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ON WEIGHTED GENERALIZATION OF THE HERMITE–HADAMARD INEQUALITY

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Abstract. The results obtained in this paper are a correction of the main results obtained in [14], for which we also give an alternative proof and improvement. We also study some new monotonic conditions under which various generalizations of the Hermite-Hadamard inequality are valid. Furthermore, we give an improvement of the obtained results.

1. Introduction

Let f be a convex function on $[a,b]\subset\mathbb{R}$. The following inequality

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)dx\leqslant\frac{f(a)+f(b)}{2} \tag{1.1}$$

is known in the literature as the integral Hermite-Hadamard inequality [10].

In [5] Fejér (see also [12, p. 138]) obtained the following weighted generalization of the Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx\leqslant\int_a^b f(x)g(x)dx\leqslant\frac{f(a)+f(b)}{2}\int_a^b g(x)dx, \tag{1.2}$$

where f is a convex function on the interval $[a,b]$, and g is positive function on the same interval such that

$$g(a+t)=g(b-t),\quad 0\leqslant t\leqslant\frac{1}{2}(a+b),$$

i.e. $y=g(x)$ is a symmetric curve with respect to the straight line which contains the point $(\frac{1}{2}(a+b),0)$ and is normal to the x-axis.

It is well known that the Hermite-Hadamard inequality plays an important role in nonlinear analysis. Over the last decade, this classical inequality has been improved and generalized in a number of ways; there has been a large number of research papers written on this subject, see [3], [4], [6], and [8] and the references therein.

G. Zabandan and A. Kiliçman [14] gave a different weighted version of the Hermite-Hadamard inequality:

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THEOREM 1.1. *Let* $f\colon [a,b] \to \mathbb{R}$ *be a differentiable convex function and let* $g\colon [a,b] \to [0,\infty]$ *be a continuous function.*

(i) If g *is decreasing on* $[a,b]$ *,then*

$$\frac{1}{\int_a^b g(x)dx}\int_a^b f(x)g(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1.3}$$

(ii) If g *is increasing on* $[a,b]$, *then*

$$f\Big(\frac{a+b}{2}\Big) \leqslant \frac{1}{\int_a^b g(x)dx}\int_a^b f(x)g(x)dx. \tag{1.4}$$

Unfortunately, inequalities (1.3) and (1.4) are not valid under the given assumptions. Functions $f(x) = b-x$ and $g(x) = b-x$ satisfy the conditions of Theorem 1.1 (i). However, for those functions the inequality (1.3) is reversed.

Analogously, functions $f(x) = b-x$ and $g(x) = x-a$ satisfy the conditions of Theorem 1.1.(ii), but for them the inequality (1.4) is also reversed.

The aim of this paper is to correct the conditions under which the inequalities (1.3) and (1.4) are valid and to give new monotonic conditions under which some generalizations of the Hermite-Hadamard inequality hold. We will also give a generalization of the obtained results for positive linear functionals. Furthermore, we will give improvements of the obtained results.

2. Results

DEFINITION 2.1. We say that a function $f\colon [a,b] \to \mathbb{R}$ is increasing (decreasing) in mean with respect to a weight function $g\colon [a,b] \to \langle 0,\infty\rangle$ if the arithmetic mean $A(f;a,x)$ is an increasing (decreasing) function, where

$$A(f;a,x) = \frac{1}{\int_a^x g(t)dt}\int_a^x f(t)g(t)dt.$$

THEOREM 2.2. *Let* $f\colon [a,b] \to \mathbb{R}$ *be a differentiable function and let* $g\colon [a,b] \to \langle 0,\infty\rangle$ *be an integrable function. If* $\frac{f'(x)}{g(x)}$ *is increasing in mean with respect to the function* g, *then the inequality* (1.3) *is valid. If* $\frac{f'(x)}{g(x)}$ *is decreasing in mean with respect to the function* g, *then the inequality in* (1.3) *is reversed.*

Proof. Let's denote

$$H(x) = \int_a^x f(t)g(t)dt - \frac{1}{2}\Big(f(a)+f(x)\Big)\int_a^x g(t)dt.$$

We assume that the function $\frac{f'}{g}$ is increasing in mean. We will show that $H'(x) \leqslant 0$. We have

$$H'(x) = f(x)g(x) - \frac{1}{2}f'(x)\int_a^x g(t)dt - \frac{1}{2}\Big(f(a)+f(x)\Big)g(x)$$
$$= \frac{1}{2}\Big[g(x)(f(x)-f(a)) - f'(x)\int_a^x g(t)dt\Big]. \tag{2.1}$$

On the other side we have

$$A\Big(\frac{f'}{g};a,x\Big)' = \left[\frac{\int_a^x g(t)\frac{f'(t)}{g(t)}dt}{\int_a^x g(t)dt}\right]' = \left[\frac{f(x)-f(a)}{\int_a^x g(t)dt}\right]'$$
$$= \frac{f'(x)\int_a^x g(t)dt - (f(x)-f(a))g(x)}{\Big(\int_a^x g(t)dt\Big)^2} \geqslant 0 \tag{2.2}$$

because $\frac{f'}{g}$ is increasing in mean with respect to the weight function g.

