ON MORE ACCURATE REVERSE MULTIDIMENSIONAL
HALF–DISCRETE HILBERT–TYPE INEQUALITIES

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Abstract. By using the methods of weight functions and Hermite-Hadamard's inequality, two kinds of more accurate equivalent reverse multidimensional half-discrete Hilbert-type inequalities with the kernel of hyperbolic cotangent function are given. The constant factor related to the Riemann zeta function is proved to be the best possible.

1. Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x)$, $g(y) \geqslant 0$, $0 < ||f||_p = \{\int_0^\infty f^p(x)dx\}^{\frac{1}{p}} < \infty$, $0 < ||g||_q < \infty$, then we have the following Hardy-Hilbert's integral inequality (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} ||f||_p ||g||_q, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Assuming that $a_m, b_n \geqslant 0$, $0 < ||a||_p = \{\sum_{m=1}^\infty a_m^p\}^{\frac{1}{p}} < \infty$, $0 < ||b||_q < \infty$, we still have the following discrete Hardy-Hilbert's inequality with the best constant $\frac{\pi}{\sin(\pi/p)}$ (cf. [1]):

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} ||a||_p ||b||_q. \tag{2}$$

Inequalities (1) and (2) are important in analysis and its applications (cf. [1], [2], [3], [4], [5], [22]). The more accurate of (2) is as follows:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n-\alpha} < \frac{\pi}{\sin(\pi/p)} ||a||_p ||b||_q$$

$(0 \leqslant \alpha \leqslant 1)$, which is an extension and a refinement of (2) (cf. [1]).

In 1998, by introducing an independent parameter $\lambda \in (0,1]$, Yang [6] gave an extension of (1) for $p = q = 2$. In 2009–2011, Yang [3] and [4] gave some extensions of (1) and (2) as follows: If $\lambda_1, \lambda_2 \in \mathbf{R}$, $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda(x,y) \geqslant 0$ is a homogeneous function of degree $-\lambda$, with $k(\lambda_1) = \int_0^\infty k_\lambda(t,1)t^{\lambda_1 - 1}dt \in \mathbf{R}_+$, $\phi(x) = x^{p(1-\lambda_1)-1}$, $\psi(y) = y^{q(1-\lambda_2)-1}$, $f(x), g(y) \geqslant 0$, $0 < ||f||_{p,\phi} := \{\int_0^\infty \phi(x)f^p(x)dx\}^{\frac{1}{p}} < \infty 0 < ||g||_{q,\psi} < \infty$, then we have

$$\int_0^\infty \int_0^\infty k_\lambda(x,y)f(x)g(y)dxdy < k(\lambda_1)||f||_{p,\phi}||g||_{q,\psi}, \tag{3}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover, if $k_\lambda(x,y)$ is finite and $k_\lambda(x,y)x^{\lambda_1-1}(k_\lambda(x,y)y^{\lambda_2-1})$ is decreasing with respect to $x > 0$ $(y > 0)$, then for $a_m, b_n \geqslant 0$, $0 < ||a||_{p,\phi} := \{\sum_{m=1}^\infty \phi(m)a_m^p\}^{\frac{1}{p}} < \infty$, $0 < ||b||_{q,\psi} < \infty$, we have

$$\sum_{m=1}^\infty \sum_{n=1}^\infty k_\lambda(m,n)a_m b_n < k(\lambda_1)||a||_{p,\phi}||b||_{q,\psi}, \tag{4}$$

where, the constant factor $k(\lambda_1)$ is still the best possible.

Clearly, for $\lambda = 1$, $k_1(x,y) = \frac{1}{x+y}$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, (3) reduces to (1), while (4) reduces to (2). Some other results including multidimensional Hilbert-type inequalities and the reverses are provided by [7]-[14]. On half-discrete Hilbert-type inequalities with the non-homogeneous kernels, Hardy et al. provided a few results in Theorem 351 of [1]. But they did not prove that the the constant factors are the best possible. However, Yang [15] gave a result with the kernel $\frac{1}{(1+nx)^\lambda}$ by introducing a variable and proved that the constant factor is the best possible. In 2011, Yang [16] gave a half-discrete Hardy-Hilbert's inequality with the best possible constant factor $B(\lambda_1, \lambda_2)$. Zhong et al ([17]–[19]) investigated several half-discrete Hilbert-type inequalities. A half-discrete Hilbert-type inequality with a general homogeneous kernel of degree $-\lambda \in \mathbf{R}$ and a best constant factor $k(\lambda_1)$ was obtained as follows:

$$\int_0^\infty f(x) \sum_{n=1}^\infty k_\lambda(x,n)a_n dx < k(\lambda_1)||f||_{p,\phi}||a||_{q,\psi}, \tag{5}$$

which is an extension of Yang's paper [16] (cf. [20]). Also a half-discrete Hilbert-type inequality with a general non-homogeneous kernel and a best constant factor was given by Yang [21].

In this paper, by using the methods of weight functions and Hermite-Hadamard's inequality, two kinds of more accurate equivalent reverse multidimensional half-discrete Hilbert-type inequalities with the kernel of hyperbolic cotangent function are given. The constant factor related to the Riemann zeta function is proved to be the best possible.

2. Some lemmas

LEMMA 1. *Suppose that* $(-1)^i h^{(i)}(t) > 0$ $(t > 0;\ i = 0, 1, 2)$. *Then (i) for* $b > 0$, $c > 1$, $0 < \alpha \leqslant 1$, *we have*

$$(-1)^i \frac{d^i}{dx^i} h((b + \ln^\alpha cx)^{\frac{1}{\alpha}}) > 0 \qquad (x > 1;\ i = 1, 2); \tag{6}$$

(ii) for $\int_{\frac{1}{2}}^\infty h(t)dt < \infty$, *we have*

$$\int_1^\infty h(t)dt < \sum_{n=1}^\infty h(n) < \int_{\frac{1}{2}}^\infty h(t)dt. \tag{7}$$

Proof. (i) We obtain

$$\frac{d}{dx} h((b + \ln^\alpha cx)^{\frac{1}{\alpha}}) = \frac{1}{x} h'((b + \ln^\alpha cx)^{\frac{1}{\alpha}})(b + \ln^\alpha cx)^{\frac{1}{\alpha}-1} \ln^{\alpha-1} cx < 0,$$

$$\frac{d^2}{dx^2} h((b + \ln^\alpha cx)^{\frac{1}{\alpha}}) = \frac{d}{dx} \left[\frac{1}{x} h'((b + \ln^\alpha cx)^{\frac{1}{\alpha}})(b + \ln^\alpha cx)^{\frac{1}{\alpha}-1} \ln^{\alpha-1} cx \right]$$

$$= -\frac{1}{x^2} h'((b + \ln^\alpha cx)^{\frac{1}{\alpha}})(b + \ln^\alpha cx)^{\frac{1}{\alpha}-1} \ln^{\alpha-1} cx$$

$$+ \frac{1}{x^2} h''((b + \ln^\alpha cx)^{\frac{1}{\alpha}})(b + \ln^\alpha cx)^{\frac{2}{\alpha}-2} \ln^{2\alpha-2} cx$$

$$+ \alpha(\frac{1}{\alpha} - 1)\frac{1}{x^2} h'((b + \ln^\alpha cx)^{\frac{1}{\alpha}})(b + \ln^\alpha cx)^{\frac{1}{\alpha}-2} \ln^{2\alpha-2} cx$$

$$+ (\alpha - 1)\frac{1}{x^2} h'((b + \ln^\alpha cx)^{\frac{1}{\alpha}})(b + \ln^\alpha cx)^{\frac{1}{\alpha}-1} \ln^{\alpha-2} cx$$

$$= [-h'((b + \ln^\alpha cx)^{\frac{1}{\alpha}})(b + \ln^\alpha cx) \ln cx + h''((b + \ln^\alpha cx)^{\frac{1}{\alpha}})$$

$$\times (b + \ln^\alpha cx)^{\frac{1}{\alpha}} \ln^\alpha cx + b(\alpha - 1)h'((b + \ln^\alpha cx)^{\frac{1}{\alpha}})]$$

$$\times \frac{1}{x^2}(b + \ln^\alpha cx)^{\frac{1}{\alpha}-2} \ln^{\alpha-2} cx > 0.$$

