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ON THE FAN AND LIDSKII MAJORISATIONS OF POSITIVE SEMIDEFINITE MATRICES

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Abstract. Let A,B be two positive definite matrices. The Fan-Lidskii majorisations can be subsumed as the symmetric norm rearrangement inequalities

$$\|A^{\downarrow}+B^{\uparrow}\| \leqslant \|A+B\| \leqslant \|A^{\downarrow}+B^{\downarrow}\|$$

where the up/down arrows on A,B mean the diagonal matrices with the same eigenvalues in decreasing/increasing order down to the diagonal. We refine these relations with sums of type $A+UBU^*$ and $A+VBV^*$ for two unitary matrices U,V associated in a quite natural way to A and B. Stronger results than majorisation are given by using simple averages in the unitary orbits of $A+B$, $A+UBU^*$ and $A+VBV^*$. Proofs rely on an orbital decomposition for positive block-matrices.

Some well-known rearrangement inequalities in classical analysis say that a given norm is maximized/minimized when two functions or sequences are arranged in the same/opposite way. It seems meaningless to discuss more on such rearrangement inequalities. However, in the non-commutative world of matrices, or Hilbert space operators, the situation is different. It is the concern of this article.

1. Majorisation

This note focuses on a refinement of two classical majorisations, the fundamental Fan and Lidskii rearrangement norm inequalities. Majorisation plays a central role in matrix analysis and its applications. This section briefly summarizes it with an emphasis on norm or anti-norm relations on the positive (semidefinite) cone. Section 2 contains a new result, a refinement of the Fan-Lidskii inequalities, or intermediate inequalities. The proof, in Section 3, depends on a quite interesting decomposition technique for block-matrices.

$\mathbb{M}_n$ denotes the space of $n\times n$ matrices with complex entries and $\mathbb{M}_n^+$ its positive semidefinite part.

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1.1. Norms and anti-norms

A symmetric norm $\|\cdot\|$ on $\mathbb{M}_n$, also called a unitarily invariant norm, is a norm such that $\|UAV\|$ for all $A \in \mathbb{M}_n$ and all unitaries $U, V \in \mathbb{M}_n$. Hence, from the polar decomposition, a symmetric norm can be defined by its restriction to the positive cone $\mathbb{M}_n^+$ by three properties as follows.

DEFINITION 1.1. A *symmetric norm* $\|\cdot\|$ on $\mathbb{M}_n^+$ is a non-zero functional such that

1. $\|\lambda A\| = \lambda\|A\|$ for all $A \in \mathbb{M}_n^+$ and all reals $\lambda \geqslant 0$,
2. $\|A\| = \|UAU^*\|$ for all $A \in \mathbb{M}_n^+$ and all unitaries U,
3. $\|A\| \leqslant \|A+B\| \leqslant \|A\| + \|B\|$ for all $A, B \in \mathbb{M}_n^+$.

Symmetric norms are homogeneous, convex, and unitarily invariant on the positive semidefinite cone. The concave counterpart notion is that of symmetric anti-norms.

DEFINITION 1.2. A *symmetric anti-norm* $\|\cdot\|_!$ on $\mathbb{M}_n^+$ is a non-negative continuous functional such that

1. $\|\lambda A\|_! = \lambda\|A\|_!$ for all $A \in \mathbb{M}_n^+$ and all reals $\lambda \geqslant 0$,
2. $\|A\|_! = \|UAU^*\|_!$ for all $A \in \mathbb{M}_n^+$ and all unitaries U,
3. $\|A+B\|_! \geqslant \|A\|_! + \|B\|_!$ for all $A, B \in \mathbb{M}_n^+$.

EXAMPLE 1.3. For all $p \geqslant 1$, the Schatten p-norms,

$$A \mapsto \{\mathrm{Tr} A^p\}^{1/p}$$

constitute a well-known family of symmetric norms on $\mathbb{M}_n^+$. When $p \in (0,1]$, these functionals are anti-norms. If $p \in (-\infty, 0)$, these functional are well defined on the invertible part of $\mathbb{M}_n$ and can be extended in a continuous way with the zero value to the non-invertible part. These also constitute a family of anti-norms.

EXAMPLE 1.4. The Minkowski functional

$$A \mapsto \det{}^{1/n} A$$

is an anti-norm on $\mathbb{M}_n^+$. This is reflected by the Minkowski inequality

$$\det{}^{1/n}(A+B) \geqslant \det{}^{1/n} A + \det{}^{1/n} B$$

for all $A, B \in \mathbb{M}_n^+$.

