_{L^p}d\tau\\
&\leqslant C\sum_{|j-j'|\leqslant 4} 2^{-j}\|\nabla S_{j'-1}\tilde{v}^k\|_{L^{p_1}}\|\partial_k\Delta_{j'}f\|_{L^{p_2}}\\
&\leqslant C\|\nabla v\|_{L^{p_1}}\sum_{|j-j'|\leqslant 4} 2^{j'-j}\|\Delta_{j'}f\|_{L^{p_2}}. \qquad (4.4)
\end{aligned}
$$

For $\mathscr{R}_j^2$, we have

$$
\mathscr{R}_j^2 = \sum_{j'\geqslant j-3} S_{j'-1}\partial_k\Delta_j f\Delta_{j'}\tilde{v}^k. \qquad (4.5)
$$

Then using Hölder inequality gives

$$
\begin{aligned}
\|\mathscr{R}_j^2\|_{L^p} &\leqslant \sum_{j'\geqslant j-3}\|\Delta_{j'}\tilde{v}^k\|_{L^{p_1}}\|S_{j'-1}\partial_k\Delta_j f\|_{L^{p_2}}\\
&\leqslant C\sum_{j'\geqslant j-3} 2^{j-j'}\|\Delta_{j'}\nabla v\|_{L^{p_1}}\|S_{j'-1}\Delta_j f\|_{L^{p_2}}\\
&\leqslant C\|\nabla v\|_{L^{p_1}}\|\Delta_j f\|_{L^{p_2}}. \qquad (4.6)
\end{aligned}
$$

For $\mathscr{R}_j^3$, we proceed as follows:

$$
\begin{aligned}
\mathscr{R}_j^3 &= -\sum_{|j-j'|\leqslant 4}\Delta_j\Big(S_{j'-1}\partial_k f\Delta_{j'}\tilde{v}^k\Big) \qquad (4.7)\\
&= -\sum_{|j-j'|\leqslant 4}\sum_{j''\leqslant j'-2}\Delta_j\Big(\Delta_{j''}\partial_k f\Delta_{j'}\tilde{v}^k\Big). \qquad (4.8)
\end{aligned}
$$

From (4.1), we get

$$\begin{aligned}\|\mathscr{R}_j^3\|_{L^p} &\leqslant C\sum_{|j-j'|\leqslant 4}\|S_{j'-1}\partial_k f\|_{L^{p_4}}\|\Delta_{j'}\tilde{v}^k\|_{L^{p_3}}\\ &\leqslant C\|\nabla f\|_{L^{p_4}}\sum_{|j-j'|\leqslant 4}2^{-j'}\|\Delta_{j'}\nabla v\|_{L^{p_3}}.\end{aligned}\tag{4.9}$$

with $1/p=1/p_3+1/p_4$. Besides, starting from (4.8), we can alternatively get

$$\begin{aligned}&2^{j\sigma}\|\mathscr{R}_j^3\|_{L^p}\\ &\leqslant C\sum_{|j-j'|\leqslant 4}\sum_{j''\leqslant j'-2}2^{q\sigma}\|\Delta_{j''}\partial_k f\|_{L^{p_4}}\|\Delta_{j'}\tilde{v}^k\|_{L^{p_3}}\\ &\leqslant C\sum_{|j-j'|\leqslant 4}\sum_{j''\leqslant j'-2}2^{(j-j'')(\sigma-1-\frac{d}{p_3})}2^{j''\sigma}\|\Delta_{j''}f\|_{L^p}2^{j'\frac{d}{p_3}}\|\Delta_{j'}\nabla v\|_{L^{p_3}}.\end{aligned}\tag{4.10}$$

Therefore, if $\sigma<1+d/p_3$, then

$$\begin{aligned}2^{j\sigma}\|\mathscr{R}_j^3\|_{L^p} &\leqslant C\|\nabla v\|_{B^{d/p_3}_{p_3,\infty}}\sum_{j''\leqslant j+2}2^{(j-j'')(\sigma-1-\frac{d}{p_3})}2^{j''\sigma}\|\Delta_{j''}f\|_{L^p}\\ &\leqslant Cc_j\|\nabla v\|_{B^{d/p_3}_{p_3,\infty}}\|f\|_{B^\sigma_{p,r}},\end{aligned}\tag{4.11}$$

where we have used the Minkowski's inequality.

