Mathematical
Inequalities
& Applications
Volume 18, Number 2 (2015), 607–614 doi:10.7153/mia-18-45

A REVERSE HILBERT–LIKE OPTIMAL INEQUALITY

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(*Communicated by J. Pečarić*)

Abstract. We prove an inequality on positive real numbers, that looks like a reverse to the well-known Hilbert inequality, and we use some unusual techniques from Fourier analysis to prove that this inequality is optimal.

1. Introduction and notation

To explain the title of this note recall that the discrete Hilbert inequality reads as follows:

$$\sum_{j,k=1}^{n} \frac{x_k y_j}{j+k} \leqslant \pi \left(\sum_{j,k=1}^{n} x_k^2 y_j^2 \right)^{1/2}$$

for every positive integer n and every real numbers $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, see [4, Ch. IX]. This inequality was intensively investigated, generalized, and extended. Furthermore, a multitude of similar inequalities, called "Hilbert-type inequalities" have been studied ([3], [8]). An excellent account of recent developments in Hilbert-type inequalities can be found in [5] and the references therein.

This research was initiated by a similar but reversed inequality that made the object of a proposed problem to the American Mathematical Monthly [2]. It was asked to prove that

$$\left(\sum_{j=1}^{n} \frac{a_j}{b_j} \right)^2 - 2 \left(\sum_{j,k=1}^{n} \frac{a_j a_k}{(b_j + b_j)^2} \right)^2 \leqslant 2 \left(\sum_{j,k=1}^{n} \frac{a_j a_k}{(b_j + b_k)} \sum_{l,m=1}^{n} \frac{a_l a_m}{(b_l + b_m)^3} \right)^{1/2}$$

for positive real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$. Our aim is not to prove or to discuss this inequality, but to notice that its form suggests the possibility of a typographic error in the denominator of the second term on the left, should it be $(b_j + b_k)^2$ instead of $(b_j + b_j)^2$?

In this note we show that the rectified version of this inequality does not hold, but rather another one with a larger constant on the right side, and we will show that this constant is the best possible. So, let us fix some notation and describe this work.

Mathematics subject classification (2010): 42A16, 42A38, 42B20.
Keywords and phrases: Inequalities, Fourier series, Fourier transform.

For a positive integer n and two vectors $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ of positive real numbers we consider the quantities

$$T_{\mathbf{a},\mathbf{b}} = \sum_{k=1}^{n} \frac{a_k}{b_k}, \tag{1}$$

and

$$S^{(m)}_{\mathbf{a},\mathbf{b}} = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{a_k a_l}{(b_k + b_l)^m} \quad \text{for } m = 1, 2, 3. \tag{2}$$

In Proposition 1 we prove that, for every positive integer n and every vectors $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ of positive real numbers, we have

$$\left(T_{\mathbf{a},\mathbf{b}}\right)^2 \leqslant 2S^{(2)}_{\mathbf{a},\mathbf{b}} + 2\sqrt{2}\sqrt{S^{(1)}_{\mathbf{a},\mathbf{b}} S^{(3)}_{\mathbf{a},\mathbf{b}}} \tag{3}$$

The difficulty does not reside in the proof of (3) but, in fact, it resides in showing that it is optimal in the sense that $2\sqrt{2}$ is the best possible constant. Precisely, we will prove in Theorem 1 that if for every positive integer n and every vectors $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ of positive real numbers, we have $\left(T_{\mathbf{a},\mathbf{b}}\right)^2 \leqslant 2S^{(2)}_{\mathbf{a},\mathbf{b}} + \lambda\sqrt{S^{(1)}_{\mathbf{a},\mathbf{b}} S^{(3)}_{\mathbf{a},\mathbf{b}}}$ then $\lambda \geqslant 2\sqrt{2}$.

This appears a difficult task, and requires tools from approximation theory and Fourier analysis. Indeed, we will prove in Proposition 2 that, for every $h > 0$ there exists two families of positive numbers $(a_j(h))_{j\in\mathbb{Z}}$ and $(b_j(h))_{j\in\mathbb{Z}}$ such that

$$\forall t \geqslant 0, \qquad \left| \frac{1}{(1+t)^2} - \sum_{j\in\mathbb{Z}} a_j(h) e^{-b_j(h)t} \right| \leqslant \frac{\delta(h)}{(1+t)^2}$$

with $\lim_{h\to 0^+} \delta(h) = 0$, and this will be exploited in proving the announced optimality result.

