Mathematical
Inequalities
& Applications
Volume 18, Number 2 (2015), 629–638 doi:10.7153/mia-18-47

RADIAL P–TH MOMENT OF A RANDOM VECTOR

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(*Communicated by N. Elezović*)

Abstract. In this paper, we first introduce a new concept – radial p-th moment of a random vector for star body, which is a general form of the standard p-th moment. Further we establish some properties of the radial p-th moment and give some related applications.

1. Introduction

There is a close connection among information theory, probability and convex geometry. During the past two decades, it has attracted increased interest (see e.g., [1, 4, 5, 6, 8, 11, 12, 13, 19, 20, 21, 22, 23, 24, 26, 27]).

E. Lutwak, D. Yang, and G. Zhang et al. initiated the interdisciplinary research of convex geometric analysis and information theory. Their articles [11, 19] stated geometry and information theory interact in a wide of ways over both theory and applications. O. G. Guleryuz, E. Lutwak, D. Yang, and G. Zhang [11] showed that geometric inequalities for bodies in Euclidean space can be obtained by applying information theoretic inequalities to the larger class of contoured distributions. Further, they [11] showed that new reverse information-theoretic inequalities for convex contoured distribution can be obtained from recently established reverse geometric inequalities for convex bodies [16]. Moreover, E. Lutwak, D. Yang, and G. Zhang [20, 21] extended the moment-entropy inequality, Stam's inequality and Cramer-Rao inequality (also, see [18]), and established the new moment-entropy inequality. Following their work, we introduce a new concept – radial p-th moment of a random vector for star body, which is a general form of the standard p-th moment. Then we establish some properties of the radial p-th moment and give some related applications.

The plan of the paper is as follows. First, in the next section is devoted to setting notation and preliminaries. Then, in Section 3, radial p-th moment of a random vector for star body is defined, elementary properties and some related applications of radial p-th moment are pointed out.

Mathematics subject classification (2010): 94A17, 60D05, 52A40.

Keywords and phrases: Radial p-th moment, Renyi entropy, star body, contour body, generalized Gaussians.

The authors would like to acknowledge the support from the National Natural Science Foundation of China (No. 11271244), Supported by Educational Commission of Hunan Province of China (12A033) and the Hunan Provinial Natural Science Foundation of China (14JJ2122).

2. Notations and preliminaries

We consider the n-dimensional Euclidean space $\mathbb{R}^n$ equipped with its usual inner product $\langle,\rangle$, norm $\|\cdot\|$, the unit sphere S^{n-1} and the unit ball B.

If $K\subset\mathbb{R}^n$ is nonempty, compact, and star-shaped, its radial function ρ_k is defined by

$$\rho_k(x)=\max\{\lambda:\lambda x\in K\}, \tag{1}$$

for all $x\in\mathbb{R}^n\backslash\{0\}$. If ρ_k is both positive and continuous, we will call K a star body (see e.g., [9, 10, 25]). Note that throughout, star bodies are assumed to contain the origin in their interiors.

Obviously, for star body K_1,K_2

$$K_1\subset K_2 \text{ if and only if } \rho_{k_1}\leqslant\rho_{k_2}. \tag{2}$$

Hence a star body is uniquely determined by its radial function.

A star body that is convex is called a convex body. An origin symmetric convex body K defines a norm $\|\cdot\|_K$, whose unit ball is K, that is

$$\|x\|_K=\frac{1}{\rho_K(x)},\quad x\in\mathbb{R}^n. \tag{3}$$

Conversely, the unit ball of any n-dimensional normed space $(\mathbb{R}^n,\|\cdot\|)$ is an origin-symmetric convex body.

We shall use $\tilde{\delta}$ to denote the radial metric [9]: If K and L are star bodies, then $\tilde{\delta}(K,L)$ is defined by

$$\tilde{\delta}(K,L)=\sup_{u\in S^{n-1}}|\rho_K(u)-\rho_L(u)|=\|\rho_K-\rho_L\|_\infty. \tag{4}$$

For a star body K, we define $\triangledown_pK$ to be the origin symmetric star body given by:

$$2\rho_{\triangledown_pK}(x)^{-p}=\rho_K(x)^{-p}+\rho_{-K}(x)^{-p}, \tag{5}$$

where $-K=\{-x:x\in K\}$.