Since for $p \in (0,1)$, $A \mapsto \{\mathrm{Tr} A^p\}^{1/p}$ and $A \mapsto \{\mathrm{Tr} A^{p-1}\}^{1/(p-1)}$ are symmetric anti-norms on $\mathbb{M}_n^+$ and since the weighted geometric operation $a^p b^{1-p}$ is a concave operation on pairs of positive numbers a, b we obtain the next example of symmetric anti-norms.

EXAMPLE 1.5. Let $p \in [0,1]$. An example of anti-norm on $\mathbb{M}_n^+$ is given by the functional defined for invertible positive operators by

$$A \mapsto \frac{\mathrm{Tr} A^p}{\mathrm{Tr} A^{p-1}} \tag{1.1}$$

and vanishing on the non-invertible part of $\mathbb{M}_n^+$. This continuous functional is clearly increasing and homogeneous on $\mathbb{M}_n^+$, thus that it is an anti-norm is equivalent to the super-additivity property

$$\frac{\mathrm{Tr}\,(A+B)^p}{\mathrm{Tr}\,(A+B)^{p-1}} \geqslant \frac{\mathrm{Tr}\,A^p}{\mathrm{Tr}\,A^{p-1}} + \frac{\mathrm{Tr}\,B^p}{\mathrm{Tr}\,B^{p-1}}.$$

for all $A,B \in \mathbb{M}_n^+$.

Besides these examples, a remarkable family of symmetric anti-norms arise from symmetric norms as follows.

EXAMPLE 1.6. Let $\|\cdot\|$ be a symmetric norm on $\mathbb{M}_n$ and $p > 0$. If $A \in \mathbb{M}_n^+$ is invertible, set

$$\|A\|_! := \|A^{-p}\|^{-1/p}$$

and extend this functional to the non-invertible part by taking the value zero. Then $\|\cdot\|_!$ is a symmetric anti-norm. This is reflected by the superadditive inequality

$$\|\,(A+B)^{-p}\,\|^{-1/p} \geqslant \|A^{-p}\|^{1/p} + \|B^{-p}\|^{-1/p} \tag{1.2}$$

for all $A,B \in \mathbb{M}_n^+$ (with the vanishing condition for non-invertible matrices).

Another functional of interest on $\mathbb{M}_n^+$ is the diameter of the numerical range, i.e., the difference between the largest and the smallest eigenvalues. It can be expressed in terms of the operator norm $\|\cdot\|_\infty$ as

$$A \mapsto \|A\|_\infty - \|A^{-1}\|_\infty^{-1}$$

where we still adhere to the natural continuity convention for non-invertible matrices. This suggests to extend this notion. Given a symmetric norm $\|\cdot\|$, the associated diameter is the functional on $\mathbb{M}_n^+$,

$$A \mapsto \mathrm{diam}_{\|\cdot\|}(A) = \|A\| - \|A^{-1}\|^{-1}. \tag{1.3}$$

From (1.2) we infer the following proposition.

PROPOSITION 1.7. *For any symmetric norm with $\|E\| = 1$ on rank one projections E, the associated diameter (1.3) is a non-negative functional, and it is a weakly symmetric semi-norm, that is:*

1. $\mathrm{diam}_{\|\cdot\|}(\lambda A) = \lambda \mathrm{diam}_{\|\cdot\|}(A)$ *for all $A \in \mathbb{M}_n^+$ and all reals $\lambda \geqslant 0$,*

2. $\mathrm{diam}_{\|\cdot\|}(A) = \mathrm{diam}_{\|\cdot\|}(UAU^*)$ *for all* $A \in \mathbb{M}_n^+$ *and all unitaries* U,

3. $\mathrm{diam}_{\|\cdot\|}(A+B) \leqslant \mathrm{diam}_{\|\cdot\|}(A) + \mathrm{diam}_{\|\cdot\|}(B)$ *for all* $A, B \in \mathbb{M}_n^+$.