For $\mathscr{R}_j^4$, it is clear that

$$\begin{aligned}\mathscr{R}_j^4 &= \sum_{|j-j'|\leqslant 2}\partial_k(\Delta_{j'}\tilde{v}^k\Delta_j\widetilde{\Delta}_{j'}f)\\ &= \sum_{|j-j'|\leqslant 2}\sum_{|j'-j''|\leqslant 1}\partial_k(\Delta_{j'}\tilde{v}^k\Delta_j\Delta_{j''}f)\\ &= \sum_{|j-j'|\leqslant 2}\sum_{|j'-j''|\leqslant 1}\Big(\Delta_{j'}\partial_k\tilde{v}^k\Delta_j\Delta_{j''}f+\Delta_{j'}\tilde{v}^k\partial_k\Delta_j\Delta_{j''}f\Big)\end{aligned}\tag{4.12}$$

which leads to

$$\begin{aligned}\|\mathscr{R}_j^4\|_{L^p} &\leqslant C\sum_{|j-j'|\leqslant 2}\sum_{|j'-j''|\leqslant 1}\Big(\|\Delta_{j'}\partial_k\tilde{v}^k\|_{L^{p_1}}\|\Delta_j\Delta_{j''}f\|_{L^{p_2}}\\ &\qquad +2^{j''-j'}\|\Delta_{j'}\nabla\tilde{v}^k\|_{L^{p_1}}\|\Delta_j\Delta_{j''}f\|_{L^{p_2}}\Big)\\ &\leqslant C\|\nabla v\|_{L^{p_1}}\sum_{|j-j''|\leqslant 3}\|\Delta_{j''}f\|_{L^{p_2}}.\end{aligned}\tag{4.13}$$

Note that $\mathscr{R}_j^5=0$ if $\mathrm{div}v=0$, otherwise, a similar bound as $\mathscr{R}_j^4$ holds for $\mathscr{R}_j^5$.

For $\mathscr{R}_j^6$, we first consider the case where $1/p+1/p_3\leqslant 1$. Let $\tilde{p}_3$ satisfy $1/\tilde{p}_3=$

$1/p+1/p_3$. Then, note that the embedding $B^{\sigma+\frac{d}{p_3}}_{\tilde{p}_3,r} \hookrightarrow B^{\sigma}_{p,r}$, it yields

$$\begin{aligned} 2^{j\sigma}\|\mathscr{R}^6_j\|_{L^p} &\leqslant C2^{j(\sigma+1)}\|\Delta_j R(\tilde{v}^k,f)\|_{L^p} \\ &\leqslant Cc_j\|R(\tilde{v},f)\|_{B^{\sigma+1}_{p,r}} \\ &\leqslant Cc_j\|R(\tilde{v},f)\|_{B^{\sigma+1+\frac{d}{p_3}}_{\tilde{p}_3,r}} \\ &\leqslant Cc_j\|\tilde{v}\|_{B^{\frac{d}{p_3}+1}_{p_3,\infty}}\|f\|_{B^{\sigma}_{p,r}} \\ &\leqslant Cc_j\|\nabla v\|_{B^{d/p_3}_{p_3,\infty}}\|f\|_{B^{\sigma}_{p,r}}, \end{aligned} \tag{4.14}$$

where the standard continuity results for the remainder has been used and $\sigma > -1-d/p_3$ is required. For the case of $1/p+1/p_3>1$, the argument can be applied with p' $(1/p'+1/p=1)$ instead of p_3, and we still get

$$2^{j\sigma}\|\mathscr{R}^6_j\|_{L^p} \leqslant Cc_j\|\nabla v\|_{B^{d/p_3}_{p_3,\infty}}\|f\|_{B^{\sigma}_{p,r}} \tag{4.15}$$

provided that $\sigma > -1-d/p'$, where the embedding $B^{d/p_3}_{p_3,\infty} \hookrightarrow B^{d/p'}_{p',\infty} (p'>p_3)$ has been used.