2. The main results

In the next proposition, we give a proof of (3).

PROPOSITION 1. *For every positive integer n and every vectors $\boldsymbol{a} = (a_1, \dots, a_n)$ and $\boldsymbol{b} = (b_1, \dots, b_n)$ of positive real numbers, we have*

$$\left(T_{\boldsymbol{a},\boldsymbol{b}}\right)^2 \leqslant 2S^{(2)}_{\boldsymbol{a},\boldsymbol{b}} + 2\sqrt{2}\sqrt{S^{(1)}_{\boldsymbol{a},\boldsymbol{b}} S^{(3)}_{\boldsymbol{a},\boldsymbol{b}}}.$$

Proof. Consider the function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, defined by

$$f(t) = \sum_{j=1}^{n} a_j e^{-b_j t}$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left(\int_0^\infty f(t)dt\right)^2 &= \left(\int_0^\infty \frac{1}{1+t}(1+t)f(t)dt\right)^2\\ &\leqslant \left(\int_0^\infty \frac{dt}{(1+t)^2}\right)\left(\int_0^\infty f^2(t)dt + 2\int_0^\infty tf^2(t)dt + \int_0^\infty t^2 f^2(t)dt\right)\\ &= \int_0^\infty f^2(t)dt + 2\int_0^\infty tf^2(t)dt + \int_0^\infty t^2 f^2(t)dt. \end{aligned} \tag{4}$$

Noting that $\int_0^\infty t^m e^{-bt}dt = \frac{m!}{b^{m+1}}$ for $m = 0,1,2$, we obtain

$$\int_0^\infty f(t)\,dt = T_{\mathbf{a},\mathbf{b}}, \qquad \int_0^\infty t^m f^2(t)\,dt = m!\,S^{(m)}_{\mathbf{a},\mathbf{b}},$$

and (4) becomes

$$(T_{\mathbf{a},\mathbf{b}})^2 \leqslant S^{(1)}_{\mathbf{a},\mathbf{b}} + 2S^{(2)}_{\mathbf{a},\mathbf{b}} + 2S^{(3)}_{\mathbf{a},\mathbf{b}}. \tag{5}$$

Applying (5) to $\lambda\,\mathbf{a} = (\lambda a_1,\dots,\lambda a_n)$ and $\lambda\,\mathbf{b} = (\lambda b_1,\dots,\lambda b_n)$ for some $\lambda > 0$, we obtain

$$(T_{\mathbf{a},\mathbf{b}})^2 \leqslant \lambda S^{(1)}_{\mathbf{a},\mathbf{b}} + 2S^{(2)}_{\mathbf{a},\mathbf{b}} + \frac{2}{\lambda}S^{(3)}_{\mathbf{a},\mathbf{b}}$$

and the desired inequality follows by choosing $\lambda = \sqrt{2S^{(3)}_{\mathbf{a},\mathbf{b}}/S^{(1)}_{\mathbf{a},\mathbf{b}}}$. $\square$

Analyzing the preceding proof, we see that in order to prove the optimality of (3) and to realize equality we need the function $t \mapsto f(t)$ to be proportional to $t \mapsto 1/(1+t)^2$, but this is impossible since the first has an exponential decay at $+\infty$. This remark holds the idea of what we will do next!. We will look for “almost” equality by approximating $t \mapsto 1/(1+t)^2$ by a linear combination of decreasing exponentials with positive coefficients. The next Proposition 2 provides us with the desired conclusion. Before we proceed, we will need the next two technical lemmas.

LEMMA 1. *The necessary and sufficient condition, on the positive parameter λ, for the following inequality to hold, for $x \in \mathbb{R}$,*

$$\frac{\pi x(1+x^2)}{\sinh(\pi x)} \leqslant \frac{1}{\cosh^2(\lambda x)}$$

is that $\lambda \leqslant \lambda_0 \overset{\text{def}}{=} \sqrt{\frac{\pi^2}{6} - 1} \approx 0.803078$.

