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ON THE q–BETA FUNCTION INEQUALITIES

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Abstract. In this paper, we provide some inequalities for the q-beta function with $q>0$ by means of calculus and q-Čebyšev-Grüss type integral inequalities. When letting $q\to 1$, some of our results approach to results obtained by some authors and the others are shown to be new.

1. Introduction

In the recent past, numerous papers were published presenting remarkable inequalities involving the q-gamma function (see [1, 3, 7, 8, 9, 12, 13, 14, 15] and the extensive list of references given therein). But only few inequalities for the q-beta function and its relatives have been found by Fitouhi and Brahim [4]). Their results were established via some q-integral inequalities obtained by Gauchman [6]. It is the aim of this article to provide to the list of the q-beta-function inequalities. At the first, the monotonicity properties of some functions involving the q-gamma function are used to establish inequalities for the q-beta function with $q>0$. At the last, the results obtained by Yang [17] on the weighted q-Čebyšev-Grüss type integral inequalities are exploited to provide some q-beta function inequalities.

2. Basic notations and definitions in q-calculus

For the convenience of the readers, this section is provided as a summary of the mathematical notations and definitions used in this paper. All of these definitions can be found in [5, 11].

For all $a\in\mathbb{C}$, the basic number, factorial function and shifted factorial are defined, respectively, as

$$[a]_q=\frac{1-q^a}{1-q},\quad [n]_q!=[n]_q[n-1]_q\cdots[1]_q,\quad n\in\mathbb{N}\quad\text{with}\quad [0]_q!=1, \tag{2.1}$$

$$(a;q)_n=\prod_{k=0}^{n-1}(1-aq^n),\quad n\in\mathbb{N},\quad\text{with}\quad (a;q)_0=1. \tag{2.2}$$

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When letting $n \to \infty$, we need the restriction $0 < q < 1$.

For an arbitrary function $f(x)$, the q-derivative is defined as

$$(D_q f)(t) = \frac{f(t) - f(tq)}{(1-q)t}, \qquad q \neq 1. \tag{2.3}$$

Clearly, if $f(x)$ is differentiable, then $(D_q f)(t)$ approaches to $f'(t)$ as $q \to 1$.

The q-Jackson integral in a generic interval $[a,b]$ is defined as

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t \tag{2.4}$$

where

$$\int_0^a f(t) d_q t = a(1-q) \sum_{n=0}^{\infty} q^n f(aq^n) \tag{2.5}$$

provided the sum is convergent.

The exponential function e^z has different q-analogues, here we recall its q-analogue

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{[n]_q!} z^n = (-(1-q)z; q)_\infty \tag{2.6}$$

The q-beta function is defined for $0 < q < 1$ as

$$B_q(x,y) = \int_0^1 t^{x-1} \frac{(tq;q)_\infty}{(tq^y;q)_\infty} d_q t = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}, \qquad x > 0,\ y \notin \mathbb{Z}_0^- \tag{2.7}$$

where Γ_q is the q-gamma function defined as

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{k=0}^{\infty} \frac{1-q^{n+1}}{1-q^{n+x}}, \qquad 0 < q < 1, \tag{2.8}$$

and

$$\Gamma_q(x) = (q-1)^{1-x} q^{\frac{x(x-1)}{2}} \prod_{k=0}^{\infty} \frac{1-q^{-(n+1)}}{1-q^{-(n+x)}}, \qquad q > 1. \tag{2.9}$$

Furthermore, it has the q-integral representation

$$\Gamma_q(x) = \int_0^{\frac{1}{1-q}} t^{x-1} E_q(-tq) d_q t, \qquad 0 < q < 1 \tag{2.10}$$

The logarithmic derivative $\psi_q(x)$ of the q-gamma function is known as the q-psi or q-digamma function, that is, it is given by

$$\psi_q(x) = \frac{d}{dx}(\log \Gamma_q(x)) = \frac{\Gamma_q'(x)}{\Gamma_q(x)} \tag{2.11}$$

which appeared in the work of Krattenthaler and Srivastava [10] when they studied the summations for basic hypergeometric series. Some of its properties presented and proved in them work, see also [15]. From (2.8), for $0<q<1$ and for all real variable $x>0$, we get

$$\psi_q(x) = -\log(1-q) + \log q \sum_{k=1}^{\infty} \frac{q^{xk}}{1-q^k}, \tag{2.12}$$