(ii) Since $h(t)$ is a decreasing convex function, by the decreasing property and Hermite-Hadamard's inequality (cf. [23]), we have $\int_n^{n+1} h(t)dt < h(n) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} h(t)dt$ $(n \in \mathbf{N})$, and then

$$\int_1^\infty h(t)dt = \sum_{n=1}^\infty \int_n^{n+1} h(t)dt < \sum_{n=1}^\infty h(n)$$

$$< \sum_{n=1}^\infty \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} h(t)dt = \int_{\frac{1}{2}}^\infty h(t)dt.$$

Hence, (7) follows. $\quad\square$

REMARK 1. The hyperbolic cotangent function $h(t) = \coth(t) := \frac{e^t + e^{-t}}{e^t - e^{-t}}$ (cf. [24]) satisfies the condition of $(-1)^i h^{(i)}(t) > 0$ $(t > 0; i = 0, 1, 2)$. In fact, $h(t) = \frac{e^t + e^{-t}}{e^t - e^{-t}} > 0$ $(t > 0)$, we find

$$h'(t) = -\frac{4}{(e^t - e^{-t})^2} < 0, h''(t) = \frac{8(e^t + e^{-t})}{(e^t - e^{-t})^3} > 0 \quad (t > 0).$$

If $i_0, j_0 \in \mathbf{N}$ ($\mathbf{N}$ is the set of positive integers), $\alpha, \beta > 0$, we put

$$||x||_\alpha := \left(\sum_{k=1}^{i_0} |x_k|^\alpha \right)^{\frac{1}{\alpha}} \quad (x = (x_1, \cdots, x_{i_0}) \in \mathbf{R}^{i_0}),$$

$$||y||_\beta := \left(\sum_{k=1}^{j_0} |y_k|^\beta \right)^{\frac{1}{\beta}} \quad (y = (y_1, \cdots, y_{j_0}) \in \mathbf{R}^{j_0}). \tag{8}$$

LEMMA 2. *If $s \in \mathbf{N}$, γ, $M > 0$, $\Psi(u)$ is a non-negative measurable function in $(0,1]$, and*

$$D_M := \left\{ x \in \mathbf{R}_+^s; 0 < u = \sum_{i=1}^{s} \left(\frac{x_i}{M} \right)^\gamma \leqslant 1 \right\},$$

then we have

$$\int \cdots \int_{D_M} \Psi(\sum_{i=1}^{s} \left(\frac{x_i}{M} \right)^\gamma) dx_1 \cdots dx_s = \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_0^1 \Psi(u) u^{\frac{s}{\gamma}-1} du. \tag{9}$$

Proof. It follows from Lemma 42.1 in (cf. [22]) (in P. 776).   □

Applying Lemma 2, in view of (9) and the condition, it follows:
(i) For $\mathbf{R}_+^s = \left\{ x \in \mathbf{R}_+^s; 0 < u = \sum_{i=1}^{s} \left(\frac{x_i}{M} \right)^\gamma \leqslant 1 (M \to \infty) \right\}$, we have

$$\int \cdots \int_{\mathbf{R}_+^s} \Psi \left(\sum_{i=1}^{s} \left(\frac{x_i}{M} \right)^\gamma \right) dx_1 \cdots dx_s$$

$$= \lim_{M \to \infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_0^1 \Psi(u) u^{\frac{s}{\gamma}-1} du; \tag{10}$$

(ii) for $\{x \in \mathbf{R}_+^s; ||x||_\gamma \geqslant 1\} = \left\{ x \in \mathbf{R}_+^s; \frac{1}{M^\gamma} < u = \sum_{i=1}^{s} \left(\frac{x_i}{M} \right)^\gamma \leqslant 1 (M \to \infty) \right\}$, setting $\Psi(u) = 0 (u \in (0, \frac{1}{M^\gamma}))$, we have

$$\int \cdots \int_{\{||x||_\gamma \geqslant 1\}} \Psi \left(\sum_{i=1}^{s} \left(\frac{x_i}{M} \right)^\gamma \right) dx_1 \cdots dx_s$$

$$= \lim_{M \to \infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{\frac{1}{M^\gamma}}^1 \Psi(u) u^{\frac{s}{\gamma}-1} du; \tag{11}$$

(iii) for $\{x \in \mathbf{R}_+^s; ||x||_\gamma \leqslant 1\} = \left\{x \in \mathbf{R}_+^s; 0 < u = \sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \leqslant \frac{1}{M^\gamma}\right\}$, setting $\Psi(u) = 0 \ (u \in (\frac{1}{M^\gamma}, \infty))$, we have

$$\int \cdots \int_{\{||x||_\gamma \leqslant 1\}} \Psi\left(\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma\right) dx_1 \cdots dx_s$$
$$= \lim_{M \to \infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_0^{\frac{1}{M^\gamma}} \Psi(u) u^{\frac{s}{\gamma}-1} du.$$

REMARK 2. For $\delta \in \{-1, 1\}$, $E_\delta = \{u > 0; \frac{1}{M^{\gamma\delta}} \leqslant u^\delta \leqslant 1\}$, in view of (ii) and (iii), it follows that

$$\int \cdots \int_{\{||x||_\gamma^\delta \geqslant 1\}} \Psi\left(\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma\right) dx_1 \cdots dx_s$$
$$= \lim_{M \to \infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{E_\delta} \Psi(u) u^{\frac{s}{\gamma}-1} du. \tag{12}$$

(iv) For $\{x \in \mathbf{R}_+^s; x_i \geqslant 1\} = \left\{x \in \mathbf{R}_+^s; \frac{s}{M^\gamma} < u = \sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \leqslant 1 (M^\gamma > s; M \to \infty)\right\}$, setting $\Psi(u) = 0 \ (u \in (0, \frac{s}{M^\gamma}))$, we have

$$\int \cdots \int_{\{x_i \geqslant 1\}} \Psi\left(\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma\right) dx_1 \cdots dx_s$$
$$= \lim_{M \to \infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{\frac{s}{M^\gamma}}^1 \Psi(u) u^{\frac{s}{\gamma}-1} du; \tag{13}$$

LEMMA 3. *For $s \in \mathbf{N}$, $\gamma > 0$, $\varepsilon > 0$, $\delta \in \{-1, 1\}$, $c = (c_1, \cdots, c_s) \in [2, \mathbf{e}]^s$, we have*

$$\int \cdots \int_{\{x \in \mathbf{R}_+^s; ||x||_\gamma^\delta \geqslant 1\}} ||x||_\gamma^{-s-\delta\varepsilon} dx_1 \cdots dx_s = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon \gamma^{s-1} \Gamma(\frac{s}{\gamma})}, \tag{14}$$

$$A_s(\varepsilon) := \sum_m \frac{||\ln cm||_\gamma^{-s-\varepsilon}}{\prod_{i=1}^s m_i} = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma} \gamma^{s-1} \Gamma(\frac{s}{\gamma})} + O(1)(\varepsilon \to 0^+). \tag{15}$$

