

SOME INEQUALITIES FOR QUANTUM TSALLIS ENTROPY RELATED TO THE STRONG SUBADDITIVITY

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Abstract. In this paper we investigate the inequality $S_q(\rho_{123}) + S_q(\rho_2) \leq S_q(\rho_{12}) + S_q(\rho_{23})$ (*) where ρ_{123} is a state on a finite dimensional Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, and S_q is the Tsallis entropy. It is well-known that the strong subadditivity of the von Neumann entropy can be derived from the monotonicity of the Umegaki relative entropy. Now, we present an equivalent form of (*), which is an inequality of relative quasi-entropies. We derive an inequality of the form $S_q(\rho_{123}) + S_q(\rho_2) \leq S_q(\rho_{12}) + S_q(\rho_{23}) + f_q(\rho_{123})$, where $f_1(\rho_{123}) = 0$. Such a result can be considered as a generalization of the strong subadditivity of the von Neumann entropy. One can see that (*) does not hold in general (a picturesque example is included in this paper), but we give a sufficient condition for this inequality, as well.

1. Introduction

If $0 \leq D \in B(\mathcal{H})$ is a state on a Hilbert space (or $0 \leq D = \sum_i \lambda_i P_i \in M_n(\mathbb{C})$ with $\text{Tr} D = 1$), then the von Neumann entropy is

$$S(D) = -\text{Tr} D \log D = -\sum_i \lambda_i \log \lambda_i \geq 0, \quad (1)$$

see in [5, 13, 18]. If D_{123} is a state on a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, then it has reduced states D_{12}, D_2, D_{23} on the spaces $\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{H}_2$ and $\mathcal{H}_2 \otimes \mathcal{H}_3$, respectively, and the strong subadditivity is

$$S(D_{123}) + S(D_2) \leq S(D_{12}) + S(D_{23}).$$

This result was made by E. Lieb and M. B. Ruskai in 1973 [16, 18]. Now we want to make some extensions and the idea is $\ln_q x$. For any real q , one can define the deformed logarithm (or q -logarithm) function $\ln_q : \mathbb{R}_+ \rightarrow \mathbb{R}$ by

$$\ln_q x = \int_1^x t^{q-2} dt = \begin{cases} \frac{x^q - 1}{q-1} & \text{if } q \neq 1, \\ \ln x & \text{if } q = 1. \end{cases}$$

The corresponding entropy

$$S_q(D) = -\text{Tr} D \ln_q D$$

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is called Tsallis entropy [1, 7]. It is reasonable to restrict ourselves to the $0 < q$ case, because $\lim_{x \rightarrow 0^+} -x \ln_q x = 0$ if and only if $0 < q$. If we introduce the notation $\text{Ln}_q x = -x \ln_q x$ we can write $S_q(D) = \text{Tr Ln}_q D$.

In this paper we present some inequalities on the Tsallis entropy which generalize or are related to the strong subadditivity of the von Neumann entropy. We consider the case of the classical probability theory, as well.

2. The Tsallis entropy is subadditive, but not strongly subadditive

If D is a state on a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, then it has reduced states D_1 and D_2 on the spaces \mathcal{H}_1 and \mathcal{H}_2 . The subadditivity of the Tsallis entropy is

$$S_q(D) \leq S_q(D_1) + S_q(D_2), \tag{2}$$

and it has been proved for $q > 1$ by Audenaert in 2007 [3]. In the $q > 1$ case (2) can be written as

$$\text{Tr} D_1^q + \text{Tr} D_2^q = \|D_1\|_q^q + \|D_2\|_q^q \leq 1 + \|D\|_q^q = 1 + \text{Tr} D^q. \tag{3}$$

The following example shows that in a special case the subadditivity can be verified by simple computations that are suggested by [9].

EXAMPLE 1. If $0 \leq a, b, c, d$ and $a + b + c + d = 1$, then we can take the density $D = \text{Diag}(a, b, c, d)$, and the reduced densities are

$$D_1 = \text{Diag}(a + b, c + d), \quad D_2 = \text{Diag}(a + c, b + d).$$

The inequality (2) is rather simple in this example.

For $1 \leq q$ the above (2) is in this form:

$$(a + b)^q + (c + d)^q + (a + c)^q + (b + d)^q \leq 1 + a^q + b^q + c^q + d^q$$

The case $q = 2$ is rather trivial. For general q , we use the function $f(x) = x - x^q$ ($x \in [0, 1]$) and the identities $f(x) = x(1 - x^{q-1})$ and $1 - x^{q-1}y^{q-1} = 1 - x^{q-1} + x^{q-1}(1 - y^{q-1})$. So

$$\begin{aligned} & f(a) + f(b) + f(c) + f(d) \\ &= a \left(1 - (a+c)^{q-1} \left(\frac{a}{a+c} \right)^{q-1} \right) + b \left(1 - (b+d)^{q-1} \left(\frac{b}{b+d} \right)^{q-1} \right) \\ &+ c \left(1 - (a+c)^{q-1} \left(\frac{c}{a+c} \right)^{q-1} \right) + d \left(1 - (b+d)^{q-1} \left(\frac{d}{b+d} \right)^{q-1} \right) \\ &= (a+c) \left(1 - (a+c)^{q-1} \right) + (b+d) \left(1 - (b+d)^{q-1} \right) + a(a+c)^{q-1} \left(1 - \left(\frac{a}{a+c} \right)^{q-1} \right) \end{aligned}$$