Proof. The homogeneity and unitary invariance properties are obvious. The subadditivity property follows from the subadditivity of $A \mapsto \|A\|$ and $A \mapsto -\|A^{-1}\|^{-1}$ (due to Example 1.6). To prove that $\mathrm{diam}_{\|\cdot\|}$ is a non-negative functional, that is

$$\|A\|^{-1} \leqslant \|A^{-1}\| \tag{1.4}$$

on the non-invertible part of $\mathbb{M}_n^+$, we use the normalization condition $\|E\| = 1$ on rank one projections E. Let $\|\cdot\|_D$ be the dual norm: for $Z \in \mathbb{M}_n^+$,

$$\|Z\|_D = \max \mathrm{Tr}\, AZ$$

where the maximum runs over all $A \in \mathbb{M}_n^+$ with $\|A\| = 1$. Since the dual of $\|\cdot\|_D$ is $\|\cdot\|$ itself, we infer that we also have the normalization condition $\|E\|_D = 1$ for all rank one projections E. Hence, if $M \in \mathbb{M}_n^+$ satisfies $\|M\|_D = 1$, then $\mathrm{Tr}\, M \geqslant 1$. In particular with such an M such that

$$\|A\| = \mathrm{Tr}\, MA$$

we have, by using Jensen inequality for $t \mapsto t^{-1}$,

$$\begin{aligned} \|A\|^{-1} &= \{\mathrm{Tr}\, MA\}^{-1} \\ &\leqslant \{\{\mathrm{Tr}\, M\}^{-2} \mathrm{Tr}\, MA^{-1} \\ &\leqslant \mathrm{Tr}\, MA^{-1} \\ &\leqslant \|A^{-1}\| \end{aligned}$$

hence (1.4) holds. □

REMARK 1.8. Symmetric (or unitarily invariant) norms is a basic notion in Matrix and Operator Theory whose first examples are the operator-, trace-, and Hilbert-Schmidt norms. The general study goes back to von Neumann in the thirties. Details can be found in most books on matrices or operators, for instance [2], [8]. In [2], weakly symmetric (unitarily invariant) norms are also discussed. The notion of symmetric anti-norms on $\mathbb{M}_n^+$ is due to Bourin and Hiai, see [3], [4] where a lot of examples and properties are given. The nice family of anti-norms of Example 1.6 appears in [4] and are called derived anti-norms; a lot of inequalities related to these anti-norms are obtained in this paper. A simple proof that these functionals are anti-norms is also given in [5]. Example 1.5 is new, though implicit in [3].

1.2. Majorisation

A few notation is necessary in order to state the various forms of a majorisation relation. If $A \in \mathbb{M}_n^+(\mathbb{C})$, denote by $\lambda_j^{\downarrow}(A)$, resp. $\lambda_j^{\uparrow}(A)$, the sequence of eigenvalues of A arranged in decreasing, resp. increasing, order. Also, denote by $A^{\downarrow}$, resp. $A^{\downarrow}$, the diagonal matrix with the $\lambda_j^{\downarrow}(A)$, resp. the $\lambda_j^{\uparrow}(A)$, down to the diagonal.

MAJORISATION. Let $A, B \in \mathbb{M}_n^+(\mathbb{C})$ such that $\mathrm{Tr}A = \mathrm{Tr}B$. The following five conditions are equivalent.

(1) For all $k = 1, 2, ...n$,

$$\sum_{j=1}^{k} \lambda_j^{\downarrow}(A) \leqslant \sum_{j=1}^{k} \lambda_j^{\downarrow}(B).$$

(2) For all symmetric norms, $\|A\| \leqslant \|B\|$.

(3) For all $k = 1, 2, ...n$,

$$\sum_{j=1}^{k} \lambda_j^{\uparrow}(A) \geqslant \sum_{j=1}^{k} \lambda_j^{\uparrow}(B).$$

(4) For all symmetric anti-norms, $\|A\|_! \geqslant \|B\|_!$.

(5) There exist some unitaries $U_i \in \mathbb{M}_n$, $i = 1, \ldots, n$ such that

$$A = \sum_{i=1}^{n} \alpha_i U_i B U_i^*$$

for some nonnegative scalars α_i, $\sum_{i=1}^{n} \alpha_i = 1$.

These conditions are five forms of the *majorisation* relation: A is majorised by B.

REMARK 1.9. Majorisation relations lie at the heart of matrix analysis and the literature is quite vast; a nice sample is Ando's survey [1]. The implication $(1) \Rightarrow (2)$ is the Fan principle for symmetric norms. Its equivalent anti-norm counterpart $(3) \Rightarrow (4)$ is pointed out in [3]. The condition (5) shows that $A \prec B$ means that A is the convex hull of the unitary orbit of B. This has been known for decades in a weak form by using a family of $n!$ unitaries instead of only n unitaries. The number $n!$ is related to another classical form of majorisation involving doubly stochastic matrices. Surprisingly enough, it is only in 2003 that, in a short note [9], Zhan noticed the sharp number n by a simple application of Caratheodory's theorem.