Proof. Suppose that the proposed inequality is satisfied for some $\lambda > 0$ then we must have

$$\frac{(1+x^2)\cosh^2(\lambda x) - 1}{x^2} \leqslant \frac{1}{x^2}\left(\frac{\sinh(\pi x)}{\pi x} - 1\right)$$

for every nonzero x. Letting x tend to 0 we obtain $1+\lambda^2 \leqslant \pi^2/6$.

Conversely, let $\lambda_0=\sqrt{\frac{\pi^2}{6}-1}$, and consider the function

$$f(x)=\frac{\sinh(\pi x)}{\pi x}-(1+x^2)\cosh^2(\lambda_0 x)=\frac{\sinh(\pi x)}{\pi x}-\frac{1}{2}(1+x^2)(1+\cosh(2\lambda_0 x)).$$

The power series expansion of f is given by

$$f(x)=\sum_{n=2}^{\infty}(1-a_n)\frac{(\pi x)^{2n}}{(2n+1)!}$$

where

$$a_n=(2n+1)\left(\frac{2\lambda_0}{\pi}\right)^{2n-2}\left(\frac{1}{3}+\frac{2n^2-n-2}{\pi^2}\right)$$

Now,

$$\frac{a_{n+1}}{a_n}=\left(\frac{2}{3}-\frac{4}{\pi^2}\right)\left(1+\frac{2}{2n+1}\right)\left(1+\frac{12n+3}{6n^2-3n+\pi^2-6}\right)$$

From this, it is straightforward to see that the sequence $\left(\frac{a_{n+1}}{a_n}\right)_{n\geqslant 2}$ is decreasing, and that $\frac{a_3}{a_2}\approx 0.8177<1$. Thus, $a_n\leqslant a_2\approx 0.96531<1$ for every $n\geqslant 2$. This proves that $f(x)\geqslant 0$ for every real number x, and the proposed inequality follows for $\lambda\in[0,\lambda_0]$. $\square$

LEMMA 2. *For $t\geqslant 0$, let $f_t:\mathbb{R}\to\mathbb{R}$ be the function defined by*

$$f_t(x)=e^{2x-(1+t)e^x}.$$

Then the Fourier transform $\widehat{f_t}=\int_{\mathbb{R}}f_t(x)e^{ix(\cdot)}\,dx$ of f_t satisfies

$$\left|\widehat{f_t}(w)\right|=\frac{1}{(1+t)^2}\sqrt{\frac{\pi w(1+w^2)}{\sinh(\pi w)}}\leqslant\frac{1}{(1+t)^2}\cdot\frac{1}{\cosh(\lambda_0 w)},$$

where λ_0 was defined in Lemma 1.

Proof. Indeed we have

$$\begin{aligned}\widehat{f_t}(w)&=\int_{-\infty}^{\infty}f_t(x)e^{-iwx}\,dx\\&=\int_{-\infty}^{\infty}e^{(2-iw)x}e^{-(1+t)e^x}\,dx,\qquad\text{setting } s\leftarrow(1+t)e^x,\\&=\frac{1}{(1+t)^{2-iw}}\int_0^{\infty}s^{1-iw}e^{-s}\,ds=\frac{\Gamma(2-iw)}{(1+t)^{2-iw}},\end{aligned}$$

where Γ is the well-known Eulerian Gamma function [7].

Thus

$$\begin{aligned}\left|\widehat{f_t}(w)\right|^2 &= \frac{\Gamma(2-iw)}{(1+t)^{2-iw}}\cdot\frac{\Gamma(2+iw)}{(1+t)^{2+iw}}\\ &= \frac{1}{(1+t)^4}(1-iw)(1+iw)iw\Gamma(1-iw)\Gamma(iw)\\ &= \frac{1}{(1+t)^4}\cdot(1+w^2)\cdot\frac{i\pi w}{\sin(i\pi w)}\\ &= \frac{1}{(1+t)^4}\cdot(1+w^2)\cdot\frac{\pi w}{\sinh(\pi w)}.\end{aligned}$$

Here we used Euler's reflection formula for the Gamma function: $\Gamma(z)\Gamma(1-z)=\frac{\pi z}{\sin(\pi z)}$, (see [1, Chapter 6, formula 6.1.17]). Finally

$$\left|\widehat{f_t}(w)\right| = \frac{1}{(1+t)^2}\sqrt{\frac{\pi w(1+w^2)}{\sinh(\pi w)}}, \tag{6}$$

and the proposed inequality follows from Lemma 1. □

In the next proposition we prove the announced approximation result. In fact, the approach consists of approximating the function $t\mapsto 1/(1+t)^2$, expressed as an integral of a positive function of exponential type, using the trapezoidal quadrature rule, and then we use Poisson's formula to obtain good control on the absolute value of the error. For more details on this approach, we refer the reader to [6] and the references therein.