$$\begin{aligned}
 &+c(a+c)^{q-1} \left(1 - \left(\frac{c}{a+c}\right)^{q-1}\right) + b(b+d)^{q-1} \left(1 - \left(\frac{b}{b+d}\right)^{q-1}\right) \\
 &+d(b+d)^{q-1} \left(1 - \left(\frac{d}{b+d}\right)^{q-1}\right) = f(a+c) + f(b+d) \\
 &+ (a+c)^q \left(f\left(\frac{a}{a+c}\right) + f\left(\frac{c}{a+c}\right)\right) + (b+d)^q \left(f\left(\frac{b}{b+d}\right) + f\left(\frac{d}{b+d}\right)\right). \quad (4)
 \end{aligned}$$

The concavity of f gives

$$\begin{aligned}
 &(a+c)^q f\left(\frac{a}{a+c}\right) + (b+d)^q f\left(\frac{b}{b+d}\right) \\
 &\leq (a+c)f\left(\frac{a}{a+c}\right) + (b+d)f\left(\frac{b}{b+d}\right) \leq f(a+b)
 \end{aligned}$$

and

$$\begin{aligned}
 &(a+c)^q f\left(\frac{c}{a+c}\right) + (b+d)^q f\left(\frac{d}{b+d}\right) \\
 &\leq (a+c)f\left(\frac{c}{a+c}\right) + (b+d)f\left(\frac{d}{b+d}\right) \leq f(c+d).
 \end{aligned}$$

So we get

$$\begin{aligned}
 &(a+c)^q \left(f\left(\frac{a}{a+c}\right) + f\left(\frac{c}{a+c}\right)\right) + (b+d)^q \left(f\left(\frac{b}{b+d}\right) + f\left(\frac{d}{b+d}\right)\right) \\
 &\leq f(a+b) + f(c+d),
 \end{aligned}$$

hence from (4) we get

$$f(a) + f(b) + f(c) + f(d) \leq f(a+c) + f(b+d) + f(a+b) + f(c+d).$$

This is our statement. \square

The aim of this paper is to investigate the inequality

$$S_q(\rho_{123}) + S_q(\rho_2) \leq S_q(\rho_{12}) + S_q(\rho_{23}), \quad (5)$$

where ρ_{123} is a state on a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ (all components are finite dimensional), and $\rho_2, \rho_{12}, \rho_{23}$ are the appropriate reduced states. For $q = 1$, this is the well-known strong subadditivity (or with the usual abbreviation: SSA) inequality (1), which is a central result of the quantum information theory [16].

First, we have to note an easy consequence of Audenaert’s theorem. The subadditivity implies that

$$S_q(\rho_{123}) \leq S_q(\rho_{12}) + S_q(\rho_3) \text{ and } S_q(\rho_{123}) \leq S_q(\rho_1) + S_q(\rho_{23}).$$

It follows that

$$S_q(\rho_{123}) + S_q(\rho_2) - S_q(\rho_{12}) - S_q(\rho_{23}) \leq \min\{S_q(\rho_1) + S_q(\rho_2) - S_q(\rho_{12}), S_q(\rho_2) + S_q(\rho_3) - S_q(\rho_{23})\}.$$

The Tsallis entropy is nonnegative and takes its maximum at the completely mixed state, and the maximal value is $-\ln_q \frac{1}{d}$, where d is the dimension of the underlying Hilbert space. Therefore,

$$S_q(\rho_{123}) + S_q(\rho_2) - S_q(\rho_{12}) - S_q(\rho_{23}) \leq -\ln_q \frac{1}{d_2} - \ln_q \frac{1}{\min\{d_1, d_2\}},$$

where d_i is the dimension of \mathcal{H}_i . However, the strong subadditivity does not hold in general.

PROPOSITION 1. *The only strongly subadditive Tsallis entropy is the von Neumann entropy, that is, the strong subadditivity of the Tsallis entropy holds if and only if $q = 1$.*

Proof. It is known that for product states the relation

$$S_q(\rho_X \otimes \rho_Y) = S_q(\rho_X) + S_q(\rho_Y) + (1 - q)S_q(\rho_X)S_q(\rho_Y) \tag{6}$$

holds, hence the Tsallis entropy can not be subadditive for $q < 1$. In fact, it is neither subadditive, nor superadditive [9]. Therefore, the Tsallis entropy is not strongly subadditive for $q < 1$. On the other hand, the next examples show that (5) does not hold for $1 < q$.

