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ON γ–QUASICONVEXITY, SUPERQUADRACITY AND TWO–SIDED REVERSED JENSEN TYPE INEQUALITIES

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Abstract. In this paper we deal with γ-quasiconvex functions when $-1 \leqslant \gamma \leqslant 0$, to derive some two-sided Jensen type inequalities. We also discuss some Jensen-Steffensen type inequalities for 1-quasiconvex functions. We compare Jensen type inequalities for 1-quasiconvex functions with Jensen type inequalities for superquadratic functions and we extend the result obtained for γ-quasiconvex functions to more general classes of functions.

1. Introduction

In [4], [5], [6], and [7] Jensen's type inequalities for γ-quasiconvex functions where $\gamma \geqslant 0$ and for γ-superquadratic functions are derived and discussed from which Hardy type inequalities were proved.

In this paper we first deal with γ-quasiconvex functions when $-1 \leqslant \gamma \leqslant 0$ (see Section 2). In Section 3 we derive some two-sided Jensen type inequalities. In Section 4 some new Jensen-Steffensen type inequalities for 1-quasiconvex functions are derived. In Section 5 we compare Jensen type inequalities for 1-quasiconvex functions with Jensen type inequalities for superquadratic functions and finally, we extend the result obtained for γ-quasiconvex functions to a more general class of functions.

We start with some definitions and results that are relevant for our discussions in the sequel.

A convex function φ on $[0,b)$, $0 < b \leqslant \infty$, is characterized by the following inequality:

$$\varphi(y) - \varphi(x) \geqslant C_\varphi(x)(y-x), \forall x,y \in (0,b]. \tag{1.1}$$

The function $K(x) = x^\gamma \varphi(x)$, where φ is convex, is called γ-quasiconvex.

In [6] we proved:

LEMMA 1. [6, Lemma 1] *Let* $K(x) = x^\gamma \varphi(x)$, $\gamma \in \mathbb{R}_+$, *where* φ *is convex on* $[0,b)$. *Then*

$$K(y) - K(x) \geqslant \varphi(x)(y^\gamma - x^\gamma) + C_\varphi(x)\, y^\gamma (y-x), \tag{1.2}$$

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holds for $x \in [0,b)$, $y \in [0,b)$, *where* $C_\varphi(x)$ *is defined by (1.1). Moreover, the Jensen type inequality*

$$\int_\Omega K(f(s))\,d\mu(s) - K\left(\int_\Omega f(s)\,d\mu(s)\right) \geqslant \int_\Omega \left[\varphi(x)\left((f(s))^\gamma - x^\gamma\right) + C_\varphi(x)(f(s))^\gamma (f(s)-x)\right] d\mu(s) \tag{1.3}$$

holds, where f *is any nonnegative function,* f *and* $K(f(s))$ *are* μ*-integrable functions on the probability measure space* (Ω,μ) *and* $x = \int_\Omega f(s)\,d\mu(s) > 0$.

If φ *is concave, then the reverse inequalities of (1.1), (1.2), and (1.3) hold, in particular*

$$\int_\Omega K(f(s))\,d\mu(s) - K\left(\int_\Omega f(s)\,d\mu(s)\right) \leqslant \int_\Omega \left[\varphi(x)\left((f(s))^\gamma - x^\gamma\right) + C_\varphi(x)(f(s))^\gamma (f(s)-x)\right] d\mu(s). \tag{1.4}$$

EXAMPLE 1. Inequalities (1.1), (1.2), and (1.3) are satisfied by $K(x) = x^p$, $p \geqslant \gamma+1$. For $\gamma < p \leqslant \gamma+1$ the reverse inequalities hold. They reduce to equalities for $p = \gamma+1$ e.g., it yields when $x = \int_\Omega f(s)\,d\mu(s) > 0$ that

$$\int_\Omega (f(s))^{1+\gamma} d\mu(s) - \left(\int_\Omega f(\sigma)\,d\mu(\sigma)\right)^{1+\gamma} = x\int_\Omega \left((f(s))^\gamma - x^\gamma\right) d\mu(s) + \int_\Omega (f(s))^\gamma (f(s)-x)\,d\mu(s). \tag{1.5}$$

THEOREM 1. [6, Theorem 1] *Let* $\gamma \in \mathbb{R}_+$, *and* f *be nonnegative function. Let* f *and* $\varphi \circ f$ *be* μ*-integrable functions on the probability measure space* (Ω,μ) *and* $x = \int_\Omega f(s)\,d\mu(s) > 0$. *If* φ *is a differentiable, nonnegative, convex, increasing function on* $[0,b)$, $0 < b \leqslant \infty$ *and* $\varphi(0) = \lim_{z\to 0^+} z\varphi'(z) = 0$, *then the Jensen type inequalities*

$$\int_\Omega \varphi(f(s))(f(s))^\gamma\, d\mu(s) - \varphi(x)x^\gamma \geqslant \varphi(x)\int_\Omega \left((f(s))^\gamma - x^\gamma\right) d\mu(s) + \varphi'(x)\int_\Omega (f(s))^\gamma (f(s)-x)\,d\mu(s) \geqslant 0 \tag{1.6}$$

hold.

REMARK 1. Note that when $\gamma = 0$ the second inequality in (1.6) reduces to equality, so that then (1.6) coincides with the usual Jensen's inequality.