For star bodies $K,L,$ and $a,b\geqslant 0$ (not both zero), the L_p-harmonic radial combination $a\cdot K\tilde{+}b\cdot L$ is the star body defined (see e.g., [15, 17]) by

$$\rho_{a\cdot K\tilde{+}b\cdot L}(x)^{-p}=a\rho_K(x)^{-p}+b\rho_L(x)^{-p}. \tag{6}$$

Roughly speaking, a distribution is called contoured, if there is a set in $\mathbb{R}^n$ such that any level set of the probability density function is a dilate of this set. We call this set the contour body of the distribution. Any reasonable star-shaped set in $\mathbb{R}^n$ that contains the origin in its interior can be realized as the contour body of a contoured distribution (see [11]).

In this paper, we will denote the standard Lebesgue density on $\mathbb{R}^n$ by dx. The notation $|A|$ stands for determinant of an $n\times n$ matrix A. If X is a random vector in $\mathbb{R}^n$, then the associated probability measure on $\mathbb{R}^n$ will be denoted by m_X. If the measure m_X is absolutely continuous with respect to Lebesgue measure, then the corresponding Radon-Nikodym derivative is called the density function of the random vector X and denoted by f_X. Further if $A \in GL(n)$ and $y \in \mathbb{R}^n$, then

$$f_{AX}(y) = |A|^{-1} f_X(A^{-1}y). \tag{7}$$

Given $t \in \mathbb{R}$, let $t_+ = \max\{t, 0\}$. Let

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx \quad \text{and} \quad B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx \ (p,q > 0)$$

denote the Gamma function and Beta function respectively.

For each $p \in (0,\infty)$ and $\lambda \in (\frac{n}{n+p}, \infty)$, let X be random vector in $\mathbb{R}^n$ whose density function $G : \mathbb{R}^n \to [0,\infty)$ is given by

$$G(x) = \begin{cases} a_{p,\lambda}(1 + (1-\lambda)|x|^p)_+^{\frac{1}{\lambda-1}}, & \text{if } \lambda \neq 1, \\ a_{p,1} e^{-|x|^p}, & \text{if } \lambda = 1, \end{cases} \tag{8}$$

where

$$a_{p,\lambda} = \begin{cases} \frac{(1-\lambda)^{\frac{n}{p}+1}}{\omega_n B(\frac{n}{p}+1, \frac{1}{1-\lambda} - \frac{n}{p})}, & \text{if } \lambda < 1, \\ \frac{1}{\omega_n \Gamma(\frac{n}{p}+1)}, & \text{if } \lambda = 1, \\ \frac{(\lambda-1)^{\frac{n}{p}+1}}{\omega_n B(\frac{n}{p}+1, \frac{1}{\lambda-1})}, & \text{if } \lambda > 1, \end{cases}$$

where ω_n denotes the volume of the n-dimensional unit ball B. Any random vector Y in $\mathbb{R}^n$ that can be written as $Y = AX$, for some $A \in GL(n)$ is called a generalized Gaussian.

3. The radial p-th moment

It will be convenient to re-define the radial function so that it is defined not only for $x \neq 0$ but for $x = 0$ as well. Define

$$\rho_K(x)^{-1} = \begin{cases} 0, & \text{if } x = 0, \\ \rho_K(x)^{-1}, & \text{if } x \neq 0. \end{cases} \tag{9}$$

As usual, for $p \in (0,\infty)$, the standard p-th moment of a random vector X is given in [21] by

$$E(|X|^p) = \int_{\mathbb{R}^n} |x|^p dm_X(x).$$

More generally, we define the radial p-th moment as follows.

DEFINITION 1. The radial p-th moment of a random vector X for star body K is defined by

$$E_K|X|^p = \int_{\mathbb{R}^n} \rho_K(x)^{-p} dm_X(x). \tag{10}$$

REMARK 1. (1). The idea of radial p-th moment exists in [19].

(2). If K is an origin symmetric convex body, using (3), (9) and (10), it follows that

$$E_K|X|^p = \int_{\mathbb{R}^n} \|x\|_K^p dm_X(x).$$

(3). In particular, when $K = B$, we have

$$E_B|X|^p = E|X|^p,$$

which shows that the radial p-th moment is a general form of the p-th moment.

Further, some simple properties of the radial p-th moment will be discussed in the following Proposition 1.