Set $q > 1$ and consider the matrix

$$\rho_{123} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

ρ_{123} is positive and $\text{Tr } \rho_{123} = 1$. We have

$$\rho_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \rho_{23} = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}, \quad \rho_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

One can compute that

$$S_q(\rho_{123}) + S_q(\rho_2) = \frac{1}{q-1} \left(1 - 2 \left(\frac{1}{2} \right)^q + 1 - 2 \left(\frac{1}{2} \right)^q \right) = \frac{1}{q-1} \left(2 - 4 \left(\frac{1}{2} \right)^q \right)$$

and

$$S_q(\rho_{12}) + S_q(\rho_{23}) = S_q(\rho_{23}) = \frac{1}{q-1} \left(1 - 4 \left(\frac{1}{4} \right)^q \right).$$

By the inequality of geometric and arithmetic means, we have $2 \cdot 2^{1-q} < 1 + 4^{1-q}$, which immediately shows that $S_q(\rho_{123}) + S_q(\rho_2) > S_q(\rho_{12}) + S_q(\rho_{23})$. \square

This is not so surprising, if we consider a bit more general example.

EXAMPLE 2. If ρ_{12} is an *entangled* pure state, $1 < \text{rank}(\rho_3)$ and $\rho_{123} = \rho_{12} \otimes \rho_3$, then

$$S_q(\rho_{123}) + S_q(\rho_2) > S_q(\rho_{12}) + S_q(\rho_{23})$$

holds for every $1 < q$. Indeed, if we use (6) we get

$$S_q(\rho_{123}) + S_q(\rho_2) = S_q(\rho_{12}) + S_q(\rho_3) + (1-q)S_q(\rho_{12})S_q(\rho_3) + S_q(\rho_2) = S_q(\rho_2) + S_q(\rho_3)$$

and

$$S_q(\rho_{12}) + S_q(\rho_{23}) = S_q(\rho_{23}) = S_q(\rho_2) + S_q(\rho_3) + (1-q)S_q(\rho_2)S_q(\rho_3),$$

because $S_q(\rho_{12}) = 0$. ρ_{12} is entangled, hence $S_q(\rho_2) > 0$, and this verifies the statement.

However, for classical probability distributions, Tsallis entropy is strongly sub-additive for $1 \leq q$ [9], and this result has a elegant and short proof. The only thing necessary for the proof is that for any positive x, y and q , the identity

$$\ln_q x - \ln_q y = -\ln_q \left(\frac{y}{x} \right) x^{q-1} \tag{7}$$

holds. Now we restate the proof of Furuichi.

Proof. If $\rho_{123} = \text{Diag}(\{p_{jkl}\})$, then by (7),

$$\begin{aligned} S_q(\rho_{123}) - S_q(\rho_{12}) &= - \sum_{j,k,l} p_{jkl} (\ln_q(p_{jkl}) - \ln_q(p_{jk})) = \sum_{j,k,l} p_{jkl}^q \ln_q \left(\frac{p_{jkl}}{p_{jk}} \right) \\ &= \sum_{j,k,l} p_{jk}^q \left(\frac{p_{jkl}}{p_{jk}} \right)^q \ln_q \left(\frac{p_{jkl}}{p_{jk}} \right). \end{aligned}$$

Observe that $x^q \ln_q \left(\frac{1}{x} \right) = \text{Ln}_q x$, hence this expression can be written as

$$\sum_{j,k,l} p_{jk}^q \text{Ln}_q \left(\frac{p_{jkl}}{p_{jk}} \right) = \sum_{j,k,l} \left(\frac{p_{jk}}{p_k} \right)^q p_k^q \text{Ln}_q \left(\frac{p_{jkl}}{p_{jk}} \right) \leq \sum_{k,l} p_k^q \left(\sum_j \frac{p_{jk}}{p_k} \text{Ln}_q \left(\frac{p_{jkl}}{p_{jk}} \right) \right).$$

Ln_q is concave, hence

$$\begin{aligned} \sum_{k,l} p_k^q \left(\sum_j \frac{p_{jk}}{p_k} \text{Ln}_q \left(\frac{p_{jkl}}{p_{jk}} \right) \right) &\leq \sum_{k,l} p_k^q \text{Ln}_q \left(\sum_j \frac{p_{jk}}{p_k} \frac{p_{jkl}}{p_{jk}} \right) = \sum_{k,l} p_k^q \text{Ln}_q \left(\frac{p_{kl}}{p_k} \right) \\ &= \sum_{k,l} p_k^q \left(\frac{p_{kl}}{p_k} \right)^q \text{Ln}_q \left(\frac{p_k}{p_{kl}} \right) = \sum_{k,l} p_{kl}^q \text{Ln}_q \left(\frac{p_k}{p_{kl}} \right) = - \sum_{k,l} p_{kl} (\text{Ln}_q(p_{kl}) - \text{Ln}_q(p_k)) \\ &= S_q(\rho_{23}) - S_q(\rho_2). \quad \square \end{aligned}$$

3. Relative entropy and monotonicity

Let f be a $(0, \infty) \rightarrow \mathbb{R}$ function, let $\rho, \sigma \in M_n(\mathbb{C})$ be positive definite matrices and $A \in M_n(\mathbb{C})$. We define the relative quasi-entropy by

$$S_f^A(\rho \parallel \sigma) := \left\langle A\rho^{\frac{1}{2}}, f(\Delta(\sigma/\rho)) \left(A\rho^{\frac{1}{2}} \right) \right\rangle, \tag{8}$$

where $\langle A, B \rangle = \text{Tr} A^*B$ is the Hilbert-Schmidt inner product and $\Delta(\sigma/\rho)$ is the relative modular operator introduced by Araki [2]:

$$\Delta(\sigma/\rho) : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}), \quad X \mapsto \sigma X \rho^{-1}.$$

If $A = I$, we simply write $S_f(\rho \parallel \sigma)$.