EXAMPLE 2. By applying (1.3) with $\mu(s) = \sum_{i=1}^N a_i\delta_i$ with $\sum_{i=1}^N a_i = 1$ and δ_i unit masses at $x = x_i$, $y_i = f(x_i)$, $i = 1,\ldots,N$, $N \in \mathbb{Z}_+$, we obtain that the following special case of (1.3) yields the inequality

$$\sum_{i=1}^{N}\alpha_i K(y_i)-K\left(\sum_{i=1}^{N}\alpha_i y_i\right) \tag{1.7}$$
$$\geqslant \varphi\left(\sum_{j=1}^{N}\alpha_j y_j\right)\left(\sum_{i=1}^{N}\alpha_i y_i^{\gamma}-\left(\sum_{j=1}^{N}\alpha_j y_j\right)^{\gamma}\right)+C_{\varphi}\left(\sum_{j=1}^{N}\alpha_j y_j\right)\sum_{i=1}^{N}\alpha_i y_i^{\gamma}\left(y_i-\sum_{j=1}^{N}\alpha_j y_j\right),$$

which holds for $x_i\in[0,b)$, $y_i\in[0,b)$, $0\leqslant\alpha_i\leqslant 1$, $i=1,\dots,n$, and $\sum_{i=1}^{N}\alpha_i=1$. Moreover, under the conditions on φ in Theorem 1, as φ is differentiable so that $C_{\varphi}=\varphi'$, then the right handside of (1.7) is nonnegative and therefore we get that (1.7) is a genuine scale of refined discrete Jensen type inequalities.

DEFINITION 1. [3] A function $\varphi:[0,b)\to\mathbb{R}$ is called superquadratic provided that for all $0\leqslant x<b$ there exists a constant $C_{\varphi}(x)\in\mathbb{R}$ such that

$$\varphi(y)-\varphi(x)\geqslant C_{\varphi}(x)(y-x)+\varphi(|y-x|) \tag{1.8}$$

for every y, $0\leqslant y<b$. The function φ is called subquadratic if $-\varphi$ is superquadratic.

From the definition of superquadracity we easily get:

LEMMA A. [3] *The function φ is superquadratic on $[0,b)$ if and only if*

$$\frac{1}{A_n}\sum_{i=1}^{n}a_i\varphi(x_i)-\varphi(\overline{x})\geqslant\frac{1}{A_n}\sum_{i=1}^{n}a_i\varphi(|x_i-\overline{x}|) \tag{1.9}$$

holds for all $x_i\in[0,b)$, $i=1,\dots,n$, and $a_i\geqslant 0$, $i=1,\dots,n$, are such that $A_n=\sum_{i=1}^{n}a_i>0$ and $\overline{x}=\frac{1}{A_n}\sum_{i=1}^{n}a_ix_i$.

The function φ is superquadratic on $[0,b)$ if and only if

$$\int_{\Omega}\varphi(f(s))\,d\mu(s)-\varphi\left(\int_{\Omega}f(s)\,d\mu(s)\right)\geqslant\int_{\Omega}\varphi\left(\left|f(s)-\int_{\Omega}f(\sigma)\,d\mu(\sigma)\right|\right)d\mu(s), \tag{1.10}$$

where f is any non-negative μ-integrable function on a probability measure space (Ω,μ) and $\int_{\Omega}f(s)\,d\mu(s)>0$.

LEMMA B. [3] *Let φ be a superquadratic function with $C_{\varphi}(x)$ as in Definition 1. Then:*

(i) $\varphi(0)\leqslant 0$,

(ii) *if $\varphi(0)=\varphi'(0)=0$, then $C_{\varphi}(x)=\varphi'(x)$ whenever φ is differentiable at $0<x<b$.*

(iii) *if $\varphi\geqslant 0$, then φ is convex and $\varphi(0)=\varphi'(0)=0$.*

LEMMA C. [3] *Suppose that* $\varphi : [0,b) \to \mathbb{R}$ *is continuously differentiable and* $\varphi(0) \leqslant 0$. *If* $\varphi\prime$ *is superadditive or* $\frac{\varphi\prime(x)}{x}$ *is nondecreasing, then* φ *is superquadratic.*

The power functions $\varphi(x) = x^p$, $x \geqslant 0$ are superquadratic when $p \geqslant 2$ and subquadratic when $1 \leqslant p \leqslant 2$. When $\varphi(x) = x^2$ (1.8) reduces to equality and therefore the same holds for (1.9) and (1.10).

2. γ-quasiconvex functions, $-1 \leqslant \gamma \leqslant 0$

The definition of γ- quasiconvex function $K(x)$, $K(x) = x^\gamma \varphi(x)$ can be meaningful even if $\gamma < 0$. We can for example state the following complement of Theorem 1:

THEOREM 2. *Let* $-1 \leqslant \gamma \leqslant 0$, *and let* f *be a* μ*-integrable and non-negative function on the probability measure space* (Ω, μ) *and* $x = \int_\Omega f(s)\, d\mu(s) > 0$. *If* φ *is a differentiable, non-negative, convex increasing function and* $\varphi(0) = \lim_{z\to 0^+} z\varphi'(z) = 0$, *then*

$$\int_\Omega \varphi(f(s))(f(s))^\gamma d\mu(s) - \varphi(x)x^\gamma \tag{2.1}$$
$$\geqslant \varphi(x)\int_\Omega \left((f(s))^\gamma - x^\gamma\right) d\mu(s) + \varphi'(x)\int_\Omega (f(s))^\gamma (f(s)-x)\, d\mu(s)$$

holds and the right hand-side expression of (2.1) is non-positive.

Proof. We start with the proof of (2.1): First we note that inequality (1.2) holds for every γ. Therefore when φ is differentiable ($C_\varphi(x) = \varphi'(x)$) then (1.3) holds too when the involved integrals exist. Hence, inequality (2.1) holds. Now we have to show that when $-1 \leqslant \gamma \leqslant 0$ the right hand-side of (2.1) is non-positive.