1.3. Fan and Lidskii

The famous Ky Fan and Lidskii majorisations can be written as a double inequality for pairs of positive operators.

FAN-LIDSKII. *Let $A, B \in \mathbb{M}_n^+$. Then for all symmetric norms,*

$$\|A^{\uparrow} + B^{\downarrow}\| \leqslant \|A + B\| \leqslant \|A^{\downarrow} + B^{\downarrow}\| \tag{1.5}$$

and these inequalities reverse for symmetric anti-norms.

When $AB = BA$, this is relevant to the Hardy-Littlewood-Polya inequalities for rearrangement invariant sequence spaces. Thanks to the above fifth form of the majorisation, (1.5) can be written in a more precise statement by making use of unitary orbits as follows:

$$A+B = \sum_{i=1}^{n} \alpha_i U_i (A^{\downarrow} + B^{\downarrow}) U_i^* \tag{1.6}$$

for n unitary $U_i \in \mathbb{M}_n$ pondered with a non-negative weight α_i, $\sum_{i=1}^{n} \alpha_i = 1$, and similarly,

$$A^{\downarrow} + B^{\uparrow} = \sum_{i=1}^{n} \beta_i V_i (A+B) V_i^* \tag{1.7}$$

for n unitary $V_i \in \mathbb{M}_n$ pondered with a non-negative weight β_i, $\sum_{i=1}^{n} \beta_i = 1$.

REMARK 1.10. The second inequality in (1.5) is equivalent the Fan majorisation principle $(1) \Rightarrow (2)$ given in the above five forms of majorisation. The second inequality in (1.5) is the Lidskii majorisation. It is a much more subtle majorisation. It is often stated as a fundamental variational inequality. In 1999, Li and Mathias [7] obtained a remarkably simple proof of the Lidskii majorisation.

QUESTION 1.11. Let $\mathscr{U}(A)$ denote the unitary orbit of A. Since $A^{\downarrow}, A^{\uparrow} \in \mathscr{U}(A)$ and $B^{\downarrow}, B^{\uparrow} \in \mathscr{U}(B)$, the expression (1.6) of the Fan majorisation raises the question whether there exist natural $A' \in \mathscr{U}(A)$ and $B' \in \mathscr{U}(B)$ such that

$$A+B = \frac{W_1(A'+B')W_1^* + W_2(A'+B')W_2^*}{2}.$$

for some unitary $W_1, W_2 \in \mathbb{M}_n$. Of course, we may consider a similar question for the expression (1.7) of the Lidskii majoristaion.

2. Average of unitary orbits

The following two theorems are our main results. The first one provides a rather simple answer to Question 1.11 for the Fan majorisation.

THEOREM 2.1. *Let $A, B \in \mathbb{M}_n^+$ and let U be the unitary in the polar decompositions $A^{1/2}B^{1/2} = U|A^{1/2}B^{1/2}|$. Then,*

$$A+B = \frac{\Lambda_1(A+UBU^*)\Lambda_1^* + \Lambda_2(A+UBU^*)\Lambda_2^*}{2}$$

for some unitaries $\Lambda_1, \Lambda_2 \in \mathbb{M}_n$.

Here, of course, if A or B is not invertible, the unitary U in the polar decomposition is not unique. The matrix $A^{1/2}B^{1/2}$ can be regarded as a geometric mean and hence U is quite naturally attached to the pair A,B. It is a rather surprising result that $A+B$ is the average of two elements in the unitary orbit of $A+UBU^*$.

To deal with Lidskii form of Question 1.11 we need to consider the ”right” geometric mean in $\mathbb{M}_n^+$. Given two positive matrices A,B, their geometric mean is defined as

$$A\sharp B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$$

and satisfies several natural properties. It is the only positive solution of the matrix Ricatti equation

$$XA^{-1}X = B,$$

and hence $A\sharp B = B\sharp A$. This algebraic definition via an equation for invertible matrices in $\mathbb{M}_n^+$ is equivalent to the two natural requirements:

(i) $A\sharp B = A^{1/2}B^{1/2}$ whenever $AB = BA$, and

(ii) $Z^*(A\sharp B)Z = (Z^*AZ)\sharp(Z^*BZ)$ for all invertible $Z \in \mathbb{M}_n$.