Whenever an expression $S_f^A(\rho \parallel \sigma)$ appears, we implicitly assume that ρ is invertible. The following statement appeared in [21] and it makes the relative quasi-entropy easy to compute in some cases.

LEMMA 1. *Let the spectral decomposition of the positive definite matrices ρ and σ be given by*

$$\rho = \sum_j \lambda_j |\varphi_j\rangle \langle \varphi_j| \text{ and } \sigma = \sum_k \mu_k |\psi_k\rangle \langle \psi_k|.$$

Then we have

$$S_f^A(\rho \parallel \sigma) = \sum_{j,k} \lambda_j f \left(\frac{\mu_k}{\lambda_j} \right) |\langle \psi_k | A | \varphi_j \rangle|^2. \tag{9}$$

Proof. $\{|\psi_k\rangle \langle \varphi_j|\}_{j,k=1}^n$ form an orthonormal basis of $M_n(\mathbb{C})$ (with respect to the Hilbert-Schmidt inner product.) It is easy to check that with the notation $v_{jk} := |\psi_k\rangle \langle \varphi_j|$ we can write the relative modular operator as

$$\Delta(\sigma/\rho) = \sum_{j,k} \frac{\mu_k}{\lambda_j} |v_{jk}\rangle \langle v_{jk}|, \tag{10}$$

where $|v_{jk}\rangle \langle v_{jk}| : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is defined by $|v_{jk}\rangle \langle v_{jk}|(X) := v_{jk} \text{Tr } v_{jk}^* X$. Therefore, $f(\Delta(\sigma/\rho)) = \sum_{j,k} f \left(\frac{\mu_k}{\lambda_j} \right) |v_{jk}\rangle \langle v_{jk}|$. Direct computation shows that

$$\left\langle A\rho^{\frac{1}{2}}, |v_{jk}\rangle \langle v_{jk}| \left(A\rho^{\frac{1}{2}} \right) \right\rangle =$$

$$\begin{aligned}
 &= \text{Tr} \left(\sum_a \lambda_a^{\frac{1}{2}} |\varphi_a\rangle \langle \varphi_a| A^* |\psi_k\rangle \langle \varphi_j| \text{Tr} \left(|\varphi_j\rangle \langle \varphi_j| A \sum_b \lambda_b^{\frac{1}{2}} |\varphi_b\rangle \langle \varphi_b| \right) \right) \\
 &= \sum_{a,b} \lambda_a^{\frac{1}{2}} \lambda_b^{\frac{1}{2}} \text{Tr} (|\varphi_a\rangle \langle \varphi_a| A^* |\psi_k\rangle \langle \varphi_j| \text{Tr} (\langle \varphi_b | \varphi_j \rangle \langle \psi_k | A | \varphi_b \rangle)) \\
 &= \sum_{a,b} \lambda_a^{\frac{1}{2}} \lambda_b^{\frac{1}{2}} \delta_{bj} \langle \psi_k | A | \varphi_b \rangle \text{Tr} (|\varphi_a\rangle \langle \varphi_a| A^* |\psi_k\rangle \langle \varphi_j|) \\
 &= \sum_{a,b} \lambda_a^{\frac{1}{2}} \lambda_b^{\frac{1}{2}} \delta_{bj} \delta_{aj} \langle \psi_k | A | \varphi_b \rangle \langle \varphi_a | A^* | \psi_k \rangle = \lambda_j |\langle \psi_k | A | \varphi_j \rangle|^2,
 \end{aligned}$$

therefore

$$S_f^A(\rho || \sigma) = \left\langle A\rho^{\frac{1}{2}}, \sum_{j,k} f \left(\frac{\mu_k}{\lambda_j} \right) |v_{jk}\rangle \langle v_{jk}| \left(A\rho^{\frac{1}{2}} \right) \right\rangle = \sum_{j,k} \lambda_j f \left(\frac{\mu_k}{\lambda_j} \right) |\langle \psi_k | A | \varphi_j \rangle|^2. \quad \square$$

A short and elegant proof of the monotonicity of the relative entropy is given by Nielsen and Petz in [17]. The statement is that if $A, B \in M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ are positive definite matrices and we set $A_1 = \text{Tr}_2 A$, $B_1 = \text{Tr}_2 B$, then for an operator convex function f the following inequality holds:

$$S_f(A||B) \geq S_f(A_1||B_1). \tag{11}$$

There is a useful extension of this monotonicity in [21]. Now we restate Sharma’s proof, which is essentially the same as the proof of Nielsen and Petz [17].

LEMMA 2. *Suppose that $A, B \in M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ are positive definite matrices and let us use the notations $A_1 = \text{Tr}_2 A$, $B_1 = \text{Tr}_2 B$. If f is an operator convex function then for any $T \in M_m(\mathbb{C})$ the following inequality holds:*

$$S_f^{T \otimes I_2}(A||B) \geq S_f^T(A_1||B_1), \tag{12}$$

where I_2 is the identity in $M_n(\mathbb{C})$.