For the case $-1 < \gamma \leqslant 0$ the fact that the right hand-side expression in (2.1) is non-positive follows similarly as in the proof of Theorem 1 in [6] because now

$$\int_\Omega (f(s))^\gamma f(s)\, d\mu(s) - \int_\Omega f(s)\, d\mu(s)\int_\Omega (f(s))^\gamma d\mu(s) \leqslant 0$$

holds by Chebychev's inequality and also

$$\int_\Omega (f(s))^{1+\gamma} d\mu(s) - \left(\int_\Omega f(\sigma)\, d\mu(\sigma)\right)^{1+\gamma} = \int_\Omega (f(s))^{1+\gamma} d\mu(s) - x^{1+\gamma}$$
$$= x\int_\Omega \left((f(s))^\gamma - x^\gamma\right) d\mu(s) + \int_\Omega (f(s))^\gamma (f(s)-x)\, d\mu(s) \leqslant 0.$$

holds for $-1 < \gamma \leqslant 0$.

For the case $\gamma = -1$ we must show that

$$\varphi(x)\int_\Omega \left(\frac{1}{f(s)} - \frac{1}{x}\right) d\mu(s) + \varphi'(x)\int_\Omega \frac{1}{f(s)}(f(s)-x)\, d\mu(s) \leqslant 0, \tag{2.2}$$

in other words we must prove that

$$I_0 = \int_\Omega \left[\varphi\left(\int_\Omega f(s)\,d\mu(s)\right)\left(\frac{1}{f(s)} - \frac{1}{\int_\Omega f(s)\,d\mu(s)}\right) + \varphi'\left(\int_\Omega f(s)\,d\mu(s)\right)\left(1 - \frac{\int_\Omega f(s)\,d\mu(s)}{f(s)}\right)\right] d\mu(s) \leqslant 0. \tag{2.3}$$

By using Jensen's inequality for the convex function $\phi(x) = x^{-1},\ x > 0,$ we find that

$$\int_\Omega f(s)\,d\mu(s) \int_\Omega \frac{1}{f(s)} d\mu(s) = \int_\Omega (f(s))\,d\mu(s) \int_\Omega (f(s))^{-1}\,d\mu(s) \geqslant \int_\Omega f(s)\,d\mu(s)\left(\int_\Omega f(s)\,d\mu(s)\right)^{-1} = 1 \tag{2.4}$$

(this inequality follows also from the Harmonic-Arithmetic mean inequality).

Moreover, since φ is convex and $\varphi(0) = 0 = \lim_{z\to 0} z\varphi'(z)$ it follows that

$$x\varphi'(x) - \varphi(x) \geqslant 0. \tag{2.5}$$

By using (2.4) and (2.5) we have that

$$\begin{aligned} I_0 &= \int_\Omega \left[\varphi\left(\int_\Omega f(s)\,d\mu(s)\right)\left(\frac{1}{f(s)} - \frac{1}{\int_\Omega f(s)\,d\mu(s)}\right) + \varphi'\left(\int_\Omega f(s)\,d\mu(s)\right)\left(1 - \frac{\int f(s)\,d\mu(s)}{f(s)}\right)\right] d\mu(s) \\ &= \int_\Omega \left[\frac{1}{\int_\Omega f(s)\,d\mu(s)} \varphi(f(s)\,d\mu(s)) + \varphi'\left(\int_\Omega f(s)\,d\mu(s)\right)\right] \cdot \\ &\quad \cdot \left(1 - \frac{\int_\Omega f(s)\,d\mu(s)}{f(s)}\right) d\mu(s) \\ &= \left[-\frac{\varphi(x)}{x} + \varphi'(x)\right]\left(1 - \int_\Omega f(s)\,d\mu(s) \int_\Omega \frac{1}{f(s)} d\mu(s)\right) \leqslant 0. \end{aligned}$$

Hence (2.3), and thus also (2.2) is proved. The proof is complete □

REMARK 2. From the case $\gamma = -1$ it follows that when φ is convex and $\varphi(0) = 0 = \lim_{z\to 0^+}(z\varphi'(z))$ and $\frac{\varphi(x)}{x}$ is concave we get a negative lower bound to our Jensen's type difference. This important fact is further developed in the next section.

3. Some two-sided reversed Jensen type inequalities

First we state the following consequence of Theorem 2:

PROPOSITION 1. *Let the conditions in Theorem 2 be satisfied and assume in addition that* $\frac{\varphi(x)}{x}$ *is concave. then the following two sided Jensen type inequality holds:*

$$\varphi(x)\int_\Omega\left((f(s))^{-1}-x^{-1}\right)d\mu(s)+\varphi'(x)\int_\Omega(f(s))^{-1}(f(s)-x)\,d\mu(s) \tag{3.1}$$
$$\leqslant\int_\Omega\frac{\varphi(f(s))}{f(s)}d\mu(s)-\frac{\varphi(x)}{x}\leqslant 0.$$

Proof. The left handside inequality is just Theorem 2 applied for $\gamma=-1$ and the right hand side inequality is just the reversed Jensen's inequality. □

The following consequence of Proposition 1 is useful for applications:

COROLLARY 1. *Let* $0<p\leqslant 1$, *and let* f *be a* μ*-measurable and positive function on the probability measure space* (μ,Ω) *and* $x=\int_\Omega f(s)\,d\mu(s)>0$. *Then*

$$-I_1+\left(\int_\Omega f(s)\,d\mu(s)\right)^p\leqslant\int_\Omega(f(s))^p\,d\mu(s)\leqslant\left(\int_\Omega f(s)\,d\mu(s)\right)^p,$$

where

$$I_1=p\left(\int_\Omega f(s)\,d\mu(s)\right)^p\left(1-\int_\Omega f(s)\,d\mu(s)\int_\Omega(f(s))^{-1}\,d\mu(s)\right)>0.$$