From the previous definitions, for all positive x,y and $q \geqslant 1$, we get

$$\Gamma_q(x) = q^{\frac{(x-1)(x-2)}{2}} \Gamma_{q^{-1}}(x). \tag{2.13}$$

$$B_q(x,y) = q^{1-xy} B_{q^{-1}}(x,y). \tag{2.14}$$

$$\psi_q(x) = \frac{2x-3}{2} \log q + \psi_{q^{-1}}(x) \tag{2.15}$$

3. Inequalities via calculus

In this section, we use the monotonicity properties for some functions related to the q-gamma function to establish some q-beta function inequalities with $q>0$. Now, we shall prove the following results:

THEOREM 3.1. *Let x,y and q be positive real numbers. Then we have*

$$\frac{q^{(1-x)(y-1)H(q-1)}}{[x]_q[y]_q} - B_q(x,y) \geqslant 0 \quad \textit{if} \quad (x-1)(y-1) \geqslant 0 \tag{3.1}$$

and

$$\frac{q^{(1-x)(y-1)H(q-1)}}{[x]_q[y]_q} - B_q(x,y) \leqslant 0 \quad \textit{if} \quad (x-1)(y-1) \leqslant 0. \tag{3.2}$$

Furthermore, we have

$$B_q(x,y) \geqslant (\leqslant) \frac{q^{x(1-y)H(q-1)}}{[x]_q}, \quad \textit{for all} \quad x>0 \quad \textit{and} \quad y \leqslant (\geqslant) 1 \tag{3.3}$$

and

$$B_q(x,y) \geqslant (\leqslant) \frac{q^{y(1-x)H(q-1)}}{[y]_q}, \quad \textit{for all} \quad y>0 \quad \textit{and} \quad x \leqslant (\geqslant) 1 \tag{3.4}$$

where $H(\cdot)$ denotes the Heaviside step function.

Proof. Let the function of two variables

$$f_q(x,y) = \log\Gamma_q(x+1) + \log\Gamma_q(y+1) - \log\Gamma_q(x+y) + (x-1)(y-1)H(q-1)\log q$$

Partial differentiation with respect to x gives

$$\frac{\partial}{\partial x} f_q(x,y) = \psi_q(x+1) - \psi_q(x+y) + (y-1)H(q-1)\log q$$

which can be represented by (2.12), when $0<q<1$, as

$$\frac{\partial}{\partial x} f_q(x,y) = \sum_{k=1}^{\infty} \frac{q^{x(k+1)}\log q}{1-q^k}\left(1-q^{k(y-1)}\right)$$

It is obvious that $1-q^{k(y-1)}$ is greater than zero if $y>1$ and less than zero if $y<1$ and so the function $x \mapsto f_q(x,y)$ is increasing on $(0,\infty)$ if $y<1$ and decreasing on $(0,\infty)$ if $y>1$. Since $f_q(1,y)=0$ for all $y>0$, then $f_q(x,y) \leqslant 0$ if $x,y \in (0,1]$ or $x,y \in [1,\infty)$ and $f_q(x,y)>0$ if $x<1$ and $y>1$ or $x>1$ and $y<1$. When $q \geqslant 1$, one can easily prove that $f_q(x,y)=f_{q^{-1}}(x,y)$ and so, we conclude that $f_q(x,y) \leqslant 0$ if $x,y \in (0,1]$ or $x,y \in [1,\infty)$ and $f_q(x,y)>0$ if $x<1$ and $y>1$ or $x>1$ and $y<1$ for all $q>0$. These are equivalent to (3.1) and (3.2). Also, we get

$$\lim_{x\to 0} f_q(x,y) = \log[y]_q + (1-y)H(q-1)\log q, \qquad y>0$$

and from the monotonicity of $f_q(x,y)$, we arrive at (3.3) and since q-beta function is symmetric for x and y, then we obtain (3.4). □

COROLLARY 3.2. *Let x and q be positive real numbers. Then we have*

$$\Gamma_q(nx) \geqslant q^{(x-1)\left(\frac{n(n-1)}{2}x-n+1\right)H(q-1)}[n-1]_{q^x}![x]_q^{2(n-1)}\Gamma_q^n(x), \qquad n\in\mathbb{N}. \tag{3.5}$$