Proof. Let us consider the linear map

$$\mathcal{U} : M_m(\mathbb{C}) \rightarrow M_m(\mathbb{C}) \otimes M_n(\mathbb{C}); X \mapsto \mathcal{U}(X) := \left(X A_1^{-\frac{1}{2}} \otimes I_2 \right) A^{\frac{1}{2}}.$$

We can check that \mathcal{U} is an isometry. For $X, Y \in M_m(\mathbb{C})$,

$$\begin{aligned}
 \langle \mathcal{U}(X), \mathcal{U}(Y) \rangle &= \text{Tr} \left(A^{\frac{1}{2}} \left(A_1^{-\frac{1}{2}} X^* \otimes I_2 \right) \left(Y A_1^{-\frac{1}{2}} \otimes I_2 \right) A^{\frac{1}{2}} \right) \\
 &= \text{Tr} \left(A \left(A_1^{-\frac{1}{2}} X^* Y A_1^{-\frac{1}{2}} \otimes I_2 \right) \right) = \text{Tr} A_1 \left(A_1^{-\frac{1}{2}} X^* Y A_1^{-\frac{1}{2}} \right) = \text{Tr} X^* Y = \langle X, Y \rangle.
 \end{aligned}$$

The short computation

$$\begin{aligned} \langle Y, \mathcal{U}(X) \rangle &= \text{Tr} Y^* \left(X A_1^{-\frac{1}{2}} \otimes I_2 \right) A^{\frac{1}{2}} = \text{Tr} \left(Y A^{\frac{1}{2}} \right)^* \left(X \otimes I_2 \right) \left(A_1^{-\frac{1}{2}} \otimes I_2 \right) \\ &= \text{Tr} \left(A_1^{-\frac{1}{2}} \otimes I_2 \right) \left(Y A^{\frac{1}{2}} \right)^* \left(X \otimes I_2 \right) = \text{Tr} \left(Y A^{\frac{1}{2}} \left(A_1^{-\frac{1}{2}} \otimes I_2 \right) \right)^* \left(X \otimes I_2 \right) \\ &= \text{Tr} \left(\text{Tr}_2 \left(Y A^{\frac{1}{2}} \left(A_1^{-\frac{1}{2}} \otimes I_2 \right) \right) \right)^* X = \left\langle \text{Tr}_2 \left(Y A^{\frac{1}{2}} \left(A_1^{-\frac{1}{2}} \otimes I_2 \right) \right), X \right\rangle \end{aligned}$$

shows that the adjoint of \mathcal{U} (which will be denoted by \mathcal{U}^*) is the map

$$Y \mapsto \text{Tr}_2 \left(Y A^{\frac{1}{2}} \left(A_1^{-\frac{1}{2}} \otimes I_2 \right) \right).$$

One can see that \mathcal{U} admits the beautiful relation

$$\mathcal{U}^* \Delta(B/A) \mathcal{U} = \Delta(B_1/A_1). \tag{13}$$

If $X \in M_m(\mathbb{C})$, then

$$\begin{aligned} \mathcal{U}^* \Delta(B/A) \mathcal{U}(X) &= \text{Tr}_2 \left(B \left(X A_1^{-\frac{1}{2}} \otimes I_2 \right) A^{\frac{1}{2}} A^{-1} A^{\frac{1}{2}} \left(A_1^{-\frac{1}{2}} \otimes I_2 \right) \right) \\ &= \text{Tr}_2 \left(B \left(X A_1^{-1} \otimes I_2 \right) \right) = B_1 X A_1^{-1} = \Delta(B_1/A_1)(X). \end{aligned}$$

By definition of the relative entropy and by (13), the right-hand-side of (12) can be written as

$$S_f^T(A_1 || B_1) = \left\langle T A_1^{\frac{1}{2}}, f(\Delta(B_1/A_1)) \left(T A_1^{\frac{1}{2}} \right) \right\rangle = \left\langle T A_1^{\frac{1}{2}}, f(\mathcal{U}^* \Delta(B/A) \mathcal{U}) \left(T A_1^{\frac{1}{2}} \right) \right\rangle.$$

The operator convexity of f implies that

$$f(\mathcal{U}^* \Delta(B/A) \mathcal{U}) \leq \mathcal{U}^* f(\Delta(B/A)) \mathcal{U}$$

(see Chapter 5 of [5]), and $\mathcal{U} \left(T A_1^{\frac{1}{2}} \right) = (T \otimes I_2) A^{\frac{1}{2}}$ is immediate. Therefore,

$$\begin{aligned} \left\langle T A_1^{\frac{1}{2}}, f(\mathcal{U}^* \Delta(B/A) \mathcal{U}) \left(T A_1^{\frac{1}{2}} \right) \right\rangle &\leq \left\langle \mathcal{U} \left(T A_1^{\frac{1}{2}} \right), f(\Delta(B/A)) \left(\mathcal{U} \left(T A_1^{\frac{1}{2}} \right) \right) \right\rangle \\ &= \left\langle (T \otimes I_2) A^{\frac{1}{2}}, f(\Delta(B/A)) \left((T \otimes I_2) A^{\frac{1}{2}} \right) \right\rangle = S_f^{T \otimes I_2}(A || B), \end{aligned}$$

and the proof is complete. \square

4. From the relative entropy to the strong subadditivity

As we have seen in Section 2, the strong subadditivity of the Tsallis entropy holds if and only if $q = 1$. Therefore, our goal is to find some formulas as

$$S_q(\rho_{123}) + S_q(\rho_2) \leq S_q(\rho_{12}) + S_q(\rho_{23}) + f_q(\rho_{123}), \tag{14}$$

where $f_1(\rho_{123}) = 0$. Such a result can be considered as a generalization of the SSA inequality.