Proof. Apply (3.1) with $\varphi(u)=u^{1+p}$, $0<p\leqslant 1$. The right hand side inequality follows directly. The left hand side expression is equal to

$$\left(\int_\Omega f(s)\,d\mu(s)\right)^{p+1}\int_\Omega\left(\frac{1}{f(s)}-\frac{1}{\int_\Omega(f(s))\,d\mu(s)}\right)d\mu(s)+$$
$$(p+1)\left(\int_\Omega f(s)\,d\mu(s)\right)^p\int_\Omega\left(1-\frac{1}{f(s)}\int_\Omega(f(s))\,d\mu(s)\right)d\mu(s)$$
$$=p\left(\int_\Omega f(s)\,d\mu(s)\right)^{p+1}\int_\Omega\left(\frac{1}{\int_\Omega(f(s))\,d\mu(s)}-\frac{1}{f(s)}\right)d\mu(s)$$
$$=p\left(\int_\Omega f(s)\,d\mu(s)\right)^{p+1}\left(\left(\int_\Omega f(s)\,d\mu(s)\right)^{-1}-\int_\Omega(f(s))^{-1}\,d\mu(s)\right)=I_1$$

The proof is complete. □

We shall now derive some more useful inequalities as those in Proposition 1 and Corollary 1 and we need the following Lemma which can be proved by obvious calculations.

LEMMA 2. *Let $\varphi:\mathbb{R}_+\to\mathbb{R}$ be differentiable function $x,y\in\mathbb{R}_+$ and $\gamma\in\mathbb{R}$. Then*

$$\begin{aligned}\varphi(x)\left(y^{\gamma}-x^{\gamma}\right)+\varphi'(x)y^{\gamma}(y-x)&\\ -\left[(x\varphi(x))\left(y^{\gamma-1}-x^{\gamma-1}\right)+(x\varphi(x))'y^{\gamma-1}(y-x)\right]&\\ =y^{\gamma-1}\varphi'(x)(y-x)^2.&\end{aligned}\tag{3.2}$$

Our next two sided reversed Jensen type inequality reads:

PROPOSITION 2. *Let f be a non-negative μ-measurable function on the probability measure space (μ,Ω) and $x=\int_{\Omega}f(s)\,d\mu(s)>0$. Assume that φ is a differentiable non-negative, convex function, $\varphi(0)=\lim_{z\to0^+}z\varphi'(z)=0$. Moreover, assume that $\frac{\varphi(x)}{x}$ is concave. Then the following two-sided Jensen type inequality holds:*

$$-\left(\frac{\varphi(x)}{x}\right)'\int_{\Omega}\frac{(f(s)-x)^2}{g(s)}d\mu(s)\leqslant\int_{\Omega}\frac{\varphi(f(s))}{f(s)}d\mu(s)-\frac{\varphi(x)}{x}\leqslant0.\tag{3.3}$$

Proof. The assumption that $\frac{\varphi(x)}{x}$ is concave implies that

$$\frac{\varphi(y)}{y}-\frac{\varphi(x)}{x}\leqslant\left(\frac{\varphi(x)}{x}\right)'(y-x).\tag{3.4}$$

From (1.3) for $\gamma=-1$ we get that

$$\frac{\varphi(y)}{y}-\frac{\varphi(x)}{x}\geqslant\varphi(x)\left(\frac{1}{y}-\frac{1}{x}\right)+\frac{\varphi'(x)}{y}(y-x)\tag{3.5}$$

Moreover, by using (3.2) with $\gamma=0$ and $\varphi(x)$ replaced by $\frac{\varphi(x)}{x}$ we find that

$$\left(\frac{\varphi(x)}{x}\right)'(y-x)=\left[\varphi(x)\left(y^{-1}-x^{-1}\right)+\varphi'(x)y^{-1}(y-x)\right]+y^{-1}\left(\frac{\varphi(x)}{x}\right)'(y-x)^2.\tag{3.6}$$

By combining (3.4), (3.5) and (3.6) we see that

$$\left(\frac{\varphi(x)}{x}\right)'(y-x)-\left(\frac{\varphi(x)}{x}\right)'\frac{1}{y}(y-x)^2\leqslant\frac{\varphi(y)}{y}-\frac{\varphi(x)}{x}\leqslant\left(\frac{\varphi(x)}{x}\right)'(y-x).\tag{3.7}$$

Next we put $y=f(s)$ and $x=\int_{\Omega}f(s)\,d\mu(s)$ and integrating (3.7) over Ω with respect to μ we obtain that (3.3) holds. Note especially that

$$\int_{\Omega}(y-x)\,d\mu(s)=\int_{\Omega}\left(f(s)-\int_{\Omega}f(s)\,d\mu(s)\right)d\mu(s)=0.$$

The proof is complete. □

By applying Proposition 2 with $\varphi(x)=x^{1+p}$, $0<p\leqslant1$ we get the following result:

COROLLARY 2. *Let* $0 < p \leqslant 1$, *let* f *be a non-negative* μ*-measurable function on the probability measure space* (Ω, μ) *and* $x = \int_\Omega f(s)\,d\mu(s) > 0$. *Then*

$$-I_2 + \left(\int_\Omega f(s)\,d\mu(s)\right)^p \leqslant \int_\Omega (f(s))^p\,d\mu(s) \leqslant \left(\int_\Omega f(s)\,d\mu(s)\right)^p,$$

where

$$I_2 = p\left(\int_\Omega f(s)\,d\mu(s)\right)^{p-1} \int_\Omega \frac{(f(s)-x)^2}{f(s)}\,d\mu(s).$$

Next we will prove the following may be even more surprising variant of Proposition 1:

PROPOSITION 3. *Let* f *be a non-negative* μ*-measurable function on the probability measure space* (Ω, μ) *and* $x = \int_\Omega f(s)\,d\mu(s) > 0$. *Assume that* φ *is a differentiable non-negative function such that* $\lim_{z\to 0_+} z\varphi'(z) = 0$, $\frac{\varphi(x)}{x^2}$ *is convex and* $\frac{\varphi(x)}{x}$ *is concave. Then the following two-sided Jensen type inequality holds:*

$$\left(\frac{\varphi(x)}{x^2}\right)' \int_\Omega (f(s)-x)^2\,d\mu(s) \leqslant \int_\Omega \frac{\varphi(f(s))}{f(s)}\,d\mu(s) - \frac{\varphi(x)}{x} \leqslant 0. \tag{3.8}$$

Proof. By discussing exactly as in the proof of Proposition 2 we find that under the assumptions of Proposition 3 (instead of (3.7)) the following inequality holds:

$$\frac{\varphi(x)}{x^2}(y-x) + \left(\frac{\varphi(x)}{x^2}\right)' y(y-x) \leqslant \frac{\varphi(y)}{y} - \frac{\varphi(x)}{x} \leqslant \left(\frac{\varphi(x)}{x}\right)'(y-x). \tag{3.9}$$

As before we put $y = f(s)$ and $x = \int_\Omega f(s)\,d\mu(s)$ and integrating (3.9) over Ω with respect to μ we find that (3.8) holds.

The proof is complete. □

By applying Proposition 3 with $\varphi(x) = x^{1+p}$, $0 < p \leqslant 1$, we obtain the following Corollary 3:

COROLLARY 3. *Let* $0 < p \leqslant 1$, *let* f *be a* μ*-measurable function on the probability measure space* (Ω, μ) *and* $x = \int_\Omega f(s)\,d\mu(s)$. *Then*

$$-I_3 + \left(\int_\Omega f(s)\,d\mu(s)\right)^p \leqslant \int_\Omega (f(s))^p\,d\mu(s) \leqslant \left(\int_\Omega f(s)\,d\mu(s)\right)^p,$$

where

$$I_3 = (1-p)\left(\int_\Omega f(s)\,d\mu(s)\right)^{p-2} \int_\Omega (f(s)-x)^2\,d\mu(s).$$

4. A Jensen-Steffensen type inequality for 1-quasiconvex functions

Let $\rho_1,\dots,\rho_n$ be Jensen-Steffensen coefficients, that is, $0 \leqslant P_j = \sum_{i=1}^{j} \rho_i \leqslant P_n$, $P_n > 0$, $\overline{P}_j = \sum_{i=j}^{n} \rho_i \geqslant 0$, $j = 1,\dots,n$, and $\mathbf{x} = (x_1,\dots,x_n) > 0$ satisfies $0 < x_1 \leqslant x_2 \leqslant \dots \leqslant x_n$. Let φ be an increasing convex function for $x \geqslant 0$, $\varphi(0) = 0$ and $\psi(x) = x\varphi(x)$. Let $\overline{x} = \frac{\sum_{i=1}^{n} \rho_i x_i}{P_n}$. We recall the known identity (see [2] and [8])

$$\sum_{i=1}^{n} \rho_i \psi(x_i) - P_n \psi(\overline{x}) \tag{4.1}$$
$$= \sum_{j=1}^{k-1} P_j \left(\psi(x_j) - \psi\left(x_{j+1}\right)\right) + P_k\left(\psi(x_k) - \psi(\overline{x})\right) + \overline{P}_{k+1}\left(\psi(x_{k+1}) - \psi(\overline{x})\right)$$
$$+ \sum_{j=k+2}^{n} \overline{P_j}\left(\psi(x_j) - \psi\left(x_{j-1}\right)\right).$$

Since φ is convex we get that

$$\psi(x) - \psi(y) \geqslant \psi'(x)(y-x) + \varphi'(x)(y-x)^2$$

and therefore, by using (4.1), we find that

$$\sum_{i=1}^{n} \rho_i \psi(x_i) - P_n \psi(\overline{x})$$
$$\geqslant \left[\sum_{j=1}^{k-1} P_j \psi'\left(x_{j+1}\right)\left(x_j - x_{j+1}\right) + P_k \psi'(\overline{x})(x_k - \overline{x})\right.$$
$$\left. + \overline{P}_{k+1} \psi'(\overline{x})(x_{k+1} - \overline{x}) + \sum_{j=k+2}^{n} \overline{P}_j \psi'\left(x_{j-1}\right)\left(x_j - x_{j-1}\right)\right]$$
$$+ \left[\sum_{j=1}^{k-1} P_j \varphi'\left(x_{j+1}\right)\left(x_j - x_{j+1}\right)^2 + P_k \varphi'(\overline{x})(x_k - \overline{x})^2\right.$$
$$\left. + \overline{P}_{k+1} \varphi'(\overline{x})(x_{k+1} - \overline{x})^2 + \sum_{j=k+2}^{n} \overline{P}_j \varphi'(x_{k-1})(x_k - x_{k-1})^2\right].$$

The first parenthesis is greater than zero because $\psi(x)$ is convex and since there is an integer k, $1 \leqslant k \leqslant n$ such that $x_k \leqslant \overline{x} \leqslant x_{k+1}$ as it was proved in [1, (2.8), (2.9) and (2.10)], see also [2], and [8]. Hence

$$\sum_{i=1}^{n} \rho_i \psi(x_i) - P_n \psi(\overline{x})$$
$$\geqslant [0] + \sum_{j=1}^{k-1} P_j \varphi'\left(x_{j+1}\right)\left(x_{j+1} - x_j\right)^2 + P_k \varphi'(\overline{x})(\overline{x} - x_k)^2$$
$$+ \overline{P}_{k+1} \varphi'(\overline{x})(x_{k+1} - \overline{x})^2 + \sum_{j=k+2}^{n} \overline{P}_j \varphi'\left(x_{j-1}\right)\left(x_j - x_{j-1}\right)^2.$$