Proof. From the inequality (3.1), we have

$$\Gamma_q(x+y) \geqslant q^{(x-1)(y-1)H(q-1)}[x]_q[y]_q\Gamma_q(x)\Gamma_q(y), \qquad (x-1)(y-1)\geqslant 0$$

When $x=y>0$, we get

$$\Gamma_q(2x) \geqslant q^{(x-1)^2H(q-1)}[x]_q^2\Gamma_q^2(x), \qquad x>0, \tag{3.6}$$

Also, we get

$$\Gamma_q(3x)=\Gamma_q(2x+x) \geqslant q^{(x-1)(2x-1)H(q-1)}[2x]_q[x]_q\Gamma_q(2x)\Gamma_q(x), \qquad x>0$$

which can be read by using the inequality (3.6) as

$$\Gamma_q(3x) \geqslant q^{(x-1)(3x-2)H(q-1)}[2x]_q[x]_q^3\Gamma_q^3(x), \qquad x>0$$

Iterating this process $(n-1)$ times gives

$$\Gamma_q(nx) \geqslant q^{(x-1)\left(\frac{n(n-1)}{2}x-n+1\right)H(q-1)}[x]_q^{n-1}\Gamma_q^n(x)\prod_{k=1}^{n-1}[kx]_q, \qquad n\in\mathbb{N}$$

which is equivalent to the desired result. □

COROLLARY 3.3. *Let x and q be positive real numbers. Then we have*

$$\Gamma_q(x) \leqslant \frac{[2]_q^{x-\frac{1}{2}}q^{-\frac{1}{2}(x-1)^2H(q-1)}}{[x]_q}\sqrt{\frac{\Gamma_{q^2}(x)\Gamma_{q^2}\left(x+\frac{1}{2}\right)}{\Gamma_{q^2}\left(\frac{1}{2}\right)}} \tag{3.7}$$

In particular, when $x=\frac{1}{2}$, *we have*

$$\Gamma_q\left(\frac{1}{2}\right) \leqslant \frac{q^{-\frac{1}{8}H(q-1)}}{[1/2]_q} = q^{-\frac{1}{8}H(q-1)}(1+\sqrt{q}). \tag{3.8}$$

Proof. Inserting the well-known q-duplication formula

$$\Gamma_{q^2}(x)\Gamma_{q^2}\left(x+\frac{1}{2}\right) = [2]_q^{1-2x}\Gamma_q(2x)\Gamma_{q^2}\left(\frac{1}{2}\right)$$

into the inequality (3.6) yields (3.7). □

REMARK 3.4. It is worth mentioning that the inequalities (3.1), (3.2) and (3.5) were obtained by Fitouhi and Brahim [4] when $0<q<1$ by means of using q-analogue of the Čebyšev's integral inequality. Here, we extend their results for all $q>0$ and introduce another proof by using the monotonicity properties for the function $f_q(x,y)$. Also, we think that the inequalities (3.3) and (3.4) are shown to be new.

THEOREM 3.5. *Let* $0<x,y\leqslant 1$ *and* $q>0$. *Then*

$$\frac{q^{(1-x)(y-1)H(q-1)}}{[x]_q[y]_q} - B_q(x,y) \geqslant \frac{[1-x]_q[1-y]_q q^{x+y-(x-1)(y-1)H(q-1)}}{[x]_q[y]_q[x+1]_q[y+1]_q} \tag{3.9}$$

Proof. Let the function

$$F_q(x,y) = \log\Gamma_q(x+y+1) - \log\Gamma_q(x+2) - \log\Gamma_q(y+2) + \log(1+q) - (x-1)(y-1)H(q-1)\log q$$

When $0<q<1$, partial differentiation gives

$$\frac{\partial}{\partial x}F_q(x,y) = \psi_q(x+y+1) - \psi_q(x+2)$$

Since $\psi_q(x)$ is increasing on $(0,\infty)$, then $\frac{\partial}{\partial x}F_q(x,y)\leqslant 0$ for all $0<x,y\leqslant 1$. It is not difficult to show that $\frac{\partial}{\partial x}F_q(x,y) = \frac{\partial}{\partial x}F_{q^{-1}}(x,y)$ which yields that $\frac{\partial}{\partial x}F_q(x,y)\leqslant 0$ for all $q>0$ and $0<x,y\leqslant 1$ and so the function $x\mapsto F_q(x,y)$ is decreasing on $(0,1]$. Since $F_q(1,y)=0$, then $F_{q^{-1}}(x,y)\geqslant 0$ for all $q>0$ and $0<x,y\leqslant 1$ which leads to the required. □