Now we collect here some elementary facts that will be used in this section.

1. For any positive x, y and q , the identity

$$\ln_q x - \ln_q y = -\ln_q\left(\frac{y}{x}\right) - (q-1)\ln_q\left(\frac{y}{x}\right)\ln_q x \tag{15}$$

holds.

2. If f and g are $\mathbb{R} \rightarrow \mathbb{R}$ functions, $\rho, \sigma \in M_n^{sa}(\mathbb{C})$ and the spectral decompositions are $\rho = \sum_j \lambda_j |\varphi_j\rangle\langle\varphi_j|$ and $\sigma = \sum_k \mu_k |\psi_k\rangle\langle\psi_k|$, and domain of f and g contains the spectrum of ρ and σ , respectively, then

$$\text{Tr}f(\rho)g(\sigma) = \sum_{j,k} f(\lambda_j)g(\mu_k) |\langle\varphi_j | \psi_k\rangle|^2. \tag{16}$$

3. If $A \in M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ and $B \in M_m(\mathbb{C})$, then

$$\text{Tr}_2(A \cdot B \otimes I_2) = (\text{Tr}_2 A) \cdot B. \tag{17}$$

The strong subadditivity of the von Neumann entropy can be derived from the monotonicity of the Umegaki relative entropy [6, 17]. Therefore, it seems to be useful to reformulate the SSA of the Tsallis entropy as an inequality of relative quasi-entropies.

THEOREM 1. *The strong subadditivity inequality of the Tsallis entropy (5) is equivalent to*

$$S_{-\ln_q}^U(\rho_{123} || \rho_{12} \otimes I_3) \geq S_{-\ln_q}^V(\rho_{23} || \rho_2 \otimes I_3), \tag{18}$$

where

$$U = \rho_{123}^{\frac{1}{2}(q-1)}, \quad V = \rho_{23}^{\frac{1}{2}(q-1)}. \tag{19}$$

Proof. By (17),

$$\text{Tr}_3(\rho_{123} \ln_q(\rho_{12} \otimes I_3)) = \rho_{12} \ln_q(\rho_{12}) \text{ and } \text{Tr}_3(\rho_{23} \ln_q(\rho_2 \otimes I_3)) = \rho_2 \ln_q(\rho_2),$$

hence the inequality

$$-S_q(\rho_{123}) + S_q(\rho_{12}) \geq -S_q(\rho_{12}) + S_q(\rho_2),$$

which is obviously equivalent to (5), can be written in the form

$$\text{Tr}\rho_{123}(\ln_q(\rho_{123}) - \ln_q(\rho_{12} \otimes I_3)) \geq \text{Tr}\rho_{23}(\ln_q(\rho_{23}) - \ln_q(\rho_2 \otimes I_3)). \tag{20}$$

By (16), if $\rho_{123} = \sum_j \lambda_j |\varphi_j\rangle\langle\varphi_j|$ and $\rho_{12} \otimes I_3 = \sum_k \mu_k |\psi_k\rangle\langle\psi_k|$, then the left hand side of (20) is

$$\sum_{j,k} \lambda_j (\ln_q \lambda_j - \ln_q \mu_k) |\langle\varphi_j | \psi_k\rangle|^2.$$

By (7), this expression can be written as

$$\sum_{j,k} \lambda_j \left(-\ln_q \left(\frac{\mu_k}{\lambda_j} \right) \lambda_j^{q-1} \right) |\langle \varphi_j | \psi_k \rangle|^2. \tag{21}$$

In addition, if $U = \rho_{123}^{\frac{1}{2}(q-1)}$ then

$$|\langle \psi_k | U | \varphi_j \rangle|^2 = \lambda_j^{q-1} |\langle \psi_k | \varphi_j \rangle|^2,$$

hence by the result of Lemma 1, we can write (21) as the following relative entropy

$$\sum_{j,k} \lambda_j \left(-\ln_q \left(\frac{\mu_k}{\lambda_j} \right) \right) |\langle \psi_k | U | \varphi_j \rangle|^2 = S_{-\ln_q}^U(\rho_{123} || \rho_{12} \otimes I_3). \tag{22}$$

The observation that the right hand side of (20) can be written as a relative entropy similarly to (22) completes the proof. \square

Note that in the special case $q = 1$, Theorem 1 states the equivalence of the monotonicity of the Umegaki relative entropy and the SSA of the von Neumann entropy.