Since φ is convex and $x_1 \leqslant \ldots \leqslant x_k \leqslant \overline{x} \leqslant x_{k+1} \leqslant \ldots \leqslant x_n$ and $P_j \geqslant 0,\ \overline{P}_j > 0,\ j = 1,\ldots,n,$ we get that

$$\begin{aligned}
&\sum_{j=1}^{k-1} P_j \varphi'\left(x_{j+1}\right)\left(x_{j+1}-x_j\right)^2 + P_k \varphi'\left(\overline{x}\right)\left(\overline{x}-x_k\right)^2 \\
&+\overline{P}_{k+1}\varphi'\left(x_{k+1}-\overline{x}\right)^2 - \sum_{j=k+2}^{n} \overline{P}_j \varphi'\left(x_{j-1}\right)\left(x_j - x_{j-1}\right)^2 \\
\geqslant\ & \varphi'\left(x_1\right)\left(\sum_{j=1}^{k} P_j + \sum_{j=k+1}^{n} \overline{P}_j\right) \cdot \\
&\cdot\left(\frac{\sum_{j=1}^{k-1} P_j\left(x_{j+1}-x_j\right) + P_k\left(\overline{x}-x_k\right) + \overline{P}_{k+1}\left(x_{k+1}-\overline{x}\right) + \sum_{j=k+2}^{n} \overline{P}_j\left(x_j - x_{j-1}\right)}{\sum_{j=1}^{k} P_j + \sum_{j=k+1}^{n} \overline{P}_j}\right)^2 \\
=\ & \varphi'\left(x_1\right)\left(\sum_{j=1}^{k} P_j + \sum_{j=k+1}^{n} \overline{P}_j\right)\left(\frac{\sum_{i=1}^{n} \rho_i \left|x_i - \overline{x}\right|}{\sum_{j=1}^{k} P_j + \sum_{j=k+1}^{n} \overline{P}_j}\right)^2.
\end{aligned}$$

The last inequality here follows from (4.1) for the function $\psi(x) = |x|$.

Since φ is convex increasing and $\varphi(0) = 0$, it yields that $\frac{\varphi(x)}{x}$ is increasing and $x\varphi\left(\frac{1}{x}\right)$ is decreasing on $(0,\infty)$. Therefore from the estimates

$$\sum_{j=1}^{n} P_j + \sum_{j=k+1}^{n} \overline{P}_j \leqslant \max\{n, n-k\} P_n \leqslant (n-1) P_n$$

it follows that

$$\begin{aligned}
&\left(\sum_{j=1}^{k} P_j + \sum_{j=k+1}^{n} \overline{P}_j\right)\left(\frac{\sum_{i=1}^{n} \rho_i \left|x_i - \overline{x}\right|}{\sum_{j=1}^{k} P_j + \sum_{j=k+1}^{n} \overline{P}_j}\right)^2 \\
\geqslant\ & P_n \max\{k, n-k\}\left(\frac{\sum_{i=1}^{n} \rho_i \left|x_i - \overline{x}\right|}{\max\{n, n+1\} P_n}\right)^2 \geqslant (n-1) P_n \left(\frac{\sum_{j=1}^{n} \rho_j \left|x_j - \overline{x}\right|}{(n-1) P_n}\right)^2.
\end{aligned}$$

By now combining all estimates above we obtain a series of estimates for N-quasiconvex functions. For simplicity and for later purposes we only formulate this result for the special case $N = 1$.

THEOREM 3. *Let $\rho_1,\ldots,\rho_n$ be Jensen-Steffensen coefficients, and suppose that $\mathbf{x} = (x_1,\ldots,x_n) > 0$ satisfies $0 < x_1 \leqslant \ldots \leqslant x_n$. Let φ be an increasing convex function for $x \geqslant 0,\ \varphi(0) = 0$ and let $\psi(x) = x\varphi(x)$. Let $\overline{x} = \sum_{i=1}^{n} \frac{\rho_i x_i}{P_n},\ 0 \leqslant P_k = \sum_{i=1}^{k} \rho_i \leqslant P_n,$ $\overline{P}_k = \sum_{i=k}^{n} \rho_i \geqslant 0,\ P_n > 0.$ Then*

$$\begin{aligned}
&\sum_{i=1}^{n} \rho_i \psi\left(x_i\right) - P_n \psi\left(\overline{x}\right) \\
&\geqslant \varphi'(x)\left(\sum_{j=1}^{k} P_j + \sum_{j=k+1} \overline{P}_j\right)\left(\frac{\sum_{i=1}^{n} \rho_i \left|x_i - \overline{x}\right|}{\sum_{j=1}^{k} P_i + \sum_{j=k+1}^{n} \overline{P}_j}\right)^2
\end{aligned}$$

$$\geqslant \varphi'(x) P_n \max\{k, n-k\} \left(\frac{\sum_{i=1}^n \rho_i |x_i - \overline{x}|^2}{P_n \max\{k, n-k\}} \right)$$

$$\geqslant \varphi'(x)(n-1) P_n \left(\frac{\sum_{i=1}^n \rho_i |x_i - \overline{x}|}{(n-1) P_n} \right)^2 .$$

5. Further results and concluding remarks

First we will compare Jensen's inequality for the convex function ψ, where $\psi(x) = x\varphi(x)$, φ is twice differentiable nonnegative increasing convex function on $x \geqslant 0$ satisfying $\varphi(0) = 0$ and the superquadratic function ψ. The function ψ is superquadratic by Lemma B because $\psi(0) = 0$ and $\left(\frac{\varphi(x)}{x}\right)' \geqslant 0$, $\varphi''(x) \geqslant 0$, so that

$$\left(\frac{\psi'(x)}{x} \right)' = \left(\frac{(x\varphi(x))'}{x} \right)' = \left(\frac{\varphi(x)}{x} \right)' + \varphi''(x) > 0.$$