REMARK 3.6. On letting $q\to 1$, we get

$$\frac{1}{xy} - B(x,y) \geqslant \frac{(1-x)(1-y)}{xy(x+1)(y+1)}, \qquad 0<x,y\leqslant 1 \tag{3.10}$$

which is the same result obtained by Alzer [2] for the classical beta function.

4. Inequalities via q-Čebyšev-Grüss integral inequalities

Yang [17] established weighted q-Čebyšev-Grüss type inequalities by using the weighted q-integral Montgomery identity. He assumed that $\omega : [a,b] \to [0,\infty)$ satisfying $\int_a^b \omega(t)d_qt = 1$ and $W(t) = \int_a^t \omega(u)d_qu$ for $t \in [a,b]$, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for $t > b$. He gave the weighted q-integral Peano kernel $P_\omega(u,t)$ defined by

$$P_\omega(u,t) = \begin{cases} W(t), & a \leqslant t \leqslant u \\ W(t) - 1, & u \leqslant t \leqslant b \end{cases} \tag{4.1}$$

For some suitable functions $\omega, f, g : [a,b] \to \mathbb{R}$, set

$$T(w,f,g) = \int_a^b \omega(t)f(tq)g(tq)d_qt - \left(\int_a^b \omega(t)f(tq)d_qt\right)\left(\int_a^b \omega(t)g(tq)d_qt\right) \tag{4.2}$$

and define $\|\cdot\|$ as $\|h\| = \sup_{t\in[a,b]} |h(t)|$ for $h \in [a,b]$. He proved five inequalities for the absolute value of $T(\omega, f, g)$, we will list his results in the following theorem.

THEOREM 4.1. ([17]) *Let $f, g : [a,b] \to \mathbb{R}$ and $\omega : [a,b] \to [0,\infty)$ satisfying $\int_a^b \omega(t)d_qt = 1$, then, for $0 < q < 1$, we have*

$$|T(w,f,g)| \leqslant \|D_qf\|\|D_qg\| \int_a^b \omega(u)H^2(uq)d_qu \tag{4.3}$$

$$|T(w,f,g)| \leqslant \frac{1}{2}\int_a^b \omega(u)[|g(uq)|\|D_qf\| + |f(uq)|\|D_qg\|]H(uq)d_qu \tag{4.4}$$

$$|T(w,f,g)| \leqslant \frac{1}{2}[\|f(tq)|\|D_qg\| + \|g(tq)|\|D_qf\|] \int_a^b \omega(u)H(uq)d_qu \tag{4.5}$$

$$|T(w,f,g)| \leqslant \|g(tq)|\|D_qf\| \int_a^b \omega(u)H(uq)d_qu \tag{4.6}$$

$$|T(w,f,g)| \leqslant \|f(tq)|\|D_qg\| \int_a^b \omega(u)H(uq)d_qu \tag{4.7}$$

where

$$H(u) = \int_a^b |P_\omega(t,u)|d_qt, \qquad u \in [a,b]. \tag{4.8}$$

In this section we exploit the results from (4.3) to (4.7) to establish inequalities for the q-beta function. On letting $q \to 1$, our results give new inequalities for the classical beta function. In order to do this, we assume that $\omega(t) = 1$ which yields $\int_0^1 \omega(t)d_qt = 1$ and $W(t) = \int_0^t \omega(u)d_qu = t \in [0,1]$, leading to

$$P_\omega(u,t) = \begin{cases} t, & 0 \leqslant t \leqslant u \\ t - 1, & u \leqslant t \leqslant 1 \end{cases} \tag{4.9}$$

and

$$H(u) = \int_0^1 |P_\omega(u,t)| d_qt = \int_0^u t\, d_qt + \int_u^1 (1-t) d_qt = \frac{2u^2-(1+q)u+q}{1+q} \tag{4.10}$$