Proof. By (12), it follows that

$$\int_{E_\delta} u^{\frac{-\delta\varepsilon}{\gamma}-1} du = \begin{cases} \int_{M^{-\gamma}}^1 u^{\frac{-\varepsilon}{\gamma}-1} du, \delta = 1, \\ \int_0^{M^{-\gamma}} u^{\frac{\varepsilon}{\gamma}-1} du, \delta = -1 \end{cases} = \begin{cases} \frac{\gamma}{\varepsilon}(M^\varepsilon - 1), & \delta = 1, \\ \frac{\gamma}{\varepsilon} M^{-\varepsilon}, & \delta = -1. \end{cases}$$

$$\int \cdots \int_{\{x \in \mathbf{R}_+^s; ||x||_\gamma^\delta \geqslant 1\}} ||x||_\gamma^{-s-\delta\varepsilon} dx_1 \cdots dx_s$$
$$= \int \cdots \int_{\{x \in \mathbf{R}_+^s; ||x||_\gamma^\delta \geqslant 1\}} \left\{M[\sum_{i=1}^s (\frac{x_i}{M})^\gamma]^{\frac{1}{\gamma}}\right\}^{-s-\delta\varepsilon} dx_1 \cdots dx_s$$

$$= \lim_{M \to \infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{E_\delta} (Mu^{1/\gamma})^{-s-\delta\varepsilon} u^{\frac{s}{\gamma}-1} du$$

$$= \lim_{M \to \infty} \frac{M^{-\delta\varepsilon} \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{E_\delta} u^{\frac{-\delta\varepsilon}{\gamma}-1} du = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon \gamma^{s-1} \Gamma(\frac{s}{\gamma})}.$$

Hence we have (14). For $M > s^{1/\gamma}$, we set

$$\Psi(u) = \begin{cases} 0, 0 < u < \frac{s}{M^\gamma}, \\ (Mu^{1/\gamma})^{-s-\varepsilon}, \frac{s}{M^\gamma} \leqslant u \leqslant 1. \end{cases}$$

Then by the decreasing property and (13), it follows

$$A_s(\varepsilon) \geqslant \int_{\{x \in \mathbf{R}_+^s; x_i \geqslant e/c_i\}} \frac{||\ln cx||_\gamma^{-s-\varepsilon}}{\prod_{i=1}^s x_i} dx \stackrel{u_i = \ln c_i x_i}{=} \int_{\{u \in \mathbf{R}_+^s; u_i \geqslant 1\}} ||u||_\gamma^{-s-\varepsilon} du$$

$$= \lim_{M \to \infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{s/M^\gamma}^1 (Mu^{1/\gamma})^{-s-\varepsilon} u^{\frac{s}{\gamma}-1} du = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma} \gamma^{s-1} \Gamma(\frac{s}{\gamma})}. \qquad (16)$$

In the following, by mathematical induction, we prove that for any $s \in \mathbf{N}$,

$$A_s(\varepsilon) \leqslant O_s(1) + \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma} \gamma^{s-1} \Gamma(\frac{s}{\gamma})} (\varepsilon \to 0^+). \qquad (17)$$

For $s = 1$, by (7), it follows that

$$A_1(\varepsilon) = \ln^{-1-\varepsilon} c_1 + \sum_{m_1=2}^\infty \frac{\ln^{-1-\varepsilon} c_1 m_1}{m_1} \leqslant \ln^{-1-\varepsilon} c_1 + \int_{\frac{3}{2}}^\infty \frac{\ln^{-1-\varepsilon} c_1 x}{x} dx$$

$$\leqslant \ln^{-1-\varepsilon} c_1 + \int_{e/c_1}^\infty \frac{\ln^{-1-\varepsilon} c_1 x}{x} dx = O_1(1) + \frac{1}{\varepsilon},$$

and then (17) is valid. Assuming that (17) is valid for $s - 1 \in \mathbf{N}$, then for s, we set

$$A_s(\varepsilon) = \sum_{\{m \in \mathbf{N}^s; \exists i_0, m_{i_0}=1\}} \frac{||\ln cm||_\gamma^{-s-\varepsilon}}{\prod_{i=1}^s m_i} + \sum_{\{m \in \mathbf{N}^s; m_i \geqslant 2\}} \frac{||\ln cm||_\gamma^{-s-\varepsilon}}{\prod_{i=1}^s m_i}.$$

There exist constants $a, b \in \mathbf{R}_+$, such that

$$\sum_{\{m \in \mathbf{N}^s; \exists i_0, m_{i_0}=1\}} \frac{||\ln cm||_\gamma^{-s-\varepsilon}}{\prod_{i=1}^s m_i} \leqslant a + b \sum_{\{m \in \mathbf{N}^{s-1}; m_i \geqslant 1\}} \frac{||\ln cm||_\gamma^{-(s-1)-(1+\varepsilon)}}{\prod_{i=1}^{s-1} m_i}.$$

By the assumption of mathematical induction for $s - 1$, we find

$$\sum_{\{m \in \mathbf{N}^{s-1}; m_i \geqslant 1\}} \frac{||\ln cm||_\gamma^{-(s-1)-(1+\varepsilon)}}{\prod_{i=1}^{s-1} m_i} \leqslant O_{s-1}(1) + \frac{\gamma^{2-s} \Gamma^{s-1}(\frac{1}{\gamma})}{(1+\varepsilon)(s-1)^{(1+\varepsilon)/\gamma} \Gamma(\frac{s-1}{\gamma})},$$

and then

$$\sum_{\{m\in\mathbf{N}^s;\exists i_0,m_{i_0}=1\}} ||\ln cm||_\gamma^{-s-\varepsilon} \frac{1}{\prod_{i=1}^s m_i} \leqslant O_s(1).$$

By Lemma 1, (7) and (16), we obtain

$$\sum_{\{m\in\mathbf{N}^s;m_i\geqslant 2\}} ||\ln cm||_\gamma^{-s-\varepsilon} \frac{1}{\prod_{i=1}^s m_i} \leqslant \int_{\{x\in\mathbf{R}_+^s;x_i\geqslant\frac{3}{2}\}} ||\ln cx||_\gamma^{-s-\varepsilon} \frac{dx}{\prod_{i=1}^s x_i}$$

$$\leqslant \int_{\{x\in\mathbf{R}_+^s;x_i\geqslant e/c_i\}} ||\ln cx||_\gamma^{-s-\varepsilon} \frac{1}{\prod_{i=1}^s x_i} dx$$

$$\overset{u_i=\ln c_i x_i}{=} \int_{\{u\in\mathbf{R}_+^s;u_i\geqslant 1\}} ||u||_\gamma^{-s-\varepsilon} du = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma}\gamma^{s-1}\Gamma(\frac{s}{\gamma})}.$$

Hence we prove that (17) is valid for $s \in \mathbf{N}$. Therefore, we have (15). $\quad\square$

3. Equivalent reverse inequalities

In the following sections, we suppose that i_0, $j_0 \in \mathbf{N}$, α, $\beta > 0$, $\tau = (\tau_1,\cdots,\tau_{j_0})$ $\in [\mathbf{2},\mathbf{e}]^{j_0}$, $\delta \in \{-1,1\}$, $\sigma > 1$, $\eta > 0$, and $p < 1$ ($p \neq 0$), $\frac{1}{p}+\frac{1}{q}=1$.

DEFINITION 1. For $x = (x_1,\cdots,x_{i_0}) \in \mathbf{R}_+^{i_0}, \ln\tau n = (\ln\tau_1 n_1,\cdots,\ln\tau_{j_0}n_{j_0})$, $n \in \mathbf{N}^{j_0}$, define two weight functions $\omega_\delta(\sigma,n)$ and $\varpi_\delta(\sigma,x)$ as follows:

$$\omega_\delta(\sigma,n) := ||\ln\tau n||_\beta^\sigma \int_{\mathbf{R}_+^{i_0}} e^{-\frac{\eta||\ln\tau n||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta||\ln\tau n||_\beta}{||x||_\alpha^\delta}\right) \frac{dx}{||x||_\alpha^{i_0+\delta\sigma}}, \tag{18}$$

$$\varpi_\delta(\sigma,x) := ||x||_\alpha^{-\delta\sigma} \sum_n e^{-\frac{\eta||\ln\tau n||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta||\ln\tau n||_\beta}{||x||_\alpha^\delta}\right) \frac{\prod_{j=1}^{j_0} n_j^{-1}}{||\ln\tau n||_\beta^{j_0-\sigma}}, \tag{19}$$

where, $\sum_n = \sum_{n_{j_0}=1}^\infty \cdots \sum_{n_1=1}^\infty$.