THEOREM 2. *For $0 < q \leq 2$ the inequality*

$$\begin{aligned} & S_q(\rho_{12}) + S_q(\rho_{23}) - S_q(\rho_{123}) - S_q(\rho_2) \\ & \geq (q-1) \left(S_{\ln_q}^{(-\ln_q \rho_{123})^{\frac{1}{2}}}(\rho_{123} || \rho_{12} \otimes I_3) - S_{\ln_q}^{(-\ln_q \rho_{23})^{\frac{1}{2}}}(\rho_{23} || \rho_2 \otimes I_3) \right) \end{aligned}$$

holds.

Remark that this statement recovers the strong subadditivity of the von Neumann entropy if $q = 1$.

Proof. We noted that if $\rho_{123} = \sum_j \lambda_j |\varphi_j\rangle\langle\varphi_j|$ and $\rho_{12} \otimes I_3 = \sum_k \mu_k |\psi_k\rangle\langle\psi_k|$, then the left hand side of (20) is

$$\sum_{j,k} \lambda_j (\ln_q \lambda_j - \ln_q \mu_k) |\langle \varphi_j | \psi_k \rangle|^2.$$

According to (15), it is equal to

$$\sum_{j,k} \lambda_j \left(-\ln_q \left(\frac{\mu_k}{\lambda_j} \right) - (q-1) \ln_q \left(\frac{\mu_k}{\lambda_j} \right) \ln_q \lambda_j \right) |\langle \varphi_j | \psi_k \rangle|^2, \tag{23}$$

and by Lemma 1, (23) has the form

$$S_{-\ln_q}(\rho_{123} || \rho_{12} \otimes I_3) + (q-1) S_{\ln_q}^{(-\ln_q \rho_{123})^{\frac{1}{2}}}(\rho_{123} || \rho_{12} \otimes I_3). \tag{24}$$

If we rewrite the right hand side of (20) similarly to (24), we get that the SSA is equivalent to

$$\begin{aligned}
 & S_{-\ln_q}(\rho_{123} \parallel \rho_{12} \otimes I_3) + (q-1) S_{\ln_q}^{(-\ln_q \rho_{123})^{\frac{1}{2}}}(\rho_{123} \parallel \rho_{12} \otimes I_3) \\
 & \geq S_{-\ln_q}(\rho_{23} \parallel \rho_2 \otimes I_3) + (q-1) S_{\ln_q}^{(-\ln_q \rho_{23})^{\frac{1}{2}}}(\rho_{23} \parallel \rho_2 \otimes I_3). \tag{25}
 \end{aligned}$$

It is easy to derive from the Löwner-Heinz theorem [5, 6, 13] that $\ln_q x$ is operator monotone, if $0 < q \leq 2$. An operator monotone function is operator concave [13], hence $-\ln_q x$ is operator convex.

By the monotonicity property (12), for $0 < q \leq 2$ we have

$$S_{-\ln_q}(\rho_{123} \parallel \rho_{12} \otimes I_3) \geq S_{-\ln_q}(\rho_{23} \parallel \rho_2 \otimes I_3),$$

and by (25), this is equivalent to

$$\begin{aligned}
 & -S_q(\rho_{123}) + S_q(\rho_{12}) - (q-1) \left(S_{\ln_q}^{(-\ln_q \rho_{123})^{\frac{1}{2}}}(\rho_{123} \parallel \rho_{12} \otimes I_3) \right) \\
 & \geq -S_q(\rho_{23}) + S_q(\rho_2) - (q-1) \left(S_{\ln_q}^{(-\ln_q \rho_{23})^{\frac{1}{2}}}(\rho_{23} \parallel \rho_2 \otimes I_3) \right). \tag{26}
 \end{aligned}$$

This is the statement of the theorem. \square

The notation (19) will be used again. Because of the monotonicity property (12), for $0 < q \leq 2$ and $f(x) = -\ln_q x, A = \rho_{123}, B = \rho_{12} \otimes I_3, T = V$ we have

$$S_{-\ln_q}^{I_1 \otimes V}(\rho_{123} \parallel \rho_{12} \otimes I_3) \geq S_{-\ln_q}^V(\rho_{23} \parallel \rho_2 \otimes I_3). \tag{27}$$

This formula is quite similar to the SSA inequality (18). By (27),

$$S_{-\ln_q}^U(\rho_{123} \parallel \rho_{12} \otimes I_3) \geq S_{-\ln_q}^{I_1 \otimes V}(\rho_{123} \parallel \rho_{12} \otimes I_3) \tag{28}$$

implies the strong subadditivity (18). We try to find a sufficient condition for (28).

THEOREM 3. *If ρ_{123} and $I_1 \otimes \rho_{23}$ commute, and (using the usual notation $\rho_{123} = \sum_j \lambda_j |\varphi_j\rangle\langle\varphi_j|$ and $\rho_{12} \otimes I_3 = \sum_k \mu_k |\psi_k\rangle\langle\psi_k|$) we have $\lambda_j \leq \mu_k$ whenever $\langle\psi_k | \varphi_j\rangle \neq 0$, then for any $1 \leq q \leq 2$ the strong subadditivity inequality*

$$S_q(\rho_{123}) + S_q(\rho_2) \leq S_q(\rho_{12}) + S_q(\rho_{23})$$

holds.

Note that if ρ_{123} is a classical probability distribution (that is, it is diagonal in a product basis), then the conditions of Theorem 3 are clearly satisfied.