From the previous assumption, we can compute the values of the q-integrals

$$\alpha(q) =: \int_0^1 \omega(u) H(uq) d_qu = \frac{2q^2}{(1+q)(1+q+q^2)} \tag{4.11}$$

and

$$\begin{aligned} \beta(q) &=: \int_0^1 \omega(u) H^2(uq) d_qu \\ &= \frac{q^3}{(1+q)^2} \left(\frac{4q}{1+q+q^2+q^3+q^4} - \frac{4}{1+q^2} + \frac{5}{1+q+q^2} \right). \end{aligned} \tag{4.12}$$

THEOREM 4.2. *Let $x, y \geqslant 2$ and $q > 0$. Then, we have*

$$\frac{q^{(1-x)(y-1)H(q-1)}}{[x]_q[y]_q} - B_q(x,y) \leqslant \hat{q}^{x-1+(xy-1)H(q-1)} [x-1]_{\hat{q}} [y-1]_{\hat{q}} \beta(\hat{q}) \tag{4.13}$$

$$\frac{q^{(1-x)(y-1)H(q-1)}}{[x]_q[y]_q} - B_q(x,y) \leqslant \frac{1}{2} q^{(1-xy)H(q-1)} h_{\hat{q}}(x,y) \tag{4.14}$$

$$\frac{q^{(1-x)(y-1)H(q-1)}}{[x]_q[y]_q} - B_q(x,y) \leqslant \frac{q^{(1-xy)H(q-1)}}{2} \left(\hat{q}^{x-1} [x-1]_{\hat{q}} + [y-1]_{\hat{q}} \right) \alpha(\hat{q}) \tag{4.15}$$

$$\frac{q^{(1-x)(y-1)H(q-1)}}{[x]_q[y]_q} - B_q(x,y) \leqslant \hat{q}^{x-1+(xy-1)H(q-1)} [x-1]_{\hat{q}} \alpha(\hat{q}) \tag{4.16}$$

$$\frac{q^{(1-x)(y-1)H(q-1)}}{[x]_q[y]_q} - B_q(x,y) \leqslant q^{(1-xy)H(q-1)} [y-1]_{\hat{q}} \alpha(\hat{q}) \tag{4.17}$$

where $\alpha(q)$ and $\beta(q)$ are defined as in (4.11) and (4.12), respectively, $\hat{q} = q$ if $0 < q \leqslant 1$ and $\hat{q} = q^{-1}$ if $q \geqslant 1$ and $h_q(x,y)$ is defined as

$$\begin{aligned} h_q(x,y) = &\frac{q^{2-x}[x-1]_q}{(1+q)[y]_q} \left(\frac{2q(1+q)}{[y+1]_q[y+2]_q} - \frac{1+q}{[y+1]_q} + 1 \right) \\ &+ \frac{q[y-1]_q}{1+q} \left(\frac{2q}{[x+2]_q} - \frac{(1+q)}{[x+1]_q} + \frac{1}{[x]_q} \right). \end{aligned} \tag{4.18}$$

Proof. Assume the functions

$$f(t) = q^{1-x} t^{x-1} \quad \text{and} \quad g(t) = \frac{(t;q)_\infty}{(tq^{y-1};q)_\infty}, \qquad t \in [0,1],\ x,y \geqslant 1 \tag{4.19}$$

On q-differentiating give

$$(D_qf)(t) = q^{1-x}[x-1]_q t^{x-2} \qquad \text{and} \qquad (D_qg)(t) = -[y-1]_q \frac{(tq;q)_\infty}{(tq^{y-1};q)_\infty}$$

It is obvious that the function $t \mapsto t^{x-\delta}, \delta = 1,2$ is increasing on $[0,1]$ if $x \geqslant \delta$ and decreasing on $[0,1]$ if $x \leqslant \delta$ which reveals that $\|f(tq)\| = 1$ if $x \geqslant 1$ and $\|f(tq)\| = \infty$ if $x \leqslant 1$ and $\|D_q f\| = q^{1-x}[x-1]_q$ if $x \geqslant 2$ and $\|D_q f\| = \infty$ if $x \leqslant 2$.