LEMMA 4. *With the assumptions of Definition 1, if $p < 0$ or $0 < p < 1$, both $\omega_\delta(\sigma,n)$ and $\varpi_\delta(\sigma,x)$ are finite, $f(x) = f(x_1,\cdots,x_{i_0}) \geqslant 0, a_n = a_{(n_1,\cdots,n_{j_0})} \geqslant 0$, then we have the following inequalities:*

$$J_1 := \left\{\sum_n \frac{\prod_{j=1}^{j_0} n_j^{-1} ||\ln\tau n||_\beta^{p\sigma-j_0}}{[\omega_\delta(\sigma,n)]^{p-1}}\right.$$

$$\times \left(\int_{\mathbf{R}_+^{i_0}} e^{-\frac{\eta||\ln\tau n||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta||\ln\tau n||_\beta}{||x||_\alpha^\delta}\right) f(x)dx\right)^p \left.\right\}^{\frac{1}{p}}$$

$$\geqslant \left\{\int_{\mathbf{R}_+^{i_0}} \varpi_\delta(\sigma,x)||x||_\alpha^{p(i_0+\delta\sigma)-i_0} f^p(x)dx\right\}^{\frac{1}{p}}, \tag{20}$$

$$J_2 := \left\{ \int_{\mathbf{R}_+^{i_0}} \frac{||x||_\alpha^{-\delta q \sigma - i_0}}{[\varpi_\delta(\sigma, x)]^{q-1}} \left(\sum_n e^{-\frac{\eta ||\ln \tau n||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta ||\ln \tau n||_\beta}{||x||_\alpha^\delta} \right) a_n \right)^q dx \right\}^{\frac{1}{q}}$$

$$\geqslant \left\{ \sum_n \omega_\delta(\sigma, n) \prod_{j=1}^{j_0} n_j^{q-1} ||\ln \tau n||_\beta^{q(j_0 - \sigma) - j_0} a_n^q \right\}^{\frac{1}{q}}. \tag{21}$$

Proof. By the reverse Hölder's inequality with weight (cf. [23]), it follows

$$\int_{\mathbf{R}_+^{i_0}} e^{-\frac{\eta ||\ln \tau n||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta ||\ln \tau n||_\beta}{||x||_\alpha^\delta} \right) f(x) dx$$

$$= \int_{\mathbf{R}_+^{i_0}} e^{-\frac{\eta ||\ln \tau n||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta ||\ln \tau n||_\beta}{||x||_\alpha^\delta} \right) \left[\frac{||x||_\alpha^{(i_0 + \delta \sigma)/q} f(x)}{||\ln \tau n||_\beta^{(j_0 - \sigma)/p} (\prod_{j=1}^{j_0} n_j)^{1/p}} \right]$$

$$\times \left[\frac{||\ln \tau n||_\beta^{(j_0 - \sigma)/p} (\prod_{j=1}^{j_0} n_j)^{1/p}}{||x||_\alpha^{(i_0 + \delta \sigma)/q}} \right] dx$$

$$\geqslant \left\{ \int_{\mathbf{R}_+^{i_0}} e^{-\frac{\eta ||\ln \tau n||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta ||\ln \tau n||_\beta}{||x||_\alpha^\delta} \right) \frac{||x||_\alpha^{(i_0 + \delta \sigma)(p-1)} f^p(x)}{||\ln \tau n||_\beta^{j_0 - \sigma} \prod_{j=1}^{j_0} n_j} dx \right\}^{\frac{1}{p}} \tag{22}$$

$$\times \left\{ \int_{\mathbf{R}_+^{i_0}} e^{-\frac{\eta ||\ln \tau n||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta ||\ln \tau n||_\beta}{||x||_\alpha^\delta} \right) \frac{||\ln \tau n||_\beta^{(j_0 - \sigma)(q-1)}}{||x||_\alpha^{i_0 + \delta \sigma} (\prod_{j=1}^{j_0} n_j)^{1-q}} dx \right\}^{\frac{1}{q}}$$

$$= [\omega_\delta(\sigma, n)]^{\frac{1}{q}} ||\ln \tau n||_\beta^{\frac{j_0}{p} - \sigma} (\prod_{j=1}^{j_0} n_j)^{\frac{1}{p}}$$

$$\times \left\{ \int_{\mathbf{R}_+^{i_0}} e^{-\frac{\eta ||\ln \tau n||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta ||\ln \tau n||_\beta}{||x||_\alpha^\delta} \right) \frac{||x||_\alpha^{(i_0 + \delta \sigma)(p-1)} f^p(x)}{||\ln \tau n||_\beta^{j_0 - \sigma} (\prod_{j=1}^{j_0} n_j)^{1-p}} dx \right\}^{\frac{1}{p}}. \tag{23}$$

Then by Lebesgue term by term integration theorem (cf. [25]), we have

$$J_1 \geqslant \left\{ \sum_n \int_{\mathbf{R}_+^{i_0}} e^{-\frac{\eta ||\ln \tau n||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta ||\ln \tau n||_\beta}{||x||_\alpha^\delta} \right) \frac{||x||_\alpha^{(i_0 + \delta \sigma)(p-1)} f^p(x)}{||\ln \tau n||_\beta^{j_0 - \sigma} \prod_{j=1}^{j_0} n_j} dx \right\}^{\frac{1}{p}}$$

$$= \left\{ \int_{\mathbf{R}_+^{i_0}} \sum_n e^{-\frac{\eta ||\ln \tau n||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta ||\ln \tau n||_\beta}{||x||_\alpha^\delta} \right) \frac{||x||_\alpha^{(i_0 + \delta \sigma)(p-1)} f^p(x)}{||\ln \tau n||_\beta^{j_0 - \sigma} \prod_{j=1}^{j_0} n_j} dx \right\}^{\frac{1}{p}}$$

$$= \left\{ \int_{\mathbf{R}_+^{i_0}} \varpi_\delta(\sigma, x) ||x||_\alpha^{p(i_0 + \delta \sigma) - i_0} f^p(x) dx \right\}^{\frac{1}{p}}. \tag{24}$$

Hence, (20) follows.

By the reverse Hölder's inequality with weight, we still can obtain

$$\sum_n e^{-\frac{\eta||\ln \tau n||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta||\ln \tau n||_\beta}{||x||_\alpha^\delta}\right) a_n$$

$$\geqslant [\varpi_\delta(\sigma,x)]^{\frac{1}{p}} ||x||_\alpha^{\frac{i_0}{q}+\delta\sigma}$$

$$\times \left\{ \sum_n e^{-\frac{\eta||\ln \tau n||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta||\ln \tau n||_\beta}{||x||_\alpha^\delta}\right) \frac{||\ln \tau n||_\beta^{(j_0-\sigma)(q-1)}}{||x||_\alpha^{i_0+\delta\sigma}} a_n^q \right\}^{\frac{1}{q}},$$

then by Lebesgue term by term integration theorem and the same way, we have (21). $\square$

LEMMA 5. *With the assumptions of Lemma 4, we have the following inequality equivalent to (20) and (21):*

$$I := \sum_n \int_{\mathbf{R}_+^{i_0}} e^{-\frac{\eta||\ln \tau n||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta||\ln \tau n||_\beta}{||x||_\alpha^\delta}\right) a_n f(x) dx$$

$$\geqslant \left\{ \int_{\mathbf{R}_+^{i_0}} \varpi_\delta(\sigma,x) ||x||_\alpha^{p(i_0+\delta\sigma)-i_0} f^p(x) dx \right\}^{\frac{1}{p}}$$

$$\times \left\{ \sum_n \omega_\delta(\sigma,n) \prod_{j=1}^{j_0} n_j^{q-1} ||\ln \tau n||_\beta^{q(j_0-\sigma)-j_0} a_n^q \right\}^{\frac{1}{q}}; \tag{25}$$