It follows that we have a decomposition

$$A\sharp B = A^{1/2}VB^{1/2} \tag{2.1}$$

where V is a unitary operator. For not necessarily invertible $A,B \in \mathbb{M}_n^+$, the geometric mean is defined by

$$A\sharp B = \lim_{r\searrow 0}(A+rI)\sharp(B+rI)$$

The factorization (2.1) for some unitaries V still exists, though not necessarily unique. We are now in position to answer to the Lidskii version of Question 1.11.

THEOREM 2.2. *Let* $A,B \in \mathbb{M}_n^+$ *and let* V *be the unitary in the geometric mean decomposition* $A\sharp B = A^{1/2}VB^{1/2}$. *Then,*

$$A+VBV^* = \frac{\Lambda_1(A+B)\Lambda_1^* + \Lambda_2(A+B)\Lambda_2^*}{2}$$

for some unitaries $\Lambda_1,\Lambda_2 \in \mathbb{M}_n$.

REMARK 2.3. These relations between $A+B$, $A+UBU^*$, $A+VBV^*$ are far from being obvious, and it is interesting to note that even for matrices $A,B \in \mathbb{M}_n(\mathbb{R})$, i.e., with *real* entries, it seems that we need to use unitaries $\Lambda_1,\Lambda_2 \in \mathbb{M}_n$, i.e, with *complex* entries.

REMARK 2.4. Operator algebraists may like to have similar statements for positive operators A,B in a von-Neumann algebra $\mathscr{M}$. One might expect that Λ_1,Λ_2 are then partial isometries in $\mathscr{M}$. This is not clear. By inspecting the proof of the matrix case, we have a version for Hilbert space operators A,B where the unitary condition on Λ_1,Λ_2 is relaxed to a contraction assumption. Details are given in Section 3. However, If A,B are nonsingular and trace class, then the situation is the same as in the matrix case.

COROLLARY 2.5. *Let $A,B \in \mathbb{M}_n^+$ be and let U,V be the unitaries in the geometric mean and polar decompositions $A\sharp B = A^{1/2}VB^{1/2}$ and $A^{1/2}B^{1/2} = U|A^{1/2}B^{1/2}|$. Then, for all symmetric norms,*

$$\|A+VBV^*\| \leqslant \|A+B\| \leqslant \|A+UBU^*\|$$

and these inequalities reverse for anti-norms.

By Example 1.4, these majorisations contain a determinantal inequality

$$\det(A+VBV^*) \geqslant \det(A+B) \geqslant \det(A+UBU^*).$$

Since the determinant is a quite special functional, it would be interesting to find more pairs of unitaries (U,V) satisfying this double inequality.

COROLLARY 2.6. *Let $A,B \in \mathbb{M}_n^+$ be and let U,V be the unitaries in the geometric mean and polar decompositions $A\sharp B = A^{1/2}VB^{1/2}$ and $A^{1/2}B^{1/2} = U|A^{1/2}B^{1/2}|$. Then, for all symmetric norms,*

$$\mathrm{diam}_{\|\cdot\|}(A+VBV^*) \leqslant \mathrm{diam}_{\|\cdot\|}(A+B) \leqslant \mathrm{diam}_{\|\cdot\|}(A+UBU^*).$$

Corollary 2.5 refines the Fan-Lidskii double majorisations (1.5). By using an elementary eigenvalue inequality for the sum of two Hermitian operators, $\lambda_{j+k+1}(S+T) \leqslant \lambda_{j+1}(S)+\lambda_{k+1}(T)$, we also infer from Theorems 2.1-2.2 the following eigenvalue relation.

COROLLARY 2.7. *Let $A,B \in \mathbb{M}_n^+$ be and let U,V be the unitaries in the geometric mean and polar decompositions $A\sharp B = A^{1/2}VB^{1/2}$ and $A^{1/2}B^{1/2} = U|A^{1/2}B^{1/2}|$. Then, for all $j,k = 0,1,\dots,$*

$$\lambda_{j+k+1}(A+VBV^*) \leqslant \lambda_{j+1}(A+B)+\lambda_{k+1}(A+B)$$

and

$$\lambda_{j+k+1}(A+B) \leqslant \lambda_{j+1}(A+UBU^*)+\lambda_{k+1}(A+UBU^*).$$

Combining the previous two theorems yields the next corollary.