Therefore, according to (2.1) and (2.2) we obtain that $H'(x) \leqslant 0$. Hence, $H(b) \leqslant H(a) = 0$, so the proof is complete. □

COROLLARY 2.3. *Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function and let $g\colon [a,b] \to \langle 0,\infty\rangle$ be an integrable function.*

(i) If the function $\frac{f'}{g}$ is increasing, then the inequality (1.3) holds.

(ii) If the function $\frac{f'}{g}$ is decreasing, then the inequality (1.3) is reversed.

Proof. It is well known that if the function $\frac{f'}{g}$ is increasing, then it is increasing in mean with respect to the weight function g. □

COROLLARY 2.4. *Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function and let $g\colon [a,b] \to \langle 0,\infty\rangle$ be an integrable function.*

(i) If the function f is convex, and f and g are monotonic in opposite directions, then the inequality (1.3) holds.

(ii) If the function f is concave, and f and g are monotonic in the same direction, then the inequality (1.3) is reversed.

Proof. If the function f is convex, then f' is increasing. Let us assume that f is an increasing function, ie. f' is positive. In this case function g is decreasing, so the function $\frac{f'}{g}$ is increasing. According to Corollary 2.3 (i), the inequality (1.3) is valid. The case where the function f is decreasing and g is increasing is proved analogously.

If f is a concave function, then f' is decreasing. If f is increasing, then f' is positive. Given that g is a positive function and monotonic in the same direction as f (which in this case is an increasing function), we see that $\frac{f'}{g}$ is decreasing. Now from Corollary 2.3 (ii) we have that the inequality (1.3) is reversed. The case where both functions f and g are decreasing is proved analogously. □

LEMMA 2.5. *Let $g\colon [a,b] \to \langle 0,\infty\rangle$ be an integrable function. If g is decreasing, then we have*

$$\frac{\int_a^b xg(x)dx}{\int_a^b g(x)dx} \leqslant \frac{a+b}{2} \tag{2.3}$$

If g is increasing, then the inequality sign in (2.3) is reversed.

Proof. Since the function $f(x) = x$ is increasing and concave, Lemma 2.5 easily follows from Corollary 2.4. □

THEOREM 2.6. *(i) Let f be a convex function. If f and g are monotonic in the same direction, then the inequality (1.4) holds.*

(ii) Let f be a concave function. If f and g are monotonic in the opposite direction, then the inequality sign in (1.4) is reversed.

Proof. Let f be a convex function. From Jensen's inequality we have

$$\frac{1}{\int_a^b g(x)dx}\int_a^b f(x)g(x)dx \geqslant f\left[\frac{\int_a^b xg(x)dx}{\int_a^b g(x)dx}\right]. \tag{2.4}$$

Now the inequality (1.4) follows directly from Lemma 2.5.

If f is a concave function, then the inequality (2.4) is reversed. We can use Lemma 2.5 again to obtain the reversed inequality in (1.4). □

A classic result due to Čebyšev is stated in the following theorem found in [12, p. 197].

THEOREM 2.7. ([12, p. 197]) *Let $f,g\colon [a,b] \to \mathbb{R}$ and $p\colon [a,b] \to \mathbb{R}_+$ be integrable functions. If f and g are monotonic in the same direction, then*

$$\int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx \geqslant \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx, \tag{2.5}$$

provided that the integrals exist. If f and g are monotonic in the opposite direction, then the reverse of the inequality in (2.5) is valid. In both cases, equality in (2.5) holds iff either g or f is constant almost everywhere.

REMARK 2.8. Corollary 2.4 and Theorem 2.6 can also be proved by using Čebyšev's inequality (2.5). Let $f\colon [a,b] \to \mathbb{R}$ be an integrable convex function and $g\colon [a,b] \to \mathbb{R}_+$ an integrable function. Let's take $p \equiv 1$ in (2.5). If f and g are monotonic in the same direction, then we have

$$(b-a)\int_a^b f(x)g(x)dx \geqslant \int_a^b f(x)dx \int_a^b g(x)dx. \tag{2.6}$$

Since $g \geqslant 0$, we can rearrange (2.6) and get:

$$\frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \geqslant \frac{1}{b-a}\int_a^b f(x)dx. \tag{2.7}$$

Inequality (1.4) now follows directly from the first part of the Hermite-Hadamard inequality (1.1). If f and g are monotonic in the opposite direction and if f is concave, then the inequality signs in (2.7) and (1.1) are reversed. The reversal of inequality (1.4) is now obtained in the same way as described above.