Proof. By the reverse Hölder's inequality (cf. [23]), it follows

$$I = \sum_n \frac{||\ln \tau n||_\beta^{\frac{j_0}{q}-(j_0-\sigma)}}{[\omega_\delta(\sigma,n)]^{\frac{1}{q}}(\prod_{j=1}^{j_0} n_j)^{\frac{1}{p}}} \left[\int_{\mathbf{R}_+^{i_0}} e^{-\frac{\eta||\ln \tau n||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta||\ln \tau n||_\beta}{||x||_\alpha^\delta}\right) f(x) dx \right]$$

$$\times \left[[\omega_\delta(\sigma,n)]^{\frac{1}{q}} (\prod_{j=1}^{j_0} n_j)^{\frac{1}{p}} ||\ln \tau n||_\beta^{(j_0-\sigma)-\frac{j_0}{q}} a_n \right]$$

$$\geqslant J_1 \left\{ \sum_n \omega_\delta(\sigma,n) \prod_{j=1}^{j_0} n_j^{q-1} ||\ln \tau n||_\beta^{q(j_0-\sigma)-j_0} a_n^q \right\}^{\frac{1}{q}}.$$

Then by (20), we have (25).

On the other hand, assuming that (25) is valid, for $n \in \mathbf{N}^{j_0}$, we set

$$b_n := \frac{||\ln \tau n||_\beta^{p\sigma-j_0}}{[\omega_\delta(\sigma,n)]^{p-1} \prod_{j=1}^{j_0} n_j} \left(\int_{\mathbf{R}_+^{i_0}} e^{-\frac{\eta||\ln \tau n||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta||\ln \tau n||_\beta}{||x||_\alpha^\delta}\right) f(x) dx \right)^{p-1}.$$

Then it follows

$$J_1^p = \sum_n \omega_\delta(\sigma,n) \prod_{j=1}^{j_0} n_j^{q-1} ||\ln \tau n||_\beta^{q(j_0-\sigma)-j_0} a_n^q.$$

If $J_1 = \infty$, then (20) is trivially valid; if $J_1 = 0$, then by (24), (20) keeps the form of equality $(= 0)$. Suppose that $0 < J_1 < \infty$. By (25), we have

$$0 < \sum_n \omega_\delta(\sigma_2, n) \prod_{j=1}^{j_0} n_j^{q-1} ||\ln \tau n||_\beta^{q(j_0-\sigma)-j_0} a_n^q = J_1^p = I$$

$$\geqslant \left\{ \int_{\mathbf{R}_+^{i_0}} \varpi_\delta(\sigma, x) ||x||_\alpha^{p(i_0+\delta\sigma)-i_0} f^p(x) dx \right\}^{\frac{1}{p}}$$

$$\times \left\{ \sum_n \omega_\delta(\sigma, n) \prod_{j=1}^{j_0} n_j^{q-1} ||\ln \tau n||_\beta^{q(j_0-\sigma)-j_0} a_n^q \right\}^{\frac{1}{q}}.$$

It follows that

$$J_1 = \left\{ \sum_n \omega_\delta(\sigma, n) \prod_{j=1}^{j_0} n_j^{q-1} ||\ln \tau n||_\beta^{q(j_0-\sigma)-j_0} a_n^q \right\}^{\frac{1}{p}}$$

$$\geqslant \left\{ \int_{\mathbf{R}_+^{i_0}} \varpi_\delta(\sigma, x) ||x||_\alpha^{p(i_0+\delta\sigma)-i_0} f^p(x) dx \right\}^{\frac{1}{p}},$$

and then (20) follows. Hence, (25) and (20) are equivalent.

By the reverse Hölder's inequality and the same way, we can obtain

$$I \geqslant \left\{ \int_{\mathbf{R}_+^{i_0}} \varpi_\delta(\sigma, x) ||x||_\alpha^{p(i_0+\delta\sigma)-i_0} f^p(x) dx \right\}^{\frac{1}{p}} J_2.$$

Then by (21), we have (25). On the other hand, assuming that (25) is valid, we set

$$f(x) = \frac{||x||_\alpha^{-q\delta\sigma-i_0}}{[\varpi_\delta(\sigma, x)]^{q-1}} \left(\sum_n e^{-\frac{\eta||\ln \tau n||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta||\ln \tau n||_\beta}{||x||_\alpha^\delta} \right) a_n \right)^{q-1} \quad (x \in \mathbf{R}_+^{i_0}).$$

Then it follows

$$J_2^q = \int_{\mathbf{R}_+^{i_0}} \varpi_\delta(\sigma, x) ||x||_\alpha^{p(i_0+\delta\sigma)-i_0} f^p(x) dx.$$

By (25) and the same way, we can obtain

$$J_2 = \left\{ \int_{\mathbf{R}_+^{i_0}} \varpi_\delta(\sigma, x) ||x||_\alpha^{p(i_0+\delta\sigma)-i_0} f^p(x) dx \right\}^{\frac{1}{q}}$$

$$\geqslant \left\{ \sum_n \omega_\delta(\sigma, n) \prod_{j=1}^{j_0} n_j^{q-1} ||\ln \tau n||_\beta^{q(j_0-\sigma)-j_0} a_n^q \right\}^{\frac{1}{q}},$$

and then (21) is equivalent to (25).

Hence (20), (21) and (25) are equivalent. $\quad \square$

LEMMA 6. *If $j_0 \in \mathbf{N} \setminus \{1\}, 1 < \widetilde{\sigma} < j_0$, $k(\widetilde{\sigma}) := \int_0^\infty e^{-v} \coth(v) v^{\widetilde{\sigma}-1} dv$, then we have*

$$k(\widetilde{\sigma}) = \left[(2 - \frac{1}{2^{\widetilde{\sigma}-1}}) \zeta(\widetilde{\sigma}) - 1 \right] \Gamma(\widetilde{\sigma}) \in \mathbf{R}_+, \tag{26}$$

$$\varpi_\lambda(\widetilde{\sigma}, n) = K_2(\widetilde{\sigma}) := \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha}) \eta^{\widetilde{\sigma}}} k(\widetilde{\sigma}) (n \in \mathbf{N}^{j_0}), \tag{27}$$

where, $\zeta(\sigma) = \sum_{k=1}^\infty k^{-\sigma}$ $(\sigma > 1)$ is the Riemann zeta function (cf. [13]).
Moreover, if $0 < \beta \leqslant 1$, then we have

$$K_1(\widetilde{\sigma})(1 - \theta_{\widetilde{\sigma}}(\|x\|_\alpha^\delta)) < \omega_\lambda(\widetilde{\sigma}, x) < K_1(\widetilde{\sigma})(x \in \mathbf{R}_+^{i_0}), \tag{28}$$

where,

$$K_1(\widetilde{\sigma}) := \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1} \Gamma(\frac{j_0}{\beta}) \eta^{\widetilde{\sigma}}} k(\widetilde{\sigma}) \in \mathbf{R}_+, \tag{29}$$

$$\theta_{\widetilde{\sigma}}(\|x\|_\alpha^\delta) := \frac{1}{k(\widetilde{\sigma})} \int_0^{\frac{1}{\|x\|_\alpha^\delta} \eta_{j_0}^{1/\beta}} e^{-v} \coth(v) v^{\widetilde{\sigma}-1} dv = O\left(\frac{1}{\|x\|_\alpha^\delta} \right) \in (0,1). \tag{30}$$