COROLLARY 2.8. *Let $A,B \in \mathbb{M}_n^+$ and let U,V be the unitaries in the geometric mean and polar decompositions $A\sharp B = A^{1/2}VB^{1/2}$ and $A^{1/2}B^{1/2} = U|A^{1/2}B^{1/2}|$. Then,*

$$A+VBV^* = \frac{1}{4}\sum_{k=1}^{4}\Lambda_k(A+UBU^*)\Lambda_k^*$$

for some unitaries $\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4 \in \mathbb{M}_n$.

3. Proof and the infinite dimension case

3.1. Proof of Theorems 2.1–2.2.

Proof. First we recall a decomposition established in [6]: Given any matrix in $\mathbb{M}_{2n}^+(\mathbb{C})$ partitioned into blocks in $\mathbb{M}_n(\mathbb{C})$ with Hermitian off-diagonal blocks, we have

$$\begin{bmatrix} Y & X \\ X & Z \end{bmatrix} = \frac{1}{2}\{J(Y+Z)J^* + K(Y+Z)K^*\}$$

for some isometries $J,K \in \mathbb{M}_{2n,n}(\mathbb{C})$. Hence the full matrix is the average of two copies in the isometry orbit of the sum of the diagonal blocks Y+Z. Here the assumption that X is Hermitian is crucial; it can not be relaxed to a normality assumption. Also, even tough we start with matrices X,Y,Z with real entries we use isometries J,K with real entries. This may be written with two unitaries $J,K \in \mathbb{M}_{2n}$ in the following way,

$$\begin{bmatrix} Y & X \\ X & Z \end{bmatrix} = \frac{1}{2}\left\{J\begin{bmatrix} Y+Z & 0 \\ 0 & 0 \end{bmatrix}J^* + K\begin{bmatrix} Y+Z & 0 \\ 0 & 0 \end{bmatrix}K^*\right\}. \tag{3.1}$$

Now, observe that

$$\begin{bmatrix} A^{1/2} & B^{1/2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A^{1/2} & 0 \\ B^{1/2} & 0 \end{bmatrix} = \begin{bmatrix} A+B & 0 \\ 0 & 0 \end{bmatrix} \tag{3.2}$$

and

$$\begin{bmatrix} A^{1/2} & 0 \\ B^{1/2} & 0 \end{bmatrix}\begin{bmatrix} A^{1/2} & B^{1/2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & A^{1/2}B^{1/2} \\ B^{1/2}A^{1/2} & B \end{bmatrix}$$

are positive semidefinite and unitarily congruent since TT^* and T^*T are positive and unitarily congruent for any square matrix T. Further, with U the unitary factor in the polar decomposition of $A^{1/2}B^{1/2} = U|A^{1/2}B^{1/2}| = |B^{1/2}A^{1/2}|U$,

$$\begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} A & A^{1/2}B^{1/2} \\ B^{1/2}A^{1/2} & B \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & U^* \end{bmatrix} = \begin{bmatrix} A & |B^{1/2}A^{1/2}| \\ |B^{1/2}A^{1/2}| & UBU^* \end{bmatrix} \tag{3.3}$$

is positive semidefinite too. From (3.1) with $Y = A$ and $Z = UBU^*$ we then infer, using that (3.3) and (3.2) are unitarily congruent,

$$\begin{bmatrix} A+B & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}\left\{J\begin{bmatrix} A+UBU^* & 0 \\ 0 & 0 \end{bmatrix}J^* + K\begin{bmatrix} A+UBU^* & 0 \\ 0 & 0 \end{bmatrix}K^*\right\}. \tag{3.4}$$

for some unitaries $J,K \in \mathbb{M}_{2n}$. It then follows that

$$A+B = \frac{\Lambda_1(A+UBU^*)\Lambda_1^* + \Lambda_2(A+UBU^*)\Lambda_2^*}{2} \tag{3.5}$$

for some contractions $\Lambda_1,\Lambda_2 \in \mathbb{M}_n$. Taking the trace in (3.5) we infer

$$\mathrm{Tr}\,A+B = \mathrm{Tr}\,(A+UBU^*)\frac{\Lambda_1^*\Lambda_1 + \Lambda_2\Lambda_2^*}{2}. \tag{3.6}$$

Assume that A,B are invertible. Then (3.6) shows that the contraction $(\Lambda_1^*\Lambda_1+\Lambda_2^*\Lambda_2)/2$ is the identity so that both Λ_1 and Λ_2 are unitaries. Indeed, if $T\in\mathbb{M}_n^+$ is non-singular and $C\in\mathbb{M}_n^+$ is a contraction such that $\operatorname{Tr}T=\operatorname{Tr}TC$, then $I-C$ is a positive contraction and $\operatorname{Tr}T(I-C)=0$ so that $C=I$. Thus Theorem 2.1 is established for non-singular $A,B\in\mathbb{M}_n^+$. The general case follows from a standard limit argument.