To determine $\|g(tq)\|$ and $\|D_q g\|$, we have

$$\log \lambda_y(t) = \sum_{n=0}^{\infty} \left\{ \log(1 - tq^{n+1}) - \log(1 - tq^{n+y}) \right\}$$

where $\lambda_y(t) = (tq;q)_\infty / (tq^{y-1};q)_\infty$. Differentiation gives

$$\lambda_y'(t) = \lambda_y(t) \sum_{n=0}^{\infty} \frac{-q^{n+1}(1-q^{y-2})}{(1-tq^{n+1})(1-tq^{n+y})} \begin{cases} \leqslant 0, & y \geqslant 2 \\ \geqslant 0, & 1 \leqslant y \leqslant 2 \end{cases}$$

which means that $\lambda_y(t)$ is decreasing on $[0,1]$ if $y \geqslant 2$ and increasing on $[0,1]$ if $1 \leqslant y \leqslant 2$. Since $\lambda_y(0) = 1$ and $\lambda_y(1) = (1-q)^{y-2}\Gamma_q(y-1)$, then we have

$$\|g(tq)\| = \sup_{t \in [0,1]} \lambda_{y+1}(t) = \begin{cases} 1, & y \geqslant 1 \\ (1-q)^{y-1}\Gamma_q(y-1), & 0 < y \leqslant 1 \end{cases} \tag{4.20}$$

and

$$\|D_q g\| = [y-1] \sup_{t \in [0,1]} \lambda_y(t) = \begin{cases} [y-1]_q, & y \geqslant 2 \\ (1-q)^{y-2}\Gamma_q(y), & 1 \leqslant y \leqslant 2 \end{cases} \tag{4.21}$$

By virtue of (4.10), we can compute

$$\begin{aligned} &\int_0^1 \omega(u)[|g(uq)|\|D_q f\| + |f(uq)|\|D_q g\|]H(uq)d_q u \\ &\quad = \frac{q^{2-x}[x-1]_q}{1+q} \int_0^1 (2qu^2 - (1+q)u + 1)\frac{(uq;q)_\infty}{(uq^y;q)_\infty} d_q u \\ &\qquad + \frac{q[y-1]_q}{1+q} \int_0^1 (2qu^2 - (1+q)u + 1)u^{x-1} d_q u \\ &\quad = \frac{q^{2-x}[x-1]_q}{1+q}(2qB_q(3,y) - (1+q)B_q(2,y) + B_q(1,y)) \\ &\qquad + \frac{q[y-1]_q}{1+q}\left(\frac{2q}{[x+2]_q} - \frac{(1+q)}{[x+1]_q} + \frac{1}{[x]_q}\right) \\ &\quad = \frac{q^{2-x}[x-1]_q}{(1+q)[y]_q}\left(\frac{2q(1+q)}{[y+1]_q[y+2]_q} - \frac{1+q}{[y+1]_q} + 1\right) \\ &\qquad + \frac{q[y-1]_q}{1+q}\left(\frac{2q}{[x+2]_q} - \frac{(1+q)}{[x+1]_q} + \frac{1}{[x]_q}\right) \end{aligned}$$

which is equivalent to $h_q(x,y)$ defined as in (4.18). From the assumptions (4.2) and (4.19) and the definition of q-beta function (2.7), we get

$$T(\omega, f, g) = B_q(x,y) - \frac{1}{[x]_q[y]_q}$$

Notice that $T(\omega,f,g)\leqslant 0$ by Theorem 3.1. and so we have to take

$$|T(\omega,f,g)|=\frac{1}{[x]_q[y]_q}-B_q(x,y). \tag{4.22}$$

Inserting the previous results into Theorem 4.1 gives the proof when $0<q<1$. We note that if $B_q(x,y)\geqslant 1/([x]_q[y]_q)-a(q)$ (say) when $0<q<1$, by (2.14) when $q\geqslant 1$, we get

$$B_q(x,y)=q^{1-xy}B_{q^{-1}}(x,y)\geqslant q^{1-xy}\left(\frac{1}{[x]_{q^{-1}}[y]_{q^{-1}}}-a(q^{-1})\right)$$

which gives the proofs when $q\geqslant 1$. This ends the proof. $\square$

It is not difficult to see that $\alpha(1)=1/3$, $\beta(1)=7/60$ and

$$\lim_{q\to 1}h_q(x,y)=\frac{(x-1)(y^2+y+2)}{2y(y+1)(y+2)}+\frac{(y-1)(x^2+x+2)}{2x(x+1)(x+2)}$$

which can be used to establish some inequalities for the classical beta function. For this proposal, we introduce the following theorem.