Proof. By Lebesgue term by term integration theorem, we obtain

$$k(\widetilde{\sigma}) = \int_0^\infty e^{-v} \frac{e^v + e^{-v}}{e^v - e^{-v}} v^{\widetilde{\sigma}-1} dv = \int_0^\infty \frac{e^{-v} + e^{-3v}}{1 - e^{-2v}} v^{\widetilde{\sigma}-1} dv$$

$$= \int_0^\infty (e^{-v} + e^{-3v}) \sum_{k=0}^\infty e^{-2kv} v^{\widetilde{\sigma}-1} dv = \sum_{k=0}^\infty \int_0^\infty (e^{-(2k+1)v} + e^{-(2k+3)v}) v^{\widetilde{\sigma}-1} dv$$

$$= \sum_{k=0}^\infty \left[\frac{1}{(2k+1)^{\widetilde{\sigma}}} + \frac{1}{(2k+3)^{\widetilde{\sigma}}} \right] \Gamma(\widetilde{\sigma}) = \left[2 \sum_{k=0}^\infty \frac{1}{(2k+1)^{\widetilde{\sigma}}} - 1 \right] \Gamma(\widetilde{\sigma})$$

$$= \left[2 \left(\sum_{k=1}^\infty \frac{1}{k^{\widetilde{\sigma}}} - \frac{1}{2^{\widetilde{\sigma}}} \sum_{k=1}^\infty \frac{1}{k^{\widetilde{\sigma}}} \right) - 1 \right] \Gamma(\widetilde{\sigma}) = \left[(2 - \frac{1}{2^{\widetilde{\sigma}-1}}) \zeta(\widetilde{\sigma}) - 1 \right] \Gamma(\widetilde{\sigma}).$$

By (10), since $\delta = \pm 1$, we find

$$\omega_\lambda(\widetilde{\sigma}, n) = \|\ln \tau n\|_\beta^{\widetilde{\sigma}} \int_{\mathbf{R}_+^{i_0}} e^{-\frac{\eta \|\ln \tau n\|_\beta}{M^\delta [\sum_{i=1}^{i_0} (\frac{x_i}{M})^\alpha]^{\frac{\delta}{\alpha}}}} \coth\left(\frac{\eta \|\ln \tau n\|_\beta}{M^\delta [\sum_{i=1}^{i_0} (\frac{x_i}{M})^\alpha]^{\frac{\delta}{\alpha}}} \right)$$

$$\times \frac{1}{M^{i_0 + \delta\widetilde{\sigma}} [\sum_{i=1}^{i_0} (\frac{x_i}{M})^\alpha]^{\frac{1}{\alpha}(i_0 + \delta\widetilde{\sigma})}} dx_1 \cdots dx_{i_0}$$

$$= \lim_{M \to \infty} \frac{M^{i_0} \Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0} \Gamma(\frac{i_0}{\alpha})} \|\ln \tau n\|_\beta^{\widetilde{\sigma}} \int_0^1 \frac{e^{-\frac{\eta \|\ln \tau n\|_\beta}{M^\delta u^{\frac{\delta}{\alpha}}}} \coth\left(\frac{\eta \|\ln \tau n\|_\beta}{M^\delta u^{\frac{\delta}{\alpha}}} \right)}{M^{i_0 + \delta\widetilde{\sigma}} u^{\frac{1}{\alpha}(i_0 + \delta\widetilde{\sigma})}} u^{\frac{i_0}{\alpha} - 1} du$$

$$= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})\eta^{\widetilde{\sigma}}} \int_0^\infty e^{-v} \coth(v) v^{\widetilde{\sigma}-1} dv (v = \frac{\eta||\ln\tau n||_\beta}{M^\delta u^{\frac{\delta}{\alpha}}}).$$

Hence, we have (26) and (27).

Moreover, by Lemma 1 and (10), we obtain

$$\varpi_\delta(\widetilde{\sigma},x) < \int_{\{y\in\mathbf{R}_+^{j_0} y_i \geqslant \frac{1}{2}\}} e^{-\frac{\eta||\ln\tau y||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta||\ln\tau y||_\beta}{||x||_\alpha^\delta}\right) \frac{||x||_\alpha^{-\delta\widetilde{\sigma}} dy}{||\ln\tau y||_\beta^{j_0-\widetilde{\sigma}} \prod_{j=1}^{j_0} y_j}$$

$$= ||x||_\alpha^{-\delta\widetilde{\sigma}} \int_{\{u\in\mathbf{R}_+^{j_0}; u_i \geqslant \frac{1}{2}\tau_i\}} e^{-\frac{\eta||u||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta||u||_\beta}{||x||_\alpha^\delta}\right) \frac{du}{||u||_\beta^{j_0-\widetilde{\sigma}}} (u_i = \ln\tau_i y_i)$$

$$\leqslant ||x||_\alpha^{-\delta\widetilde{\sigma}} \int_{\mathbf{R}_+^{j_0}} e^{-\frac{\eta||u||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta||u||_\beta}{||x||_\alpha^\delta}\right) \frac{du}{||u||_\beta^{j_0-\widetilde{\sigma}}}$$

$$= \int_{\mathbf{R}_+^{j_0}} e^{-(\frac{\eta M[\sum_{j=1}^{j_0}(\frac{u_j}{M})^\beta]^{\frac{1}{\beta}}}{||x||_\alpha^\delta})} \coth\left(\frac{\eta M[\sum_{j=1}^{j_0}(\frac{u_j}{M})^\beta]^{\frac{1}{\beta}}}{||x||_\alpha^\delta}\right) \frac{M^{\widetilde{\sigma}-j_0}||x||_\alpha^{-\delta\widetilde{\sigma}} du}{[\sum_{j=1}^{j_0}(\frac{u_j}{M})^\beta]^{\frac{j_0-\widetilde{\sigma}}{\beta}}}$$

$$= \lim_{M\to\infty} \frac{M^{j_0}\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0}\Gamma(\frac{j_0}{\beta})} ||x||_\alpha^{-\delta\widetilde{\sigma}} \int_0^1 e^{-\frac{\eta M t^{\frac{1}{\beta}}}{||x||_\alpha^\delta}} \coth\left(\frac{\eta M t^{\frac{1}{\beta}}}{||x||_\alpha^\delta}\right) \frac{t^{\frac{j_0}{\beta}-1} dt}{M^{j_0-\widetilde{\sigma}} t^{\frac{j_0-\widetilde{\sigma}}{\beta}}}$$

$$= \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})\eta^{\widetilde{\sigma}}} \int_0^\infty e^{-v} \coth(v) v^{\widetilde{\sigma}-1} dv = K_2(\widetilde{\sigma})(v = \frac{\eta M t^{\frac{1}{\beta}}}{||x||_\alpha^\delta}).$$

By Lemma 1, (13) and (26), we find

$$\varpi_\lambda(\widetilde{\sigma},x) > \int_{\{y\in\mathbf{R}_+^{j_0} y_i \geqslant e/\tau_i\}} e^{-\frac{\eta||\ln\tau y||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta||\ln\tau y||_\beta}{||x||_\alpha^\delta}\right) \frac{||x||_\alpha^{-\delta\widetilde{\sigma}} dy}{||\ln\tau y||_\beta^{j_0-\widetilde{\sigma}} \prod_{j=1}^{j_0} y_j}$$

$$= ||x||_\alpha^{-\delta\widetilde{\sigma}} \int_{\{u\in\mathbf{R}_+^{j_0}; u_i \geqslant 1\}} e^{-\frac{\eta||u||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta||u||_\beta}{||x||_\alpha^\delta}\right) \frac{du}{||u||_\beta^{j_0-\widetilde{\sigma}}} (u_i = \ln\tau_i y_i)$$

$$= ||x||_\alpha^{-\delta\widetilde{\sigma}} \lim_{M\to\infty} \frac{M^{j_0}\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} \int_{\frac{j_0}{M^\beta}}^1 e^{-\frac{\eta M t^{\frac{1}{\beta}}}{||x||_\alpha^\delta}} \coth\left(\frac{\eta M t^{\frac{1}{\beta}}}{||x||_\alpha^\delta}\right) \frac{t^{\frac{j_0}{\beta}-1} dt}{M^{j_0-\widetilde{\sigma}} t^{\frac{j_0-\widetilde{\sigma}}{\beta}}}$$

$$= \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})\eta^{\widetilde{\sigma}}} \int_{\frac{1}{||x||_\alpha^\delta} \eta j_0^{1/\beta}}^\infty e^{-v} \coth(v) v^{\widetilde{\sigma}-1} dv (v = \frac{\eta||u||_\beta}{||x||_\alpha^\delta})$$

$$= \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})\eta^{\widetilde{\sigma}}} k(\widetilde{\sigma})[1 - \theta_{\widetilde{\sigma}}(||x||_\alpha^\delta)] > 0,$$

$$0 < \theta_{\widetilde{\sigma}}(||x||_\alpha^\delta) = \frac{1}{k(\widetilde{\sigma})} \int_0^{\frac{1}{||x||_\alpha^\delta} \eta j_0^{1/\beta}} e^{-v} \coth(v) v^{\widetilde{\sigma}-1} dv.$$