PROPOSITION 1. *If X is a random vector, K and L are star body in $\mathbb{R}^n$, then*

(1) (*Continuity*). *The radial p-th moment is continuous with respect to the radial metric.*

(2) (*Monotonicity*). *It is monotone nonincreasing with respect to set inclusion, that is, if $K \subset L$, then*

$$E_K|X|^p \geqslant E_L|X|^p.$$

In particular,

$$E_{B_\infty^n}|X|^p \leqslant E_B|X|^p \leqslant E_{B_1^n}|X|^p.$$

(3) (*Change under affine transformations*). *If $A \in GL(n)$, then*

$$E_{AK}|AX|^p = E_K|X|^p.$$

(4) (*Dilate*). *If $a, b > 0$, then*

$$E_{aK}|bX|^p = \left(\frac{b}{a}\right)^p E_K|X|^p.$$

(5) (*Positive multilinearity*). *If $a, b \geqslant 0$ (not both zero), then*

$$E_{a\cdot K \tilde{+} b\cdot L}|X|^p = aE_K|X|^p + bE_K|X|^p.$$

(6) (*Symmetry*). *If the density function of a random vector X is even, then there exists an originsymmetric star body $\bigtriangledown_p K$, such that*

$$E_K|X|^p = E_{\bigtriangledown_p K}|X|^p.$$

Proof. (1). According to the definition of the radial metric (4), it follows that $E_K|X|^p$ is continuous.

(2). Combining (2) with (10), we can obtain the desired result: if $K \subset L$, then

$$E_K|X|^p \geqslant E_L|X|^p.$$

According to (10), the radial p-th moment of the cube and octahedron can be given as follows respectively.

$$E_{B_1^n}|X|^p = \int_{\mathbb{R}^n} (\sum_{i=1}^n |x_i|)^p dm_X(x), \quad E_{B_\infty^n}|X|^p = \int_{\mathbb{R}^n} \max_{i=1,\cdots,n} |x_i|^p dm_X(x).$$

It follows that

$$E_{B_\infty^n}|X|^p \leqslant E_B|X|^p \leqslant E_{B_1^n}|X|^p.$$

(3). Applying (1), (7) and (10), we can conclude that

$$\begin{aligned} E_{AK}|AX|^p &= \int_{\mathbb{R}^n} \rho_{AK}(y)^{-p} dm_{AX}(y) \\ &= \int_{\mathbb{R}^n} \rho_K(A^{-1}y)^{-p}|A|^{-1} f_X(A^{-1}y)dy \\ &= E_K|X|^p. \end{aligned}$$

Proposition 1(3) shows that mass transportation [7] and star body can interact in radial p-th moment.

(4). From (1), (7) and (10), we can deduce that

$$\begin{aligned} E_{aK}|bX|^p &= \int_{\mathbb{R}^n} \rho_{aK}(y)^{-p} dm_{bX}(y) \\ &= \left(\frac{b}{a}\right)^p \int_{\mathbb{R}^n} \rho_K(b^{-1}y)^{-p} b^{-n} f_X(b^{-1}y)dy \\ &= \left(\frac{b}{a}\right)^p E_K|X|^p. \end{aligned}$$

(5). It can be deduced easily from (6) and (10).

(6). From (5) and (10), it follows the desired result. □

Recall that if K is a convex body, the volume ratio of K is

$$vr(K) = \left(\frac{V(K)}{V(\varepsilon)}\right)^{\frac{1}{n}},$$

where ε is the John ellipsoid of K (see e.g., [2, 3]).

Similarly, for a star body K, we define the radial p-th moment ratio of K as follows.

DEFINITION 2. If K is a star body, we define the radial p-th moment ratio of K by

$$mr(K) = \left(\frac{E_\varepsilon|X|^p}{E_K|X|^p}\right)^{\frac{1}{p}},$$

where ε is the ellipsoid of maximal volume in K.

According to the well-known John theorem, we have the following result.

PROPOSITION 2. *(1) If K is a convex body in $\mathbb{R}^n$ and ε is its John ellipsoid whose center lies at the origin, then*

$$1 \leqslant mr(K) \leqslant n.$$

(2) If K is an originsymmetric convex body in $\mathbb{R}^n$ and ε is its John ellipsoid whose center lies at the origin, then

$$1 \leqslant mr(K) \leqslant \sqrt{n}.$$

Proof. If K is a convex body in $\mathbb{R}^n$ and ε is its John ellipsoid whose center lies at the origin, according to the well-known John theorem, we have

$$\varepsilon \subset K \subset n\varepsilon.$$

When K is originsymmetric, we have

$$\varepsilon \subset K \subset \sqrt{n}\varepsilon.$$

According to Proposition 1(2), Proposition 1(4) and Definition 2, these yield the desired results. □

The previous Proposition 2 implies that if K is an origin symmetric convex body in $\mathbb{R}^n$ and B is its John ellipsoid , then

$$E_K|X|^p \geqslant \left(\frac{1}{\sqrt{n}}\right)^p E|X|^p.$$

Further, we have the following result.