THEOREM 4.3. *Let $x,y\geqslant 2$. Then, we have*

$$\frac{1}{xy}-B(x,y)\leqslant\frac{7}{60}(x-1)(y-1) \tag{4.23}$$

$$\frac{1}{xy}-B(x,y)\leqslant\frac{(x-1)(y^2+y+2)}{4y(y+1)(y+2)}+\frac{(y-1)(x^2+x+2)}{4x(x+1)(x+2)} \tag{4.24}$$

$$\frac{1}{xy}-B(x,y)\leqslant\frac{1}{6}(x+y-2) \tag{4.25}$$

$$\frac{1}{xy}-B(x,y)\leqslant\frac{1}{3}(x-1) \tag{4.26}$$

$$\frac{1}{xy}-B(x,y)\leqslant\frac{1}{3}(y-1). \tag{4.27}$$

The results from (4.23) to (4.27) for beta function are shown to be new.

REFERENCES

[1] H. ALZER, A. Z. GRINSHPAN, *Inequalities for the gamma and q-gamma functions*, Journal of Approximation Theory, Vol. 144 (2007) 67–82.

[2] H. ALZER, *Some beta-function inequalities*, Proceedings of the Royal Society of Edinburgh, Vol. 133A (2003) 731–745.

[3] N. ELEZOVIC, C. GIORDANO AND J. PECARIC, *Convexity and q-gamma function*, Rendiconti del Circolo Matematico di Palermo, Vol. Serie II, Tomo XLVIII (1999) 285–298.

[4] A. FITOUHI, K. BRAHIM, *Some inequalities for the q-beta and the q-gamma functions via some q-integral inequalities*, Applied Mathematics and Computation, Vol. 204 (2008) 385–394.

[5] G. GASPER, M. RAHMAN, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge 2004.

[6] H. GAUCHMAN, *Integral inequalities in q-calculus*, Computers and Mathematics with Applications, Vol. 47 (2004) 281–300.

[7] A. Z. GRINSHPAN AND M. E. H. ISMAIL, *Completely monotonic functions involving the gamma and q-gamma functions*, Proc. Amer. Math. Soc. Vol. 134 (2006) 1153–1160.

[8] M. E. H. ISMAIL AND M. E. MULDOON, *Inequalities and monotonicity properties for gamma and q-gamma functions*, in: R. V. M. Zahar (Ed.), Approximation and Computation, International Series of Numerical Mathematics, Vol. 119 Birkhauser, Boston, MA (1994) 309–323.

[9] M. E. H. ISMAIL, L. LORCH AND M. E. MULDOON, *Completely monotonic functions associated with the gamma function and its q-analogues*, Journal of Mathematical Analysis and Applications, Vol. 116 (1986) 1–9.

[10] C. KRATTENTHALER, H. M. SRIVASTAVA, *Summations for basic hypergeometric series involving a q-analogue of the digamma function*, Computers Math. Applic., Vol. 32 (2) (1996) 73–91.

[11] D. S. MOAK, *The q-gamma function for $q > 1$*, Aequationes Math., Vol. 20 (1980) 278–285.

[12] A. SALEM, *Complete monotonicity properties of functions involving q-gamma and q-digamma functions*, Mathematical Inequalities and Applications, Vol. 17 (3) (2014), 801–811.

[13] A. SALEM, *Two classes of bounds for the q-gamma and the q-digamma functions in terms of the q-zeta functions*, Banach J. Math. Anal., Vol. 8 (1) (2014) 109–117.

[14] A. SALEM, *An infinite class of completely monotonic functions involving the q-gamma function*, Journal of Mathematical Analysis and Applications, Vol. 406 (2) (2013) 392–399.

[15] A. SALEM, *Some Properties and Expansions Associated with q-Digamma Function*, Quaestiones mathematicae, Vol. 36 (1) (2013) 67–77.

[16] A. SALEM, *A completely monotonic function involving q-gamma and q-digamma functions*, Journal of Approximation Theory, Vol. 164 (7) (2012) 971–980.

[17] W. YANG, *On weighted q-Čebyšev-Grüss type inequalities*, Computers and Mathematics with Applications, Vol. 61 (2011) 1342–1347.

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