PROPOSITION 2. *For $h>0$ and $n\in\mathbb{Z}$, let*

$$a_n(h) = h\exp\left(2nh-e^{nh}\right), \qquad b_n(h)=e^{nh},$$

then

$$\forall t\geqslant 0,\qquad \left|\frac{1}{(1+t)^2}-\sum_{n\in\mathbb{Z}}a_n(h)e^{-b_n(h)t}\right|\leqslant\frac{\delta(h)}{(1+t)^2}$$

with

$$\delta(h)=\frac{4}{\exp\left(\frac{2\pi\lambda_0}{h}\right)-1},$$

where λ_0 was defined in Lemma 1.

Proof. Note that for $t\geqslant 0$ we have

$$\frac{1}{(1+t)^2}=\int_0^\infty ue^{-(1+t)u}\,du=\int_{-\infty}^\infty f_t(x)\,dx \tag{7}$$

where f_t is the positive function defined in Lemma 2.

The function f_t is super-exponentially decreasing for positive x and exponentially decreasing for negative x. A simple upper bound for f_t is obtained as follows, for $x \geqslant 0$ we have

$$2x-(1+t)e^x \leqslant 2x-e^x \leqslant 2e^{x-1}-e^x=(2-e)e^{x-1} \leqslant (2-e)x \tag{8}$$

since $x \leqslant e^{x-1}$ for every real x. And, for $x<0$, we have

$$2x-(1+t)e^x < 2x < (e-2)x \tag{9}$$

Combining (8) and (9) we see that $f_t(x) \leqslant e^{(2-e)|x|}$, for $x \in \mathbb{R}$.

This simple upper bound shows that the series $\sum_{n\in\mathbb{Z}} f_t(\cdot+nh)$ is uniformly convergent on every compact subset of $\mathbb{R}$. Therefore, we define an h-periodic *continuous* function F_t by the formula

$$F_t(x)=\sum_{n\in\mathbb{Z}} f_t(x+nh). \tag{10}$$

Moreover, the exponential Fourier coefficients $(C_m(F_t))_{m\in\mathbb{Z}}$ of F_t are given by

$$\begin{aligned} C_m(F_t) &= \frac{1}{h}\int_0^h F_t(x)e^{-2i\pi mx/h}dx \\ &= \frac{1}{h}\sum_{n\in\mathbb{Z}}\int_0^h f_t(x+nh)e^{-2i\pi mx/h}dx \\ &= \frac{1}{h}\sum_{n\in\mathbb{Z}}\int_{nh}^{(n+1)h} f_t(x)e^{-2i\pi mx/h}dx \\ &= \frac{1}{h}\int_{-\infty}^{\infty} f_t(x)e^{-2i\pi mx/h}dx = \frac{1}{h}\widehat{f_t}\left(\frac{2\pi m}{h}\right) \end{aligned} \tag{11}$$

where $\widehat{f_t}$ is the Fourier transform of f_t. In particular, according to Lemma 2, the Fourier series of F_t is normally convergent and consequently it is equal to F_t. Taking the value at $x=0$ we get

$$h\sum_{n\in\mathbb{Z}} f_t(nh) = \sum_{m\in\mathbb{Z}} \widehat{f_t}\left(\frac{2\pi m}{h}\right) \tag{12}$$

Using (7) and Lemma 2 we find that

$$\begin{aligned} \left|\frac{1}{(1+t)^2}-h\sum_{n\in\mathbb{Z}} f_t(nh)\right| &\leqslant 2\sum_{m=1}^{\infty}\left|\widehat{f_t}\left(\frac{2\pi m}{h}\right)\right| \\ &\leqslant \frac{2}{(1+t)^2}\sum_{m=1}^{\infty}\frac{1}{\cosh(2\pi\lambda_0 m/h)} \\ &\leqslant \frac{4}{(1+t)^2}\sum_{m=1}^{\infty}\exp\left(-\frac{2\pi\lambda_0 m}{h}\right) = \frac{\delta(h)}{(1+t)^2}, \end{aligned}$$

and the proposition follows. □

Now we are ready to prove the main result of this note.