Note that in the cases $\sigma = -1-\frac{d}{p_3}$ or $\sigma = -1-\frac{d}{p'}$, Proposition 2.2 gives

$$\left\|\left(2^{j\sigma}\|\mathscr{R}^6_j\|_{L^p}\right)_j\right\|_{\ell^1} \leqslant C\|\nabla v\|_{B^{d/p_3}_{p_3,1}}\|f\|_{B^{\sigma}_{p,\infty}}. \tag{4.16}$$

Besides, if $\sigma > -1$, together with the imbedding $L^{p_1} \hookrightarrow B^0_{p_1,\infty}$, we can alternatively arrive at

$$2^{j\sigma}\|\mathscr{R}^6_j\|_{L^p} \leqslant Cc_j\|\tilde{v}\|_{B^1_{p_1,\infty}}\|f\|_{B^{\sigma}_{p_2,r}} \leqslant Cc_j\|\nabla v\|_{L^{p_1}}\|f\|_{B^{\sigma}_{p_2,r}}. \tag{4.17}$$

For $\mathscr{R}^7_j$, it follows from the same argument as $\mathscr{R}^6_j$ that

$$2^{j\sigma}\|\mathscr{R}^7_j\|_{L^p} \leqslant \begin{cases} Cc_j\|\nabla v\|_{B^{d/p_3}_{p_3,\infty}}\|f\|_{B^{\sigma}_{p,r}}, & \sigma > -d/p_3, \\ Cc_j\|\nabla v\|_{B^{d/p_3}_{p_3,\infty}}\|f\|_{B^{\sigma}_{p,r}}, & \sigma > -d/p', \\ Cc_j\|\nabla v\|_{L^{p_1}}\|f\|_{B^{\sigma}_{p_2,r}}, & \sigma > 0, \end{cases} \tag{4.18}$$

and

$$\left\|\left(2^{j\sigma}\|\mathscr{R}^7_j\|_{L^p}\right)_j\right\|_{\ell^1} \leqslant C\|\nabla v\|_{B^{d/p_3}_{p_3,1}}\|f\|_{B^{\sigma}_{p,\infty}}, \tag{4.19}$$

provided that $\sigma = -\frac{d}{p_3}$ or $\sigma = -\frac{d}{p'}$.

For the last term $\mathscr{R}_j^8$, it holds that

$$\begin{aligned}
\mathscr{R}_j^8 &= \sum_{|j-j'|\leqslant 1} [\Delta_{-1}v^k, \Delta_j]\partial_k\Delta_{j'}f \\
&= \sum_{|j-j'|\leqslant 1} \Big\{\Delta_{-1}v^k\partial_k\Delta_j\Delta_{j'}f - \Delta_j(\Delta_{-1}v^k\partial_k\Delta_{j'}f)\Big\} \\
&= \sum_{|j-j'|\leqslant 1} 2^{jd}\int_{\mathbb{R}^d}\varphi_0(2^j(x-y))\Big\{\Delta_{-1}v^k(x) - \Delta_{-1}v^k(y)\Big\}\partial_k\Delta_{j'}f(y)dy \\
&= \sum_{|j-j'|\leqslant 1} 2^{jd}\int_{\mathbb{R}^d}\varphi_0(2^j(x-y))\int_0^1((x-y)\cdot\nabla)\Delta_{-1}v^k(x+\tau(y-x))d\tau\partial_k\Delta_{j'}f(y)dy \\
&= \sum_{|j-j'|\leqslant 1} 2^{-j}\int_{\mathbb{R}^d}\varphi_0(z)\int_0^1(z\cdot\nabla)\Delta_{-1}v^k(x-\tau 2^{-j}z)d\tau\partial_k\Delta_{j'}f(x-2^{-j}z)dz. \quad (4.20)
\end{aligned}$$

Hence, by applying the Young-like inequality in Theorem 1.3 again, we obtain

$$\begin{aligned}
\|\mathscr{R}_j^8\|_{L^p} &\leqslant C\sum_{|j-j'|\leqslant 1} 2^{j'-j}\|\nabla\Delta_{-1}v^k\|_{L^{p_1}}\|\Delta_{j'}f\|_{L^{p_2}} \\
&\leqslant C\|\nabla v\|_{L^{p_1}}\sum_{|j-j'|\leqslant 1} 2^{j'-j}\|\Delta_{j'}f\|_{L^{p_2}}. \quad (4.21)
\end{aligned}$$

Therefore, combining with (4.4), (4.6), (4.9) or (4.11), (4.13)–(4.19) and (4.21), we conclude (1.6), (1.7) and (1.8), and hence complete the proof of Theorem 1.4. □

Finally, we give the corresponding commutator estimates on the framework of Chemin-Lerner spaces, whereas the additional time exponent θ behaves according to Hölder inequality.