Since $e^{-v}\coth(v)v^{\widetilde{\sigma}-1}\to 0(v\to 0^+,\text{ or } v\to\infty)$, there exists a positive constant L, such that $e^{-v}\coth(v)v^{\widetilde{\sigma}-1}\leqslant L(v\in(0,\infty))$, and then

$$\theta_{\widetilde{\sigma}}(||x||_\alpha^\delta)\leqslant \frac{L}{k(\widetilde{\sigma})}\int_0^{\frac{1}{||x||_\alpha^\delta}}\eta_{j_0}^{1/\beta}\,dv = \frac{L}{k(\widetilde{\sigma})}\frac{1}{||x||_\alpha^\delta}\eta_{j_0}^{1/\beta}.$$

Hence we have (28), (29) and (30). $\square$

4. Two kinds of equivalent reverse forms

We set $\Phi_\delta(x) := ||x||_\alpha^{p(i_0+\delta\sigma)-i_0}$, $\widetilde{\Phi}_\delta(x) := (1-\theta_\sigma(||x||_\alpha^\delta))||x||_\alpha^{p(i_0+\delta\sigma)-i_0}$ $(x\in \mathbf{R}_+^{i_0})$, and $\Psi(n) := \prod_{j=1}^{j_0}n_j^{q-1}||\ln\tau n||_\beta^{q(j_0-\sigma)-j_0}$ $(n\in\mathbf{N}^{j_0})$.

THEOREM 1. *If* $p<0$ $(0<q<1)$, $0<\beta\leqslant 1$, $j_0\in\mathbf{N}\backslash\{1\}$, $f(x)=f(x_1,\cdots,x_{i_0})$ $\geqslant 0$, $a_n=a_{(n_1,\cdots,n_{j_0})}\geqslant 0$,

$$0<||f||_{p,\Phi_\delta}=\{\int_{\mathbf{R}_+^{i_0}}\Phi_\delta(x)f^p(x)dx\}^{\frac{1}{p}}<\infty,0<||a||_{q,\Psi}=\{\sum_n\Psi(n)a_n^q\}^{\frac{1}{q}}<\infty,$$

then we have the following equivalent reverse inequalities with the best possible constant factor $K(\sigma)$:

$$I = \sum_n\int_{\mathbf{R}_+^{i_0}}e^{-\frac{\eta||\ln\tau n||_\beta}{||x||_\alpha^\delta}}\coth\left(\frac{\eta||\ln\tau n||_\beta}{||x||_\alpha^\delta}\right)a_n f(x)dx > K(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\Psi}, \quad (31)$$

$$J := \left\{\sum_n\frac{\prod_{j=1}^{j_0}n_j^{-1}}{||\ln\tau n||_\beta^{j_0-p\sigma}}\left(\int_{\mathbf{R}_+^{i_0}}e^{-\frac{\eta||\ln\tau n||_\beta}{||x||_\alpha^\delta}}\coth\left(\frac{\eta||\ln\tau n||_\beta}{||x||_\alpha^\delta}\right)f(x)dx\right)^p\right\}^{\frac{1}{p}}$$
$$> K(\sigma)||f||_{p,\Phi_\delta}, \quad\quad (32)$$

$$H := \left\{\int_{\mathbf{R}_+^{i_0}}\frac{1}{||x||_\alpha^{i_0+q\delta\sigma}}\left(\sum_n e^{-\frac{\eta||\ln\tau n||_\beta}{||x||_\alpha^\delta}}\coth\left(\frac{\eta||\ln\tau n||_\beta}{||x||_\alpha^\delta}\right)a_n\right)^q dx\right\}^{\frac{1}{q}}$$
$$> K(\sigma)||a||_{q,\Psi}, \quad\quad (33)$$

where, $k(\sigma)=[(2-\frac{1}{2^{\sigma-1}})\zeta(\sigma)-1]\Gamma(\sigma)$,

$$K(\sigma) = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}}\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}}\frac{k(\sigma)}{\eta^\sigma}.$$

Proof. By Lemma 4, Lemma 5 and Lemma 6, we have equivalent reverse inequalities (31), (32) and (33). By the reverse Hölder's inequality, we still can obtain

$$I\geqslant J\left\{\sum_n\prod_{j=1}^{j_0}n_j^{q-1}||\ln\tau n||_\beta^{q(j_0-\sigma)-j_0}a_n^q\right\}^{\frac{1}{q}}, \quad\quad (34)$$

$$I \geqslant \{ \int_{\mathbf{R}_+^{i_0}} ||x||_\alpha^{p(i_0+\delta\sigma)-i_0} f^p(x)dx \}^{\frac{1}{p}} H. \tag{35}$$

For $0 < \varepsilon < q(\sigma-1)$, we set $\tilde{a}_n, \tilde{f}(x)$ as: $\tilde{a}_n := ||\ln \tau n||_\beta^{(\sigma-\frac{\varepsilon}{q})-j_0} \prod_{j=1}^{j_0} n_j^{-1}$, $n \in \mathbf{N}^{j_0}$,

$$\tilde{f}(x) := \begin{cases} 0, & 0 < ||x||_\alpha^\delta < 1, \\ ||x||_\alpha^{-\delta\sigma-\frac{\delta\varepsilon}{p}-i_0}, & ||x||_\alpha^\delta \geqslant 1. \end{cases}$$

Then for $\tilde{\sigma} = \sigma - \frac{\varepsilon}{q}$ (> 1), in view of (14), (15) and (28), we find

$$||\tilde{f}||_{p,\Phi_\delta} ||\tilde{a}||_{q,\Psi} = \left\{ \int_{\{x \in \mathbf{R}_+^{i_0}; ||x||_\alpha^\delta \geqslant 1\}} ||x||_\alpha^{-i_0-\delta\varepsilon} dx \right\}^{\frac{1}{p}} \left\{ \sum_n \frac{||\ln \tau n||_\beta^{-j_0-\varepsilon}}{\prod_{j=1}^{j_0} n_j} \right\}^{\frac{1}{q}}$$

$$= \frac{1}{\varepsilon} \left\{ \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \right\}^{\frac{1}{p}} \left\{ \frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} + \varepsilon O(1) \right\}^{\frac{1}{q}},$$

$$\tilde{I} := \int_{\mathbf{R}_+^{i_0}} \sum_n e^{-\frac{\eta||\ln \tau n||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta||\ln \tau n||_\beta}{||x||_\alpha^\delta}\right) \tilde{a}_n \tilde{f}(x) dx$$

$$= \int_{\{x \in \mathbf{R}_+^{i_0}; ||x||_\alpha^\delta \geqslant 1\}} ||x||_\alpha^{-i_0-\delta\varepsilon} \varpi_\delta(\tilde{\sigma}, x) dx$$

$$\leqslant K_1(\tilde{\sigma}) \int_{\{x \in \mathbf{R}_+^{i_0}; ||x||_\alpha^\delta \geqslant 1\}} ||x||_\alpha^{-i_0-\delta\varepsilon} dx = \frac{K_1(\tilde{\sigma})\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}.$$

If there exists a constant $K \geqslant K(\sigma)$, such that (31) is valid when replacing $K(\sigma)$ by K, then in particular, we have

$$K_1(\tilde{\sigma})\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \geqslant \varepsilon\tilde{I} > \varepsilon K ||\tilde{f}||_{p,\Phi_\delta} ||\tilde{a}||_{q,\Psi}$$

$$= K \left\{ \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \right\}^{\frac{1}{p}} \left\{ \frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} + \varepsilon O(1) \right\}^{\frac{1}{q}},$$

and then $K(\sigma) \geqslant K(\varepsilon \to 0^+)$. Hence $K = K(\sigma)$ is the best possible constant factor of (31).