Denote $\overline{x} = \sum \alpha_i x_i$, $\sum \alpha_i = 1$, $0 \leqslant \alpha_i \leqslant 1$, $x_i \geqslant 0$, $i = 1, \ldots, n$. As $\psi(x) = x\varphi(x)$, then

$$\sum \alpha_i \psi(x_i) - \psi(\overline{x}) \geqslant \sum_{i=1}^n \alpha_i \varphi'(\overline{x})(x_i - \overline{x})^2 \tag{5.1}$$

because φ is differentiable and convex and

$$\sum \alpha_i \psi(x_i) - \psi(\overline{x}) \geqslant \sum_{i=1}^n \alpha_i \psi|x_i - \overline{x}| \tag{5.2}$$

holds since ψ is superquadratic.

When $|x_i - \overline{x}| \leqslant \overline{x} \Longleftrightarrow 0 \leqslant x_i \leqslant 2\overline{x}$, $i = 1, \ldots, n$, (which is satisfied for instance when $0 < a \leqslant x_i \leqslant 2a$, $i = 1, \ldots, n$, and for $n = 2$, $\alpha_1 = \alpha_2 = \frac{1}{2}$, $x_1, x_2 > 0$) we get that (because $\frac{\varphi(x)}{x}$ is increasing for $x > 0$)

$$\frac{\varphi(|x_i - \overline{x}|)}{(|x_i - \overline{x}|)} = \frac{\psi(|x_i - \overline{x}|)}{(x_i - \overline{x})^2} < \frac{\varphi(\overline{x})}{\overline{x}}, \qquad i = 1, \ldots, n. \tag{5.3}$$

Moreover, since $\frac{\varphi(x)}{x} < \varphi'(x)$ when φ is differentiable nonnegative convex increasing and $\varphi(0) = 0$ and $x_i \leqslant 2\overline{x}$, $i = 1, \ldots, n$, we get that $\overline{x}\varphi'(\overline{x}) - \varphi(\overline{x}) \geqslant 0$ so that, by (5.3),

$$\frac{\psi(|x_i - \overline{x}|)}{(x_i - \overline{x})^2} < \frac{\varphi(\overline{x})}{\overline{x}} = \frac{\psi(\overline{x})}{\overline{x}^2} \leqslant \varphi'(\overline{x}).$$

Hence by comparing (5.1) and (5.2) we get that (5.1) is sharper than (5.2), that is, we get:

PROPOSITION 4. *Let $\psi(x) = x\varphi(x)$, where φ is non-negative convex increasing and twice differentiable function on $[0, b)$, and $\varphi(0) = 0$. Then the inequalities*

$$\sum_{i=1}^n \alpha_i \psi(x_i) - \psi(\overline{x}) \geqslant \sum_{i=1}^n \alpha_i \varphi'(\overline{x})(x_i - \overline{x})^2 \geqslant \sum_{i=1}^n \alpha_i \psi(|x_i - \overline{x}|)$$

hold for

$$x_i \leqslant 2\overline{x}, \quad \overline{x} = \sum \alpha_i x_i, \quad 0 \leqslant \alpha_i \leqslant 1, \quad \sum_{i=1}^{n} \alpha_i = 1, \quad i = 1, \ldots, n$$

Similarly we also get that:

PROPOSITION 5. *Under the conditions of Theorem 1 for* $\psi(x) = x\varphi(x)$ *the inequalities*

$$\begin{aligned}
&\int_\Omega \psi(f(s))\,d\mu(s) - \psi\left(\int_\Omega f(s)\,d\mu(s)\right) \\
&\geqslant \int_\Omega \varphi'\left(\int_\Omega f(\sigma)\,d\mu(\sigma)\right)\left(f(s) - \int_\Omega f(\sigma)\,d\mu(\sigma)\right)^2 d\mu(s) \\
&\geqslant \int_\Omega \psi\left(\left|f(s) - \int_\Omega f(\sigma)\,d\mu(\sigma)\right|\right) d\mu(s),
\end{aligned}$$

hold when $0 < a \leqslant f(s) \leqslant 2a,\ s \in \Omega$.

Proof. The proof is similar to that of Proposition 4 and therefore omitted. □

Next we note that the results in this paper can be generalized in various directions. Here we just present the following generalization of Lemma 1.

THEOREM 4. *Let* φ *be a convex function on* $[0,b)$ *and let* $g : [0,b) \to \mathbb{R}_+$. *Then* $\varphi(x)\,g(x) = w(x)$ *satisfies*

$$w(y) - w(x) \geqslant \varphi(x)(g(y) - g(x)) + C_\varphi(x)\,g(y)(y - x), \tag{5.4}$$

where $C_\varphi(x)$ *is the constant in (1.1). Moreover,*

$$\begin{aligned}
&\int_\Omega \varphi(f(s))\,g(f(s))\,d\mu(s) - \varphi(x)\,g(x) \\
&\geqslant \varphi(x)\int_\Omega g(f(s)) - g(x)\,d\mu(s) + C_\varphi(x)\int_\Omega g(f(s))(f(s) - x)\,d\mu(s)
\end{aligned} \tag{5.5}$$

when f *is nonnegative and* f, $\varphi \circ f$, g, $g \circ f$ *are* μ*-integrable functions on a probabiliy measure space* (Ω, μ) *and*

$$x = \int_\Omega f(s)\,d\mu(s) > 0.$$

Moreover, if φ *is nonnegative, increasing, convex, and differentiable on* $[0,b)$, *and* $\varphi(0) = \lim_{z\to 0} z\varphi'(z) = 0$ *and* g *is nonnegative, increasing and* $xg(x)$ *is convex when* $0 \leqslant x < b$, *then:*