PROPOSITION 3. *If K is an origin symmetric convex body in $\mathbb{R}^n$ that contains the Euclidean unit ball B and for which*

$$\left(\frac{V(K)}{V(B)}\right)^{\frac{1}{n}} = R.$$

Then there is an orthogonal transformation A of $\mathbb{R}^n$ for which

$$\max\left\{E_K|X|^p,\ E_{A^{-1}K}|X|^p\right\} \geqslant \left(\frac{1}{8R^2}\right)^p E|X|^p.$$

Proof. K. M. Ball (Theorem 4.2, [3], also see B. S. Kasin [14]) shows that if K is a symmetric convex body in $\mathbb{R}^n$ that contains the Euclidean unit ball B and for which

$$\left(\frac{V(K)}{V(B)}\right)^{\frac{1}{n}} = R.$$

Then there is an orthogonal transformation A of $\mathbb{R}^n$ for which

$$K \cap AK \subset 8R^2 B.$$

The radius of the body $K \cap AK$ in a given direction is the minimum of the radii of K and AK in that direction. Therefore

$$\min_{\theta \in S^{n-1}} \{\rho_K(\theta),\ \ \rho_K(A\theta)\} \leqslant 8R^2.$$

According to the definition (1), it follows that

$$\min\{\rho_K(x),\ \ \rho_K(Ax)\} \leqslant \frac{8R^2}{|x|}\ \ (x \neq 0).$$

This together with (10) shows that

$$\max\{E_K|X|^p,\ \ E_K|AX|^p\} \geqslant \Big(\frac{1}{8R^2}\Big)^p E|X|^p.$$

Finally, using Proposition 1(3), we can obtain the desired result. $\square$

For $t \geqslant 0$, define $G : [0,\infty) \to [0,\infty)$ by

$$G(t) = \begin{cases} a_{p,\lambda}(1+(1-\lambda)t^p)_+^{\frac{1}{\lambda-1}} & \text{if } \lambda \neq 1 \\ a_{p,1}e^{-t^p} & \text{if } \lambda = 1, \end{cases} \tag{11}$$

then the generalized Gaussian – $bG(\frac{m}{\rho_K(x)})$, whose contour body is K, is defined in $\mathbb{R}^n$.

We are now ready to compute the radial p-th moment of the generalized Gaussian – $bG\Big(\frac{m}{\rho_K(x)}\Big)$ whose contour body is K.

PROPOSITION 4. *For each $p \in (0,\infty)$ and $\lambda \in (\frac{n}{n+p},\infty)$, if K is a star body in $\mathbb{R}^n$ and $bG(\frac{m}{\rho_K(x)})$ is a probability density of a random vector X in $\mathbb{R}^n$, then*

$$E_K|X|^p = \frac{1}{m^p}\frac{n}{p\lambda - n(1-\lambda)},$$

with some $m, b > 0$.

Proof. Using Fubini's theorem, we can deduce that

$$\begin{aligned}\int_{\mathbb{R}^n} \rho_K(x)^{-p} bG\Big(\frac{m}{\rho_K(x)}\Big)dx &= b\int_{S^{n-1}}\int_0^\infty \rho_K(ru)^{-p}G\Big(\frac{m}{\rho_K(ru)}\Big)r^{n-1}drdu \\ &= bm^{-n-p}\int_{S^{n-1}}\rho_K(u)^n du\int_0^\infty G(s)s^{n+p-1}ds \\ &= nbV(K)m^{-n-p}\int_0^\infty G(s)s^{n+p-1}ds.\end{aligned}$$

We will divide $\frac{n}{n+p} < \lambda < \infty$ into three subcases, and prove the result case and case.

(a) case $\lambda < 1$.