To prove Theorem 2.2, we proceed in a similar way. By a limit argument we may assume that A,B are non-singular. Let $A\sharp B=A^{1/2}VB^{1/2}$ be the geometric mean decomposition and note that

$$\begin{bmatrix} A & A\sharp B \\ A\sharp B & B \end{bmatrix}=\begin{bmatrix} A^{1/2} & 0 \\ 0 & B^{1/2} \end{bmatrix}\begin{bmatrix} I & V \\ V^* & I \end{bmatrix}\begin{bmatrix} A^{1/2} & 0 \\ 0 & B^{1/2} \end{bmatrix}$$
$$=\begin{bmatrix} A^{1/2} & 0 \\ 0 & B^{1/2} \end{bmatrix}\begin{bmatrix} I & 0 \\ V^* & 0 \end{bmatrix}\begin{bmatrix} I & V \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A^{1/2} & 0 \\ 0 & B^{1/2} \end{bmatrix}$$

Thus

$$\begin{bmatrix} A & A\sharp B \\ A\sharp B & B \end{bmatrix}\simeq\begin{bmatrix} I & V \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A^{1/2} & 0 \\ 0 & B^{1/2} \end{bmatrix}\begin{bmatrix} A^{1/2} & 0 \\ 0 & B^{1/2} \end{bmatrix}\begin{bmatrix} I & 0 \\ V^* & 0 \end{bmatrix}=\begin{bmatrix} A+VBV^* & 0 \\ 0 & 0 \end{bmatrix}$$

where the symbol $\simeq$ stands for the unitary congruence relation. From (3.1) we thus have

$$\begin{bmatrix} A+VBV^* & 0 \\ 0 & 0 \end{bmatrix}=\frac{1}{2}\left\{J\begin{bmatrix} A+B & 0 \\ 0 & 0 \end{bmatrix}J^*+K\begin{bmatrix} A+B & 0 \\ 0 & 0 \end{bmatrix}K^*\right\}.$$

for some isometries $J,K\in\mathbb{M}_{2n}$. Arguing as above with a trace argument completes the proof of Theorem 2.2. □

3.2. The infinite dimension case

We briefly indicate how to adapt the proof of the matrix case to the setting of operators on a general Hilbert space. We also give two applications which are not obvious consequences of the matrix case.

Let $\mathbb{B}$ denote the set of all bounded linear operators on an infinite dimensional Hilbert space, and let $\mathbb{B}^+$ stands for the positive (semidefinite) part. For a general $T\in\mathbb{B}$, the positive operators TT^* and T^*T are not necessarily unitarily congruent; however there still exists a partial isometry W such that $TT^*=WT^*TW^*$. In fact it is the partial isometry occurring in the polar decomposition $T=W|T|=|T^*|W$. Similarly, the decomposition (3.1) still holds in the setting of operators $X,Y,Z\in\mathbb{B}$ by using partial isometries $J,K\in\mathbb{M}_2(\mathbb{B})$. We may then reproduce the proof of the matrix case and obtain the identity (3.5) for some contractions $\Lambda_1,\Lambda_2\in\mathbb{B}$.

If $A,B\in\mathbb{B}^+$ are non-singular, then the partial isometry U occurring in the polar decomposition $A^{1/2}B^{1/2}=U|A^{1/2}B^{1/2}|$ is actually a unitary. If A,B are further trace class, then the trace argument given in the matrix case is still valid, so that (3.5) holds with two unitary $\Lambda_1,\Lambda_2\in\mathbb{B}$.