If f and g are monotonic in the opposite direction, after rearranging the Čebyšev inequality we have

$$\frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \leqslant \frac{1}{b-a}\int_a^b f(x)dx. \tag{2.8}$$

Inequality (1.3) now follows directly from the second part of the Hermite-Hadamard inequality (1.1). If f and g are monotonic in the same direction and if f is concave, then the inequality (2.7) is valid and the inequality signs in (1.1) are reversed. The reversal of inequality (1.3) is now obtained in the same way as described above.

REMARK 2.9. There exist several results which show that Čebyšev inequalities are valid under weaker conditions (see [12, p. 198]), so it is clear from the previous remark that Corollary 2.4 and Theorem 2.6 will also hold under the same conditions:

(i) The condition that the functions f and g are monotonic in the same (opposite) direction can be replaced by the condition that they are similarly (oppositely) ordered.

Note that the functions $f,g\colon I \to \mathbb{R}$ are said to be similarly ordered if

$$(f(x)-f(y))(g(x)-g(y)) \geqslant 0, \text{ for every } x,y \in I$$

holds, and they are said to be oppositely ordered in the reversed inequality holds.

(ii) The condition that the functions f and g are monotonic in the same (opposite) direction can also be replaced by the condition that they are monotonic in mean, i.e. that functions

$$A(f;a,x,p) = \int_a^x p(t)f(t)dt \Big/ \int_a^x p(t)dt$$

and

$$B(g;a,x,p) = \int_a^x p(t)g(t)dt \Big/ \int_a^x p(t)dt$$

are monotone.

When we take $p \equiv 1$ as we did in the previous remark, we obtain that Corollary 2.4 and Theorem 2.6 hold for a convex integrable function f and positive integrable function g for which the functions $A(f;a,x) = \int_a^x f(t)dt/(x-a)$ and $B(g;a,x) = \int_a^x g(t)dt/(x-a)$ are monotonic.

(iii) Steffensen ([13]) noted that Čebyšev's inequality is valid when f is an increasing function on $[a,b]$ and g satisfies the condition

$$\frac{\int_a^x p(t)g(t)dt}{\int_a^x p(t)dt} \leqslant \frac{\int_a^b p(t)g(t)dt}{\int_a^b p(t)dt}.$$

Again, when we take $p \equiv 1$, we easily obtain that Corollary 2.4 and Theorem 2.6 are valid for f convex and increasing on $[a,b]$ and $g \geqslant 0$ such that $\frac{\int_a^x g(t)dt}{x-a} \leqslant \frac{\int_a^b g(t)dt}{b-a}$ for every $x \in [a,b]$.

The inequality (1.3) can be improved. In order to do that, we first need to state a result which is a special case of [11, p. 717, Theorem 1] obtained for $n = 2$.

LEMMA 2.10. *Let f be a convex function on D_f, $x,y \in D_f$ and $p,q \in [0,1]$ such that $p+q=1$. Then*

$$\min\{p,q\}\left[f(x)+f(y)-2f\left(\frac{x+y}{2}\right)\right] \leqslant pf(x)+qf(y)-f(px+qy) \qquad (2.9)$$
$$\leqslant \max\{p,q\}\left[f(x)+f(y)-2f\left(\frac{x+y}{2}\right)\right].$$

THEOREM 2.11. *Let $f\colon [a,b] \to \mathbb{R}$ be a convex function and let $g\colon [a,b] \to [0,\infty\rangle$ be a continuous function. If the functions f and g are monotonic in the opposite direction, then*

$$\frac{1}{\int_a^b g(x)dx}\int_a^b f(x)g(x)dx \leqslant \frac{f(a)+f(b)}{2} - \left(\frac{1}{2} - \frac{\tilde{G}}{b-a}\right)\delta_f, \qquad (2.10)$$

where

$$\tilde{G} = \frac{1}{\int_a^b g(x)dx}\int_a^b \left|x - \frac{a+b}{2}\right| g(x)dx \text{ and } \delta_f = f(a)+f(b)-2f\left(\frac{a+b}{2}\right).$$

If f is conccave and the functions f and g are monotonic in the same direction, then the inequality (2.10) is reversed.

Proof. We will only prove the convex case since the concave case follows in a similar way. Let f be a convex function and $x \in [a,b]$. Since $\frac{x-a}{b-a}, \frac{b-x}{b-a} \in [0,1]$ and

$\frac{x-a}{b-a}+\frac{b-x}{b-a}=1$, from (2.9) it follows that

$$\begin{aligned}
f(x) &= f\Big(\frac{b-x}{b-a}a+\frac{x-a}{b-a}b\Big)\\
&\leqslant \frac{b-x}{b-a}f(a)+\frac{x-a}{b-a}f(b)-\min\Big\{\frac{x-a}{b-a},\frac{b-x}{b-a}\Big\}\Big[f(a)+f(b)-2f\Big(\frac{a+b}{2}\Big)\Big]\\
&= \frac{bf(a)-af(b)}{b-a}+\frac{f(b)-f(a)}{b-a}x-\min\Big\{\frac{x-a}{b-a},\frac{b-x}{b-a}\Big\}\delta_f\\
&= \frac{bf(a)-af(b)}{b-a}+\frac{f(b)-f(a)}{b-a}x-\Big(\frac{1}{2}-\frac{1}{b-a}\Big|x-\frac{a+b}{2}\Big|\Big)\delta_f
\end{aligned} \tag{2.11}$$