Using the definition of Beta function and the change of variable, we can infer that

$$\begin{aligned}\int_0^\infty G(s)s^{n+p-1}ds &= a\int_0^\infty (1+(1-\lambda)s^p)^{\frac{1}{\lambda-1}}s^{n+p-1}ds\\ &= \frac{a}{p}\Big(\frac{1}{1-\lambda}\Big)^{\frac{n+p}{p}}\int_0^1 x^{\frac{1}{1-\lambda}-\frac{n}{p}-2}(1-x)^{\frac{n}{p}}dx\\ &= \frac{a}{p}\Big(\frac{1}{1-\lambda}\Big)^{\frac{n+p}{p}}B\Big(\frac{\lambda}{1-\lambda}-\frac{n}{p},\ \frac{n}{p}+1\Big).\end{aligned}$$

(b) case $\lambda = 1$.

From the definition of Gama function and the change of variable, we obtain

$$\int_0^\infty G(s)s^{n+p-1}ds = a\int_0^\infty e^{-s^p}s^{n+p-1}ds = \frac{a}{p}\Gamma\Big(\frac{n}{p}+1\Big).$$

(c) case $\lambda > 1$.

Observing the definition of Beta function and the change of variable, we get

$$\begin{aligned}\int_0^\infty G(s)s^{n+p-1}ds &= a\int_0^\infty (1+(1-\lambda)s^p)_+^{\frac{1}{\lambda-1}}s^{n+p-1}ds\\ &= a\int_0^{(\frac{1}{\lambda-1})^{\frac{1}{p}}} (1+(1-\lambda)s^p)^{\frac{1}{\lambda-1}}s^{n+p-1}ds\\ &= \frac{a}{p}\Big(\frac{1}{\lambda-1}\Big)^{\frac{n+p}{p}}B\Big(\frac{\lambda}{\lambda-1},\frac{n}{p}+1\Big).\end{aligned}$$

Applying the three cases: (a), (b) and (c), we can deduce that

$$\int_{\mathbb{R}^n}\rho_K(x)^{-p}bG\Big(\frac{m}{\rho_K(x)}\Big)dx = \frac{b}{m^{n+p}}\frac{naV(K)}{p}a_1, \tag{12}$$

where $a = a_{p,\lambda}$,

$$a_1 = \begin{cases}(\frac{1}{1-\lambda})^{\frac{n+p}{p}}B(\frac{\lambda}{1-\lambda}-\frac{n}{p},\frac{n}{p}+1), & \text{if } \lambda<1,\\ \Gamma(\frac{n}{p}+1), & \text{if } \lambda=1,\\ (\frac{1}{\lambda-1})^{\frac{n+p}{p}}B(\frac{\lambda}{\lambda-1},\frac{n}{p}+1), & \text{if } \lambda>1.\end{cases}$$

Similarly, we can conclude that

$$1 = \int_{\mathbb{R}^n} bG\Big(\frac{m}{\rho_K(x)}\Big)dx = \frac{b}{m^n}\frac{naV(K)}{p}a_2, \tag{13}$$

where

$$a_2 = \begin{cases}(\frac{1}{1-\lambda})^{\frac{n}{p}}B(\frac{\lambda}{1-\lambda}-\frac{n}{p},\frac{n}{p}), & \text{if } \lambda<1,\\ \Gamma(\frac{n}{p}), & \text{if } \lambda=1,\\ (\frac{1}{\lambda-1})^{\frac{n}{p}}B(\frac{\lambda}{\lambda-1},\frac{n}{p}), & \text{if } \lambda>1.\end{cases}$$

Thus, this together with (10) and (12) gives the desired result. □

REMARK 2. (1). Proposition 4 compute the information measure: the radial p-th moment of the generalized Gaussian – $bG\left(\frac{m}{\rho_K(x)}\right)$ whose contour body is K.

(2). (13) shows that the solution of m,b is not unique. Moreover, from the above Proposition 4, we can learn that the radial p-th moment of $bG\left(\frac{m}{\rho_K(x)}\right)$ is determined only by m.

REFERENCES

[1] Z. ARTSTEIN AND R. A. VITALE, *A strong law of large numbers for random compact sets*, Ann. Probab. **5 (3)** (1975), 879–882.

[2] J. BOURGAIN AND V. D. MILMAN, *New volume ratio properties for convex symmetric bodies in R^n*, Invent. Math. **(88)** (1987), 319–340.

[3] K. M. BALL, *An elementary introduction to modern convex geometry*, Flavors of Geometry, ed. by Silvio Levy, Cambridge University Press, New York, **(31)** (1997), 1–58.