Proof. By Lemma 1, (28) is equivalent to

$$\begin{aligned} & \sum_{j,k} \left| \langle \psi_k | \rho_{123}^{\frac{1}{2}(q-1)} | \varphi_j \rangle \right|^2 \left(-\ln_q \left(\frac{\mu_k}{\lambda_j} \right) \right) \lambda_j \\ & \geq \sum_{j,k} \left| \langle \psi_k | I_1 \otimes \rho_{23}^{\frac{1}{2}(q-1)} | \varphi_j \rangle \right|^2 \left(-\ln_q \left(\frac{\mu_k}{\lambda_j} \right) \right) \lambda_j. \end{aligned}$$

If $\lambda_j \leq \mu_k$, then $\left(-\ln_q \left(\frac{\mu_k}{\lambda_j} \right) \right) \lambda_j \leq 0$. On the other hand,

$$\left| \langle \psi_k | \rho_{123}^{\frac{1}{2}(q-1)} | \varphi_j \rangle \right|^2 = \lambda_j^{q-1} |\langle \psi_k | \varphi_j \rangle|^2.$$

If ρ_{123} and $I_1 \otimes \rho_{23}$ commute, then $I_1 \otimes \rho_{23}$ is diagonal in the basis $\{ \varphi_j \}_{j \in J}$ and $\rho_{123} \leq I_1 \otimes \rho_{23}$ holds, that is, $I_1 \otimes \rho_{23} = \sum_j v_j | \varphi_j \rangle \langle \varphi_j |$ with some $v_j \geq \lambda_j$.

If $1 \leq q$, the map $t \mapsto t^{(q-1)}$ is monotone on \mathbb{R}_+ , hence we have

$$\left| \langle \psi_k | I_1 \otimes \rho_{23}^{\frac{1}{2}(q-1)} | \varphi_j \rangle \right|^2 = v_j^{q-1} |\langle \psi_k | \varphi_j \rangle|^2 \geq \lambda_j^{q-1} |\langle \psi_k | \varphi_j \rangle|^2.$$

We concluded that if the conditions of Theorem 3 are satisfied, then (28) holds, and hence the proof is complete. \square

The following example shows that one can apply Theorem 3 in essentially non-classical cases, as well.

EXAMPLE 3. Set $p, q \in [\frac{1}{2}, 1]$ such that $pq \leq 1 - q$ and $t \in \mathbb{R}$. Let us define V and Λ by

$$V = \begin{bmatrix} \cos t & 0 & 0 & -\sin t \\ 0 & \cos t & -\sin t & 0 \\ 0 & \sin t & \cos t & 0 \\ \sin t & 0 & 0 & \cos t \end{bmatrix}, \Lambda = \text{Diag}(pq, (1-p)q, p(1-q), (1-p)(1-q)).$$

V describes a family of orthonormal bases, this family can be considered as a one-parameter extension of the Bell basis. Let $\rho_1 \in M_m(\mathbb{C})$ be an arbitrary density, and $\rho_{23} \in M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ be defined by

$$\begin{aligned} \rho_{23} &= V \Lambda V^{-1} \\ &= \begin{bmatrix} pq \cos^2 t + (1-p)(1-q) \sin^2 t & 0 & 0 & (pq - (1-p)(1-q)) \sin t \cos t \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{21} & A_{22} & 0 \\ (pq - (1-p)(1-q)) \sin t \cos t & 0 & 0 & (1-p)(1-q) \cos^2 t + pq \sin^2 t \end{bmatrix}, \end{aligned}$$

where

$$A_{11} = (1-p)q \cos^2 t + p(1-q) \sin^2 t, \quad A_{12} = ((1-p)q - p(1-q)) \sin t \cos t,$$

$$A_{21} = ((1-p)q - p(1-q)) \sin t \cos t, \quad A_{22} = p(1-q) \cos^2 t + (1-p)q \sin^2 t.$$

One can easily compute that

$$\rho_2 = \text{Tr}_3 \rho_{23} = \begin{bmatrix} q \cos^2 t + (1-q) \sin^2 t & 0 \\ 0 & q \sin^2 t + (1-q) \cos^2 t \end{bmatrix}.$$

Let us take the density $\rho_{123} = \rho_1 \otimes \rho_{23}$. The spectrum of ρ_{123} is

$$\bigcup_{j=1}^m \{v_j p q, v_j (1-p) q, v_j p (1-q), v_j (1-p) (1-q)\},$$

where v_j 's are the eigenvalues of ρ_1 . The spectrum of ρ_{12} (or $\rho_{12} \otimes I_3$) is

$$\bigcup_{j=1}^m \{v_j (q \cos^2 t + (1-q) \sin^2 t), v_j (q \sin^2 t + (1-q) \cos^2 t)\}.$$

The assumption $pq \leq 1-q$ guarantees that the eigenvalues of ρ_{123} are smaller than the eigenvalues of $\rho_{12} \otimes I_3$, whenever the corresponding eigenvectors are not orthogonal. ρ_{123} and $I_1 \otimes \rho_{23}$ obviously commute, hence the conditions of Theorem 3 are satisfied by ρ_{123} despite the fact that ρ_{23} can not be diagonalized in any product basis.

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