We can multiply (2.11) by function g because it is positive and then integrate the obtained inequalities over $[a,b]$. After dividing by $\int_a^b g(x)dx>0$ we get:

$$\begin{aligned}
\frac{1}{\int_a^b g(x)dx}\int_a^b f(x)g(x)dx \leqslant \frac{bf(a)-af(b)}{b-a}+\frac{f(b)-f(a)}{b-a}\frac{\int_a^b xg(x)dx}{\int_a^b g(x)dx}\\
-\Big(\frac{1}{2}-\frac{\tilde{G}}{b-a}\Big)\delta_f
\end{aligned} \tag{2.12}$$

Now let us suppose that functions f and g are monotonic in the opposite direction. If f is increasing, then $\frac{f(b)-f(a)}{b-a}\geqslant 0$ holds. In this case function g is decreasing, so if we take $f(x)=x$ which is convex and increasing in (1.3), we directly obtain

$$\frac{\int_a^b xg(x)dx}{\int_a^b g(x)dx}\leqslant\frac{a+b}{2}. \tag{2.13}$$

Inequality (2.10) now directly follows from (2.13) and $\frac{f(b)-f(a)}{b-a}\geqslant 0$.

If f is decreasing, then $\frac{f(b)-f(a)}{b-a}\geqslant 0$ holds. In that case function g is increasing, so the inequality (2.13) is reversed and (2.10) directly follows. □

REMARK 2.12. It is easy to see that $0<\frac{1}{2}-\frac{1}{b-a}\Big|x-\frac{a+b}{2}\Big|\leqslant\frac{1}{2}$ holds for every $x\in[a,b]$. If we multiply those inequalities by $g(x)\geqslant 0$, integrate them over $[a,b]$ and then divide the obtained inequalities by $\int_a^b g(x)dx$ we easily get $0<\frac{1}{2}-\frac{\tilde{G}}{b-a}\leqslant\frac{1}{2}$. We see that Theorem 2.11 really gives an improvement of the inequality (1.3) because for a convex function f we have $f(a)+f(b)-2f\Big(\frac{a+b}{2}\Big)\geqslant 0$, so it follows that $\Big(\frac{1}{2}-\frac{\tilde{G}}{b-a}\Big)\delta_f\geqslant 0$.

3. Further generalizations

In this section we will give new monotonic conditions under which some generalizations of the Hermite-Hadamard inequality hold and we will give a generalization of the results obtained in the previous section for positive linear functionals.

First we will state two generalizations of the Hermite-Hadamard inequality.

THEOREM 3.1. ([12, p. 143]) *Let p,q be given positive numbers and $a_1 \leqslant a < b \leqslant b_1$. Then the inequalities*

$$f\Big(\frac{pa+qb}{p+q}\Big) \leqslant \frac{1}{2y}\int_{T-y}^{T+y} f(x)dx \leqslant \frac{pf(a)+qf(b)}{p+q} \tag{3.1}$$

hold for $T=(pa+qb)/(p+q)$, $y>0$, and all continuous convex functions $f\colon [a_1,b_1]\to \mathbb{R}$ iff

$$y \leqslant \frac{b-a}{p+q}\min\{p,q\}. \tag{3.2}$$

The second result that we state is a weighted generalization of the previous result and it was proved by Brenner and Alzer in [2], but in fact it is a Féjer type variant of (3.1).

THEOREM 3.2. ([2]) *Let p,q be positive real numbers and $g\colon [a,b]\to \mathbb{R}_0^+$ and integrable and symmetric function with respect to the line $x=(pa+qb)/(p+q)=T$ in the sense that $g(T-t)=g(T+t)$ for all $t\in \left[0,\frac{b-a}{p+q}\min\{p,q\}\right]$. If $f\colon [a,b]\to\mathbb{R}$ is a convex function, then*

$$f\Big(\frac{pa+qb}{p+q}\Big)\int_{T-y}^{T+y} g(x)dx \leqslant \int_{T-y}^{T+y} f(x)g(x)dx \leqslant \frac{pf(a)+qf(b)}{p+q}\int_{T-y}^{T+y} g(x)dx \tag{3.3}$$

for all $y\in\mathbb{R}$ such that (3.2) holds.

In our first two results we give new conditions, different from the ones that Féjer gave, under which the inequalities in (3.3) from Theorem 3.2 are valid.