[4] M. H. M. COSTA AND T. M. COVER, *On the similarity of the entropy power inequality and the Brunn-Minkowski inequality*, IEEE Trans. Inform. Theory. **6 (IT-30)** (1984), 837–839.

[5] T. M. COVER, A. DEMBO, AND J. A. THOMAS, *Information theoretic inequalities*, IEEE Trans. Inform. Theory. **(37)** (1991), 1501–1518.

[6] T. M. COVER AND J. A. THOMAS, *Elements of information theory*, New York: Wiley-Interscience, (1991).

[7] D. CORDERO-ERAUSQUIN, B. NAZARET, AND C. VILLANI, *A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities*, Adv. Math. **2 (182)** (2004), 307–332.

[8] A. DEMBO, *Information inequalities and uncertainty principles*, Dep. Statist., Stanford Univ., Stanford, CA, Tech. Rep. (1991).

[9] R. J. GARDNER, *Geometric tomography*, Cambridge University Press, Cambridge, (1995).

[10] P. M. GRUBER, *Convex and discrete geometry*, Springer-Verlag, New York, (2007).

[11] O. G. GULERYUZ, E. LUTWAK, D. YANG, AND G. ZHANG, *Information-theoretic inequalities for contoured probability distributions*, IEEE Trans. Inform. Theory. **8 (48)** (2002), 2377–2383.

[12] R. HE AND G. LENG, *A strong law of large numbers on the harmonic p-combination*, Geom. Dedicata, **(154)** (2011), 103–116.

[13] R. HE AND G. LENG, *A probabilistic version of quermassintegrals' Brunn-Minkowski inequality* (in Chinese), Sci. Sin. Math, **(43)** (2013), 941–952.

[14] B. S. KASIN, *The widths of certain finite-dimensional sets and classes of smooth functions*, Izv. Akad. Nauk SSSR Ser. Mat. **2 (41)** (1977), 334–351.

[15] E. LUTWAK, *The Brunn-Minkowski-Firey theory II: Affine and Geominimal Surface Areas*, Adv. Math. **(118)** (1996), 244–294.

[16] E. LUTWAK, D. YANG, AND G. ZHANG, *A new ellipsoid associated with convex bodies*, Duke Math. J. **3 (104)** (2000), 375–390.

[17] E. LUTWAK, D. YANG, AND G. ZHANG, *L_p affine isoperimetric inequalities*, J. Differential Geom. **1 (56)** (2000), 111–132.

[18] E. LUTWAK, D. YANG, AND G. ZHANG, *The Cramer-Rao inequality for star bodies*, Duke Math. J. **1 (112)** (2002), 59–81.

[19] E. LUTWAK, D. YANG, AND G. ZHANG, *Moment-entropy inequalities*, Ann. Probab. **(32)** (2004), 757–774.

[20] E. LUTWAK, D. YANG, AND G. ZHANG, *Cramer-Rao and moment-entropy inequalities for Renyi entropy and generalized Fisher information*, IEEE Trans. Inform. Theory. **(51)** (2005), 473–478.

[21] E. LUTWAK, D. YANG, AND G. ZHANG, *Moment-entropy inequalities for a random vector*, IEEE Trans. Inform. Theory. **(53)** (2007), 1603–1607.

[22] E. LUTWAK, S. LV, D. YANG, AND G. ZHANG, *Extensions of Fisher Information and Stam's Inequality*, IEEE Trans. Inform. Theory. **(58)** (2012), 1319–1327.

[23] E. LUTWAK, S. LV, D. YANG, AND G. ZHANG, *Affine moments of a random vector*, IEEE Trans. Inform. Theory. **(59)** (2013), 5592–5599.
[24] S. LV, AND X. LV, *Affine Fisher information inequalities*, J. Math. Anal. Appl. **(371)** (2010), 347–354.
[25] R. SCHNEIDER, *Convex bodies: the Brunn-Minkowski theory*, Cambridge University Press, Cambridge, 1993.
[26] R. A. VITALE, *On the volume of parallel bodies: a probabilistic derivation of the Steiner formula*, Adv. in appl. Probab. **(27)** (1995), 97–101.
[27] R. A. VITALE, *The Wills functional and Gaussian processes*, Ann. Probab. **(24)** (1996), 2172–2178.

(Received July 12, 2011)

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