PROPOSITION 4.1. *Let $\sigma\in\mathbb{R}$, $1\leqslant r\leqslant\infty$, $1\leqslant p\leqslant p_3\leqslant\infty$ and $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$. Let v be a vector field over $\mathbb{R}^d$. Assume that*

$$\sigma > -d\min\Big\{\frac{1}{p_3},\frac{1}{p'}\Big\} \quad \textit{or} \quad \sigma > -1-d\min\Big\{\frac{1}{p_3},\frac{1}{p'}\Big\} \ \textit{if}\ \operatorname{div}v = 0$$

with $1/p+1/p' = 1$. Define $\mathscr{R}_j = [v\cdot\nabla,\Delta_j]f$ (or $\mathscr{R}_j = \operatorname{div}[v,\Delta_j]f$ if $\operatorname{div}v = 0$). There exists a constant C, depending continuously on p, p_1, p_3, σ and d, such that

$$\begin{aligned}
&\Big\|\Big(2^{j\sigma}\|\mathscr{R}_j\|_{L_T^\theta(L^p)}\Big)_j\Big\|_{\ell^r} \\
&\leqslant C\Big(\|\nabla v\|_{L_T^{\theta_1}(L^{p_1})}\|f\|_{\widetilde{L}_T^{\theta_2}(B^\sigma_{p_2,r})} + \|\nabla v\|_{\widetilde{L}_T^{\theta_3}(B^{d/p_3}_{p_3,\infty})}\|f\|_{\widetilde{L}_T^{\theta_4}(B^\sigma_{p,r})}\Big) \quad (4.22)
\end{aligned}$$

if $\sigma < 1+\frac{d}{p_3}$. Furthermore, assume that $\sigma > 0$ (or $\sigma > -1$, if $\operatorname{div}v = 0$), then

$$\begin{aligned}
&\Big\|\Big(2^{j\sigma}\|\mathscr{R}_j\|_{L_T^\theta(L^p)}\Big)_j\Big\|_{\ell^r} \\
&\leqslant C\Big(\|\nabla v\|_{L_T^{\theta_1}(L^{p_1})}\|f\|_{\widetilde{L}_T^{\theta_2}(B^\sigma_{p_2,r})} + \|\nabla f\|_{\widetilde{L}_T^{\theta_4}(L^{p_4})}\|\nabla v\|_{\widetilde{L}_T^{\theta_3}(B^{\sigma-1}_{p_3,r})}\Big). \quad (4.23)
\end{aligned}$$

In the limit case that $\sigma=-\frac{d}{p_3}$ *or* $\sigma=-\frac{d}{p'}$ ($\sigma=-1-\frac{d}{p_3}$ *or* $\sigma=-1-\frac{d}{p'}$, *if* $\mathrm{div}v=0$), *we have*

$$\sup_{j\geqslant -1}\Big(2^{j\sigma}\|\mathscr{R}_j\|_{L^{\theta}_T(L^p)}\Big)_j \leqslant C\Big(\|\nabla v\|_{L^{\theta_1}_T(L^{p_1})}\|f\|_{\widetilde{L}^{\theta_2}_T(B^{\sigma}_{p_2,\infty})}+\|\nabla v\|_{\widetilde{L}^{\theta_3}_T(B^{d/p_3}_{p_3,1})}\|f\|_{\widetilde{L}^{\theta_4}_T(B^{\sigma}_{p,\infty})}\Big), \tag{4.24}$$

where

$$\frac{1}{\theta}=\frac{1}{\theta_1}+\frac{1}{\theta_2}=\frac{1}{\theta_3}+\frac{1}{\theta_4}.$$

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