THEOREM 3.3. *Let p,q be positive real numbers and $g\colon [a,b]\to\mathbb{R}_0^+$ integrable function. If $f\colon [a,b]\to\mathbb{R}$ is a convex function, then for all $y\in\mathbb{R}$ such that $0<y\leqslant \frac{b-a}{p+q}\min\{p,q\}$ we have*

$$f\Big(\frac{pa+qb}{p+q}\Big)\int_{T-y}^{T+y} g(x)dx \leqslant \int_{T-y}^{T+y} f(x)g(x)dx \tag{3.4}$$

if the functions f and g are monotonic in the same direction. If f is concave and f and g are monotonic in the opposite direction, then the inequality sign in (3.4) is reversed.

Proof. Let f be a convex function. From Jensen's inequality we have:

$$f\left(\frac{\int_a^b xg(x)dx}{\int_a^b g(x)dx}\right) \leqslant \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx}. \tag{3.5}$$

If conditions $a \leqslant T-y \leqslant b$ and $a \leqslant T+y \leqslant b$ are satisfied, we can make supstitutions $a = T-y$ and $b = T+y$ in (3.5) and get:

$$f\left(\frac{\int_{T-y}^{T+y} xg(x)dx}{\int_{T-y}^{T+y} g(x)dx}\right) \leqslant \frac{\int_{T-y}^{T+y} f(x)g(x)dx}{\int_{T-y}^{T+y} g(x)dx}. \tag{3.6}$$

Since $0 < y \leqslant \frac{b-a}{p+q}\min\{p,q\}$, conditions $a \leqslant T-y \leqslant b$ and $a \leqslant T+y \leqslant b$ are satisfied. On the other side, function $h(x) = x$ is concave and increasing, so if g is decreasing from Lemma 2.5 we have

$$\frac{\int_{T-y}^{T+y} xg(x)dx}{\int_{T-y}^{T+y} g(x)dx} \leqslant \frac{T-y+T+y}{2} = T = \frac{pa+qb}{p+q}. \tag{3.7}$$

In this case function f is also decreasing, so when we apply it to (3.7) we get a reversed inequality sign. After combining the obtained inequality with (3.6) we get (3.4). If f and g are both increasing then the inequality sign in (3.7) is reversed, but after applying f to that inequality it stays the same and (3.4) again follows from (3.6).

If f is concave, then the inequality (3.6) is reversed and the rest of the proof follows in a similar way. □

THEOREM 3.4. *Let p,q be positive real numbers and $g\colon [a,b] \to \mathbb{R}_0^+$ integrable function. If $f\colon [a,b] \to \mathbb{R}$ is a convex function, then for all $y \in \mathbb{R}$ such that $0 < y \leqslant \frac{b-a}{p+q}\min\{p,q\}$ we have*

$$\frac{1}{\int_{T-y}^{T+y} g(x)dx}\int_{T-y}^{T+y} f(x)g(x)dx \leqslant \frac{f(T-y)+f(T+y)}{2} \leqslant \frac{pf(a)+qf(b)}{p+q}, \tag{3.8}$$

if the functions f and g are monotonic in the opposite direction. Inequality (3.8) is reversed if f is concave and f and g are monotonic in the same direction.

Proof. Let f be a convex function, f and g monotonic in the opposite direction and let $h\colon [a,b] \to \mathbb{R}$ be such that $m \leqslant h(x) \leqslant M$, $m < M$, for every $x \in [a,b]$. From the Lah-Ribarič inequality we have:

$$\begin{aligned}\frac{\int_a^b f(h(x))g(x)dx}{\int_a^b g(x)dx} &\leqslant \frac{M-\bar{h}}{M-m}f(m) + \frac{\bar{h}-m}{M-m}f(M)\\ &= \frac{Mf(m)-mf(M)}{M-m} + \frac{f(M)-f(m)}{M-m}\bar{h},\end{aligned} \tag{3.9}$$

where $\bar{h} = \frac{1}{\int_a^b g(x)dx} \int_a^b h(x)g(x)dx$. Conditions $a \leqslant T - y \leqslant b$ and $a \leqslant T + y \leqslant b$ are satisfied for for all $y \in \left(0, \frac{b-a}{p+q} \min\{p,q\}\right]$, so we can make supstitutions $m = a = T - y$, $M = b = T + y$ and $h(x) = x$ in (3.9) and get:

$$\frac{\int_{T-y}^{T+y} f(x)g(x)dx}{\int_{T-y}^{T+y} g(x)dx} \leqslant \frac{(T+y)f(T-y) - (T-y)f(T+y)}{2y} + \frac{f(T+y) - f(T-y)}{2y} \frac{\int_{T-y}^{T+y} xg(x)dx}{\int_{T-y}^{T+y} g(x)dx}, \tag{3.10}$$

If f is increasing, then $\frac{f(T+y) - f(T-y)}{2y} \geqslant 0$ holds. In that case function g is decreasing, so inequality (3.7) is valid and from (3.10) we get

$$\frac{\int_{T-y}^{T+y} f(x)g(x)dx}{\int_{T-y}^{T+y} g(x)dx} \leqslant \frac{f(T+y) + f(T-y)}{2} \leqslant \left(\frac{b-T}{b-a} f(a) + \frac{T-a}{b-a} f(b)\right) = \frac{pf(a) + qf(b)}{p+q} \tag{3.11}$$

due to convexity of f. If f is decreasing, then $\frac{f(T+y) - f(T-y)}{2y} \leqslant 0$ holds. In that case function g is increasing, so inequality (3.7) is reversed and we get (3.8) in the same way as above.