THEOREM 1. *Consider a positive real constant* λ *such that, for every positive integer* n *and every vectors* $\boldsymbol{a} = (a_1, \dots, a_n)$ *and* $\boldsymbol{b} = (b_1, \dots, b_n)$ *of positive real numbers, we have*

$$\left(T_{\boldsymbol{a},\boldsymbol{b}}\right)^2 \leqslant 2S^{(2)}_{\boldsymbol{a},\boldsymbol{b}} + \lambda \sqrt{S^{(1)}_{\boldsymbol{a},\boldsymbol{b}} S^{(3)}_{\boldsymbol{a},\boldsymbol{b}}} \tag{13}$$

Then $\lambda \geqslant 2\sqrt{2}$.

Proof. Consider $h > 0$ and let the families $(a_n(h))_{n\in\mathbb{Z}}$ and $(b_n(h))_{n\in\mathbb{Z}}$ be defined as in Proposition 2. Accordingly we have

$$\frac{1-\delta(h)}{(1+t)^2} \leqslant \sum_{n\in\mathbb{Z}} a_n(h) e^{-b_n(h)t} \leqslant \frac{1+\delta(h)}{(1+t)^2}$$

we conclude that

$$1 - \delta(h) \leqslant \int_0^\infty \left(\sum_{n\in\mathbb{Z}} a_n(h) e^{-b_n(h)t} \right) dt = \sum_{n\in\mathbb{Z}} \frac{a_n(h)}{b_n(h)} \tag{14}$$

and, for $m = 0, 1, 2$,

$$\int_0^\infty t^m \left(\sum_{n\in\mathbb{Z}} a_n(h) e^{-b_n(h)t} \right)^2 dt \leqslant (1+\delta(h))^2 \int_0^\infty \frac{t^m}{(1+t)^4} dt \tag{15}$$

This yields

$$\sum_{(k,l)\in\mathbb{Z}^2} \frac{a_k(h)a_l(h)}{b_k(h)+b_l(h)} \leqslant \frac{(1+\delta(h))^2}{3} \tag{16}$$

$$\sum_{(k,l)\in\mathbb{Z}^2} \frac{a_k(h)a_l(h)}{(b_k(h)+b_l(h))^2} \leqslant \frac{(1+\delta(h))^2}{6} \tag{17}$$

$$\sum_{(k,l)\in\mathbb{Z}^2} \frac{a_k(h)a_l(h)}{(b_k(h)+b_l(h))^3} \leqslant \frac{(1+\delta(h))^2}{6} \tag{18}$$

Now, according to (14) there is a positive integer ν such that

$$1 - 2\delta(h) \leqslant \sum_{n=-\nu}^{\nu} \frac{a_n(h)}{b_n(h)} \tag{19}$$

Taking $n = 2\nu + 1$, $\mathbf{a} = (a_n(h))_{-\nu \leqslant n \leqslant \nu}$, and $\mathbf{b} = (b_n(h))_{-\nu \leqslant n \leqslant \nu}$, we obtain using (16)–(19):

$$1 - 2\delta(h) \leqslant T_{\mathbf{a},\mathbf{b}},$$

$$S^{(1)}_{\mathbf{a},\mathbf{b}} \leqslant \frac{(1+\delta(h))^2}{3}, \quad S^{(2)}_{\mathbf{a},\mathbf{b}} \leqslant \frac{(1+\delta(h))^2}{6}, \quad S^{(3)}_{\mathbf{a},\mathbf{b}} \leqslant \frac{(1+\delta(h))^2}{6}$$

and from (13) we conclude that

$$(1-2\delta(h))^2 \leqslant \frac{(1+\delta(h))^2}{3}\left(1+\frac{\lambda}{\sqrt{2}}\right).$$

Letting h tend to 0 and recalling that $\lim_{h\to 0}\delta(h)=0$ we obtain $\lambda \geqslant 2\sqrt{2}$. □

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(Received April 25, 2014)

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