By the equivalency, we can prove that the constant factor $K(\sigma)$ in (32) ((33)) is the best possible. Otherwise, we would reach a contradiction by (34) ((35)) that the constant factor $K(\sigma)$ in (31) is not the best possible. $\square$

THEOREM 2. *If $0 < p < 1$ $(q < 0)$, $0 < \beta \leqslant 1$, $j_0 \in \mathbf{N}\backslash\{1\}$, $f(x) = f(x_1, \cdots, x_{i_0})$ $\geqslant 0$, $a_n = a_{(n_1, \cdots, n_{j_0})} \geqslant 0$,*

$$0 < ||f||_{p,\tilde{\Phi}_\delta} = \{ \int_{\mathbf{R}_+^{i_0}} \tilde{\Phi}_\delta(x) f^p(x)dx \}^{\frac{1}{p}} < \infty, 0 < ||a||_{q,\Psi} = \{ \sum_n \Psi(n)a_n^q \}^{\frac{1}{q}} < \infty,$$

then we have the following equivalent reverse inequalities with the best possible constant factor $K(\sigma)$:

$$I = \sum_n \int_{\mathbf{R}_+^{i_0}} e^{-\frac{\eta||\ln \tau n||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta||\ln \tau n||_\beta}{||x||_\alpha^\delta}\right) a_n f(x) dx > K(\sigma)||f||_{p,\widetilde{\Phi}_\delta}||a||_{q,\Psi}, \quad (36)$$

$$J = \left\{\sum_n \frac{\prod_{j=1}^{j_0} n_j^{-1}}{||\ln \tau n||_\beta^{j_0-p\sigma}}\left(\int_{\mathbf{R}_+^{i_0}} e^{-\frac{\eta||\ln \tau n||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta||\ln \tau n||_\beta}{||x||_\alpha^\delta}\right) f(x) dx\right)^p\right\}^{\frac{1}{p}}$$
$$> K(\sigma)||f||_{p,\widetilde{\Phi}_\delta}, \quad (37)$$

$$H_1 := \left\{\int_{\mathbf{R}_+^{i_0}} \frac{(1-\theta_\sigma(||x||_\alpha^\delta))^{1-q}}{||x||_\alpha^{i_0+q\delta\sigma}}\left(\sum_n e^{-\frac{\eta||\ln \tau n||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta||\ln \tau n||_\beta}{||x||_\alpha^\delta}\right) a_n\right)^q dx\right\}^{\frac{1}{q}}$$
$$> K(\sigma)||a||_{q,\Psi}. \quad (38)$$

Proof. By Lemma 4, Lemma 5 and Lemma 6, we have equivalent reverse inequalities (36), (37) and (38). By the reverse Hölder's inequality, we still can obtain

$$I \geqslant J\left\{\sum_n \prod_{j=1}^{j_0} n_j^{q-1}||\ln \tau n||_\beta^{q(j_0-\sigma)-j_0} a_n^q\right\}^{\frac{1}{q}}, \quad (39)$$

$$I \geqslant \left\{\int_{\mathbf{R}_+^{i_0}} (1-\theta_\sigma(||x||_\alpha^\delta))||x||_\alpha^{p(i_0+\delta\sigma)-i_0} f^p(x) dx\right\}^{\frac{1}{p}} H_1. \quad (40)$$

For $0 < \varepsilon < |q|(\sigma-1)$, we set $\widetilde{a}_n$, $\widetilde{f}(x)$ as Theorem 1. Then for $\widetilde{\sigma} = \sigma - \frac{\varepsilon}{q}$ (>1), in view of (14), (15) and (28), we find

$$||\widetilde{f}||_{p,\widetilde{\Phi}_\delta}||\widetilde{a}||_{q,\Psi}$$

$$= \left\{\int_{\{x\in \mathbf{R}_+^{i_0};||x||_\alpha^\delta\geqslant 1\}} (1-O(\frac{1}{||x||_\alpha^\delta}))||x||_\alpha^{-i_0-\delta\varepsilon} dx\right\}^{\frac{1}{p}} \left\{\sum_n \frac{||\ln \tau n||_\beta^{-j_0-\varepsilon}}{\prod_{j=1}^{j_0} n_j}\right\}^{\frac{1}{q}}$$

$$= \frac{1}{\varepsilon}\left\{\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} - \varepsilon O_1(1)\right\}^{\frac{1}{p}} \left\{\frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} + \varepsilon O(1)\right\}^{\frac{1}{q}},$$

$$\widetilde{I} := \int_{\mathbf{R}_+^{i_0}} \sum_n e^{-\frac{\eta||\ln \tau n||_\beta}{||x||_\alpha^\delta}} \coth\left(\frac{\eta||\ln \tau n||_\beta}{||x||_\alpha^\delta}\right) \widetilde{a}_n \widetilde{f}(x) dx$$

$$= \int_{\{x\in \mathbf{R}_+^{i_0};||x||_\alpha^\delta\geqslant 1\}} ||x||_\alpha^{-i_0-\delta\varepsilon} \varpi_\delta(\widetilde{\sigma},x) dx$$

$$\leqslant K_1(\widetilde{\sigma}) \int_{\{x\in \mathbf{R}_+^{i_0};||x||_\alpha^\delta\geqslant 1\}} ||x||_\alpha^{-i_0-\delta\varepsilon} dx = \frac{K_1(\widetilde{\sigma})\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}.$$

If there exists a constant $K \geqslant K(\sigma)$, such that (36) is valid when replacing $K(\sigma)$ by K, then in particular, we have

$$K_1(\widetilde{\sigma})\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \geqslant \varepsilon\widetilde{I} > \varepsilon K||\widetilde{f}||_{p,\widetilde{\Phi}_\delta}||\widetilde{a}||_{q,\Psi}$$

$$= K\left\{\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} - \varepsilon O_1(1)\right\}^{\frac{1}{p}}\left\{\frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} + \varepsilon O(1)\right\}^{\frac{1}{q}},$$

and then $K(\sigma) \geqslant K$ ($\varepsilon \to 0^+$). Hence $K = K(\sigma)$ is the best possible constant factor of (36).

By the equivalency, we can prove that the constant factor $K(\sigma)$ in (37) ((38)) is the best possible. Otherwise, we would reach a contradiction by (39) ((40)) that the constant factor $K(\sigma)$ in (36) is not the best possible. $\square$

REMARK 3. Putting $\tau = e$ in (31) ($p < 0$), we have the following inequality:

$$\sum_n \int_{\mathbf{R}_+^{i_0}} e^{-\frac{\eta||\ln en||_\beta}{||x||_\alpha^\delta}}\cot h\left(\frac{\eta||\ln en||_\beta}{||x||_\alpha^\delta}\right)a_n f(x)dx > K(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\Psi}. \qquad (41)$$

Hence, (31) is an extension and a refinement of (41), which is a more accurate inequality of (41). In particular, for $\delta = -1$, we have the following inequality with the non-homogeneous kernel:

$$\sum_n \int_{\mathbf{R}_+^{i_0}} e^{-\eta||x||_\alpha||\ln en||_\beta}\cot h\left(\eta||x||_\alpha||\ln en||_\beta\right)a_n f(x)dx > K(\sigma)||f||_{p,\Phi_{-1}}||a||_{q,\Psi}. \qquad (42)$$

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