If the function f is concave and f and g are monotonic in the same direction, then the inequality in (3.10) is reversed, and the rest of the proof follows in the same way as in the convex case. □

REMARK 3.5. By making substitutions $a \leftrightarrow T - y$ and $b \leftrightarrow T + y$ in (1.2) it can be easily shown that (1.2) and (3.3) are equivalent (see [12, p. 144]). Therefore, it is obvious that the inequalities (3.4) and (3.8) respectively will be valid if any of the conditions under which (1.4) and (1.3) are valid holds.

Now, let E be a nonempty set and let L be a linear class of real-valued functions $f : E \to \mathbb{R}$ containing constants, that is, having the properties:

L1: $f, g \in L \Rightarrow (af + bg) \in L$ for all $a, b \in \mathbb{R}$;

L2: $\mathbf{1} \in L$, i.e., if $f(t) = 1$ for every $t \in E$, then $f \in L$.

In other words, L is a subspace of the vector space $\mathbb{R}^E$ over $\mathbb{R}$ containing $\mathbf{1}$.

We also consider positive linear functionals $A : L \to \mathbb{R}$, that is, we assume that:

A1: $A(af + bg) = aA(f) + bA(g)$ for $f, g \in L$ and $a, b \in \mathbb{R}$;

A2: $f \in L$, $f(t) \geqslant 0$ for every $t \in E \Rightarrow A(f) \geqslant 0$ (A is positive).

If additionally the condition $A(\mathbf{1}) = 1$ is satisfied, we say that A is a positive normalized linear functional.

Throughout this paper without further noticing when using $[m,M]$ we assume that $-\infty < m < M < \infty$.

Jessen [7] gave the following generalization of Jensen's inequality for convex functions (see also [12, p. 47]):

THEOREM 3.6. ([7]) *Let L satisfy properties $L1, L2$ on a nonempty set E, and assume that f is a continous convex function on an interval $I \subset \mathbb{R}$. If A is a positive linear functional with $A(\mathbf{1}) = 1$, then for all $g \in L$ such that $f(g) \in L$ we have $A(g) \in I$ and*

$$f(A(g)) \leqslant A(f(g)). \tag{3.12}$$

The following result is proved in [1] by Beesack and Pečarić (see also [12, p. 98]):

THEOREM 3.7. ([1]) *Let f be convex on $I = [m,M]$. Let L satisfy conditions $L1, L2$ on E and let A be any positive linear functional on L with $A(\mathbf{1}) = 1$. Then for every $g \in L$ such that $f(g) \in L$ (so that $m \leqslant g(t) \leqslant M$ for all $t \in E$), we have*

$$A(f(g)) \leqslant \frac{M - A(g)}{M - m} f(m) + \frac{A(g) - m}{M - m} f(M). \tag{3.13}$$

We state a generalization of the Hermite-Hadamard inequality found in [12].

THEOREM 3.8. ([12]) *Let f be a continuous convex function on an interval $I \supset [m,M]$. Suppose that $g\colon E \to \mathbb{R}$ satisfies $m \leqslant g(t) \leqslant M$ for every $t \in E$, $g \in L$ and $f(g) \in L$. Let $A\colon L \to \mathbb{R}$ be a positive linear functional with $A(\mathbf{1}) = 1$, and let $p = p_g$, $q = q_g$ be nonnegative real numbers (with $p + q > 0$) for which*

$$A(g) = \frac{pm + qM}{p + q}. \tag{3.14}$$

Then

$$f\left(\frac{pm + qM}{p + q}\right) \leqslant A(f(g)) \leqslant \frac{pf(m) + qf(M)}{p + q}. \tag{3.15}$$

The following result is a generalization of Theorem 2.2 for positive linear functionals and it gives monotonic conditions under which the second inequality in (3.15) from Theorem 3.8 is valid.

THEOREM 3.9. *Let f be a continuous convex function on an interval $I \supset [m,M]$. Suppose that $g\colon E \to \mathbb{R}$ satisfies $m \leqslant g(t) \leqslant M$ for every $t \in E$, $g \in L$ and $f(g) \in L$. Let $A\colon L \to \mathbb{R}$ be a positive linear functional with $A(\mathbf{1}) = 1$, and let $p = p_g$, $q = q_g$ be nonnegative real numbers. If the function f is increasing and $A(g) \leqslant (pm + qM)/(p + q)$ is valid, then the following inequality holds:*

$$A(f(g)) \leqslant \frac{pf(m) + qf(M)}{p + q}. \tag{3.16}$$

Inequality (3.16) also holds if f is decreasing and $A(g) \geqslant (pm+qM)/(p+q)$. If f is an increasing concave function and $A(g) \geqslant (pm+qM)/(p+q)$, or if f is a decreasing concave function and $A(g) \leqslant (pm+qM)/(p+q)$, the inequality sign in (3.16) is reversed.