$$\begin{aligned}
&\int_\Omega \varphi(f(s))\,g(f(s))\,d\mu(s) - \varphi(x)\,g(x) \\
&\geqslant \varphi(x)\int_\Omega (g(f(s)) - g(x))\,d\mu(s) + \varphi'(x)\int_\Omega g(f(s))(f(s) - x)\,d\mu(s) \geqslant 0,
\end{aligned} \tag{5.6}$$

where

$$x=\int_{\Omega}f(s)\,d\mu(s)>0.$$

The reverse of (5.4), (5.5) and (5.6) hold when φ *is concave.*

REMARK 3. When φ is convex and φg is concave, we get a lower bound of the Jensen's difference besides the zero upper bounds resulting from concavity. In other words, we get for $x=\int_{\Omega}f(s)\,d\mu(s)>0$:

$$\begin{aligned}&\varphi(x)\int_{\Omega}g(f(s))(f(s)-x)\,d\mu(s)+C_{\varphi}(x)\int_{\Omega}g(f(s))(f(s)-x)\,d\mu(s)\\ &\leqslant\int_{\Omega}\varphi(f(s))\,g(f(s))\,d\mu(s)-\varphi(x)\,g(x)\leqslant 0.\end{aligned}$$

EXAMPLE 3. To demonstrate Remark 3, take the convex function $\varphi(x)=x^{-1/2}$, $0<a\leqslant x$ and $g(x)=x$ and hence $\varphi(x)\,g(x)=x^{1/2}$ is concave and then for $x=\int_{\Omega}f(s)\,d\mu(s)>0$

$$\begin{aligned}&x^{-\frac{1}{2}}\int_{\Omega}(f(s)-x)\,d\mu(s)-\frac{1}{2}x^{-\frac{3}{2}}\int_{\Omega}(f(s)-x)\,f(s)\,d\mu(s)\\ &\leqslant\int_{\Omega}\left((f(s))^{\frac{1}{2}}-x^{\frac{1}{2}}\right)d\mu(s)\leqslant 0\end{aligned}$$

REMARK 4. It is well known that several of the classical inequalities can easily be proved by just using Jensen's inequality (see e.g., [9] and the references given there). By using instead our new scales of refined Jensen type inequalities for γ-superconvex functions it may be possible to derive also scales of refined versions of these classical inequalities The present authors aim to investigate this idea in a forthcoming paper. In particular as in [4], [5], [6], and [7] some new scales of refined Hardy type inequalities will be derived.

REMARK 5. We say that a function φ is quasi-monotone and belongs to tha class $Q(a,b)$, $a,b\in\mathbb{R}$, if $\varphi(x)x^{-a}$ is non-decreasing and $\varphi(x)x^{-b}$ is non-increasing. such classes are very important for applications (e.g., in interpolation theory, approximation theory, Function spaces, index theory, etc.), see the review article [10] and the references given there. In particular any function in the class $Q(0,1)$ are quasi-convex e.g. equivalent to a concave function .

In this paper we have seen that functions satisfying that φ is convex and $\frac{\varphi(x)}{x}$ is concave have a special importance for creating new two-sided Jensen type inequalities. Hence, it seems to be very interesting to create a similar general theory for functions satisfying two quasiconvexity/concavity conditions e.g. that $\varphi(x)x^{-a}$ is convex and $\varphi(x)x^{-b}$ is concave.

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REFERENCES

[1] S. ABRAMOVICH, S. BANIĆ, M. KLARIČIĆ BAKULA, *A Variant of Jensen-Steffensen's Inequality for Convex and for Superquadratic Functions*, J. Inequal. Pure Appl. Math. (JIPAM), 7 (2) Art 70, (2006).

[2] S. ABRAMOVICH, M. KLARIČIĆ BAKULA, M. MATIĆ, AND J. PEČARIĆ, *A Variant of Jensen-Steffensen's Inequality and Quazi Arithmetic Means*, J. Math. Anal. Appl., 307 (2005) 370–386.

[3] S. ABRAMOVICH, G. JAMESON AND G. SINNAMON, *Refining Jensen's Inequality*, Bull. Mathe. de la Soc. des Sciences Mathe de Roumanie, (Novel Series) 47 (95), (2004), 3–14.

[4] S. ABRAMOVICH AND L. E. PERSSON, *Some new refined Hardy type inequalities with breaking points* $p = 2$ *or* $p = 3$, Proceedings of IWOTA 2011, Operator Theory: Advances and Applications, Vol. 236, 1–10, 2014, Birkhäuser, Springer-Basel.

[5] S. ABRAMOVICH AND L. E. PERSSON, *Some new scales of refined Hardy type inequalities via functions related to superquadracity*, Math. Inequal. Appl. Vol. 16 (2013), 679–695.

[6] S. ABRAMOVICH, L. E. PERSSON AND N. SAMKO, *Some new scales of refined Jensen and Hardy type inequalities*, Math. Inequal. Appl., Vol. 17 (2014), 1105–1114.

[7] J. A. OGUNTUASE AND L. E. PERSSON, *Refinement of Hardy's inequalities via superquadratic and subquadratic functions*, J. Math. Anal. Appl. 339 (2008), 1305–1312.

[8] J. E. PEČARIĆ, F. PROSCHAN, Y. L. TONG, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, New York, 1992.

[9] L. E. PERSSON AND N. SAMKO, *Inequalities and convexity*, Operator Theory: Advances and Applications, Birkhäuser Verlag, Basel, Vol. 1, (2013), 1–29.

[10] L. E. PERSSON, N. SAMKO AND P. WALL, *Quasi-monotone weight functions and their characteristics and applications*, Math. Inequal. Appl. 15 (2012), no. 3, 687–705.

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