THEOREM 3.1. *Let $A,B\in\mathbb{B}^+$ and let U,V be the partial isometries in the geometric mean and polar decompositions $A\sharp B=A^{1/2}VB^{1/2}$ and $A^{1/2}B^{1/2}=U|A^{1/2}B^{1/2}|$.*

Then,

$$A+B=\frac{\Gamma_1(A+UBU^*)\Gamma_1^*+\Gamma_2(A+UBU^*)\Gamma_2^*}{2}$$

and

$$A+VBV^*=\frac{\Gamma_3(A+B)\Gamma_3^*+\Gamma_4(A+B)\Gamma_4^*}{2}$$

for some contractions $\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4\in\mathbb{B}$. *If furthermore* A,B *are trace class and non-singular, then* $\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4$ *are unitary.*

It is not possible to derive the next two corollaries for operators from the statement for matrices of Section 2.

COROLLARY 3.2. *Let* $A,B\in\mathbb{B}^+$ *and let* U,V *be the partial isometries in the geometric mean and polar decompositions* $A\sharp B=A^{1/2}VB^{1/2}$ *and* $A^{1/2}B^{1/2}=U|A^{1/2}B^{1/2}|$. *Then,*

$$\|A+VBV^*\|_{\rm ess}\leqslant\|A+B\|_{\rm ess}\leqslant\|A+UBU^*\|_{\rm ess}$$

Here the notation $\|A\|_{\rm ess}$ means the essential norm of $A\in\mathbb{B}$, $\|A\|_{\rm ess}=\inf_R\|A+R\|_\infty$ where the infimum runs overs all finite ranks perturbations R.

For our last corollary we denote by $\mathbb{T}^+$ the set of all positive trace class operators. A weakly symmetric norm $\|\cdot\|_{\rm w}$ on $\mathbb{T}^+$ is a functional with values in $[0,\infty)$ such that

1. $\|\lambda A\|_{\rm w}=\lambda\|A\|_{\rm w}$ for all $A\in\mathbb{T}^+$ and all reals $\lambda\geqslant 0$,
2. $\|A\|_{\rm w}=\|UAU^*\|_{\rm w}$ for all $A\in\mathbb{T}^+$ and all unitaries U,
3. $\|A+B\|_{\rm w}\leqslant\|A\|_{\rm w}+\|B\|_{\rm w}$ for all $A,B\in\mathbb{T}^+$.
4. $A\in\mathbb{T}^+$ and $A\neq 0$ ensure $\|A\|_{\rm w}>0$.

If the fourth condition is not required, then $\|\cdot\|_{\rm w}$ is called a weakly symmetric semi-norm. Of course, these definitions are also valid for $\mathbb{M}_n^+$. Note that a weakly symmetric (semi-)norms differ from a symmetric (semi-)norm as the monotony assumption $A\leqslant B$ in $\mathbb{M}_n^+$ does not any longer ensure $\|A\|_{\rm w}\leqslant\|B\|_{\rm w}$.

COROLLARY 3.3. *Let* $A,B\in\mathbb{T}^+$ *be non-singular and let* U,V *be the unitaries in the geometric mean and polar decompositions* $A\sharp B=A^{1/2}VB^{1/2}$ *and* $A^{1/2}B^{1/2}=U|A^{1/2}B^{1/2}|$. *Then, for all weakly symmetric semi-norm on* $\mathbb{T}^+$,

$$\|A+VBV^*\|_{\rm w}\leqslant\|A+B\|_{\rm w}\leqslant\|A+UBU^*\|_{\rm w}$$

A large family of weakly symmetric semi-norm on $\mathbb{M}_n^+$ is given in Proposition 1.7. A non-trivial example of a weakly symmetric semi-norm on $\mathbb{T}^+$ is given by the functional

$$\|A\|_{\rm w}=\sup_{\mathscr{S}}\left\{\left(\mathrm{Tr}\,(A^2)_{\mathscr{S}}\right)^{1/2}-\mathrm{Tr}A_{\mathscr{S}}\right\}$$

where the supremum runs over the set of all two dimensional subspaces $\mathscr{S}\subset\mathscr{H}$ and $A_{\mathscr{S}}$ denotes the compression of A onto $\mathscr{S}$.

4. Conclusion

Let A,B be two positive matrices. The naive geometric mean $A^{1/2}B^{1/2} = U|A^{1/2}B^{1/2}|$ and the standard geometric mean $A\sharp B = A^{1/2}VB^{1/2}$ provide two unitaries U,V quite naturally associated to A,B. It turns out that the unitary orbits of the operators $A+B$, $A+VBV^*$, and $A+UBU^*$ satisfy some remarkable simple relations. This is a rather surprising phenomenon.

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