Proof. Let f be a convex function. When we rearrange the Lah-Ribarič inequality (3.13), we have

$$A(f(g)) \leqslant \frac{M-A(g)}{M-m}f(m)+\frac{A(g)-m}{M-m}f(M) = \frac{Mf(m)-mf(M)}{M-m}+\frac{f(M)-f(m)}{M-m}A(g) \tag{3.17}$$

If f is increasing, then $\frac{f(M)-f(m)}{M-m} \geqslant 0$ holds, so if $A(g) \leqslant \frac{pm+qM}{p+q}$, from (3.17) we easily get (3.16). If f is decreasing, then $\frac{f(M)-f(m)}{M-m} \leqslant 0$ holds, so if $A(g) \geqslant \frac{pm+qM}{p+q}$, from (3.17) we again easily get (3.16).

If f is a concave function, then the inequality sign in (3.17) is reversed. The rest of the proof follows in a similar way. □

Our next result is a generalization of Theorem 2.6 for positive linear functionals and it gives monotonic conditions under which the first inequality in (3.15) from Theorem 3.8 is valid.

THEOREM 3.10. *Let f be a continuous convex function on an interval $I \supset [m,M]$. Suppose that $g\colon E \to \mathbb{R}$ satisfies $m \leqslant g(t) \leqslant M$ for every $t \in E$, $g \in L$ and $f(g) \in L$. Let $A\colon L \to \mathbb{R}$ be a positive linear functional with $A(\mathbf{1}) = 1$, and let $p = p_g$, $q = q_g$ be nonnegative real numbers. If the function f is increasing and $A(g) \geqslant (pm+qM)/(p+q)$ is valid, then we have*

$$f\left(\frac{pm+qM}{p+q}\right) \leqslant A(f(g)). \tag{3.18}$$

Inequality (3.18) also holds if f is decreasing and $A(g) \leqslant (pm+qM)/(p+q)$. If f is an increasing concave function and $A(g) \leqslant (pm+qM)/(p+q)$, or if f is a decreasing concave function and $A(g) \geqslant (pm+qM)/(p+q)$, the inequality sign in (3.18) is reversed.

Proof. We will only prove the case when f is convex. Jessen's inequality (3.12) states that $A(f(g)) \geqslant f(A(g))$.

If f is increasing, when we apply it to $A(g) \geqslant \frac{pm+qM}{p+q}$, we get $f(A(g)) \geqslant f\left(\frac{pm+qM}{p+q}\right)$, which in combination with Jessen's inequality gives us (3.18). If the function f is decreasing, when we apply it to $A(g) \leqslant \frac{pm+qM}{p+q}$, we again get $f(A(g)) \geqslant f\left(\frac{pm+qM}{p+q}\right)$, and (3.18) follows in the same way as described above.

If f is a concave function, Jessen's inequality states $A(f(g)) \leqslant f(A(g))$ and the rest of the proof follows in a similar way. □

The following two results are simple consenquences of Theorem 3.9 and Theorem 3.10 obtained by choosing $p = q = 1$.

COROLLARY 3.11. *Let f be a continuous convex function on an interval $I \supset [m,M]$. Suppose that $g\colon E \to \mathbb{R}$ satisfies $m \leqslant g(t) \leqslant M$ for every $t \in E$, $g \in L$ and $f(g) \in L$. Let $A\colon L \to \mathbb{R}$ be a positive linear functional with $A(\mathbf{1}) = 1$. If the function f is increasing and $A(g) \leqslant (m+M)/2$ is valid, then the following inequality holds.*

$$A(f(g)) \leqslant \frac{f(m)+f(M)}{2} \tag{3.19}$$

Inequality (3.19) also holds if f is decreasing and $A(g) \geqslant (m+M)/2$. If f is an increasing concave function and $A(g) \geqslant (m+M)/2$, or if f is a decreasing concave function and $A(g) \leqslant (m+M)/2$, the inequality sign in (3.19) is reversed.

COROLLARY 3.12. *Let f be a continuous convex function on an interval $I \supset [m,M]$. Suppose that $g\colon E \to \mathbb{R}$ satisfies $m \leqslant g(t) \leqslant M$ for every $t \in E$, $g \in L$ and $f(g) \in L$. Let $A\colon L \to \mathbb{R}$ be a positive linear functional with $A(\mathbf{1}) = 1$. If the function f is increasing and $A(g) \geqslant (m+M)/2$ is valid, then we have*

$$f\left(\frac{m+M}{2}\right) \leqslant A(f(g)). \tag{3.20}$$

Inequality (3.20) also holds if f is decreasing and $A(g) \leqslant (m+M)/2$. If f is an increasing concave function and $A(g) \leqslant (m+M)/2$, or if f is a decreasing concave function and $A(g) \geqslant (m+M)/2$, the inequality sign in (3.20) is reversed.

4. Improvements

In order to give an improvement of Theorem 3.9 first we will need to equip our linear class of functions with an additional property denoted by:

L3: $f,g \in L \Rightarrow \min\{f,g\} \in L \wedge \max\{f,g\} \in L$ (lattice property)

M. Klaričić Bakula, J. Pečarić and J. Perić in their paper [9] gave the following refinement of the converse Jensen inequality:

THEOREM 4.1. ([9]) *Let L satisfy L1, L2, L3 on a nonempty set E and let A be a positive normalized linear functional. If f is a convex function on $[m,M]$ then for all $g \in L$ such that $f(g) \in L$ we have $A(g) \in [m,M]$ and*

$$A(f(g)) \leqslant \frac{M-A(g)}{M-m} f(m) + \frac{A(g)-m}{M-m} f(M) - A(\tilde{g})\delta_f, \tag{4.1}$$

where

$$\tilde{g} = \frac{1}{2}\mathbf{1} - \frac{|g - \frac{m+M}{2}\mathbf{1}|}{M-m}, \quad \delta_f = f(m) + f(M) - 2f\Big(\frac{m+M}{2}\Big). \tag{4.2}$$

Let $f, g, h\colon [a,b] \to \mathbb{R}$ be integrable functions such that $g(x) \geqslant 0$ and $a \leqslant h(x) \leqslant b$ and let us define $A(f) = \frac{1}{\int_a^b g(x)dx}\int_a^b g(x)f(x)dx$. It is clear that $A(\mathbf{1}) = 1$, so A is normalized positive linear functional. Now from (4.1) we have

$$\frac{1}{\int_a^b g(x)dx}\int_a^b g(x)f(h(x))dx \leqslant \frac{b-\bar{h}}{b-a}f(a) + \frac{\bar{h}-a}{b-a}f(b) - \Big(\frac{1}{2} - \frac{\tilde{H}}{b-a}\Big)\delta_f, \tag{4.3}$$

where

$$\bar{h} = \frac{1}{\int_a^b g(x)dx}\int_a^b g(x)h(x)dx, \quad \tilde{H} = \frac{1}{\int_a^b g(x)dx}\int_a^b \Big|h(x) - \frac{a+b}{2}\Big|g(x)dx \tag{4.4}$$

and

$$\delta_f = f(a) + f(b) - 2f\Big(\frac{a+b}{2}\Big). \tag{4.5}$$

Now we can prove a refinement of Theorem 3.4.

THEOREM 4.2. *Let p, q be positive real numbers and $g\colon [a,b] \to \mathbb{R}_0^+$ integrable function. If $f\colon [a,b] \to \mathbb{R}$ is a convex function and f and g are monotonic in the opposite direction, then for all $y \in \mathbb{R}$ such that $0 < y \leqslant \frac{b-a}{p+q}\min\{p,q\}$ we have*

$$\begin{aligned}\frac{\int_{T-y}^{T+y} f(x)g(x)dx}{\int_{T-y}^{T+y} g(x)dx} &\leqslant \frac{f(T+y)+f(T-y)}{2} - \Big(\frac{1}{2} - \frac{\tilde{G}}{2y}\Big)\delta_f \\ &\leqslant \frac{pf(a)+qf(b)}{p+q} - \Big(\frac{1}{2} - \frac{\tilde{G}}{2y}\Big)\delta_f \end{aligned} \tag{4.6}$$

where $\tilde{G} = \frac{1}{\int_{T-y}^{T+y} g(x)dx}\int_{T-y}^{T+y}|x-T|g(x)dx$ and δ_f is defined by (4.5).

Inequality (4.6) is reversed if f is concave and f and g are monotonic in the same direction.

Proof. Let f be a convex function, f and g monotonic in the opposite direction and let $h\colon [a,b] \to \mathbb{R}$ be such that $a \leqslant h(x) \leqslant b$ for every $x \in [a,b]$. From (4.3) we have:

$$\begin{aligned}\frac{\int_a^b f(h(x))g(x)dx}{\int_a^b g(x)dx} &\leqslant \frac{b-\bar{h}}{b-a}f(a) + \frac{\bar{h}-a}{b-a}f(b) - \Big(\frac{1}{2} - \frac{\tilde{H}}{2y}\Big)\delta_f \\ &= \frac{bf(a)-af(b)}{b-a} + \frac{f(b)-f(a)}{b-a}\bar{h} - \Big(\frac{1}{2} - \frac{\tilde{H}}{2y}\Big)\delta_f, \end{aligned} \tag{4.7}$$

where $\bar{h}$ and $\tilde{H}$ are defined by (4.4). Conditions $a \leqslant T-y \leqslant b$ and $a \leqslant T+y \leqslant b$ are satisfied for for all $y \in \left(0, \frac{b-a}{p+q}\min\{p,q\}\right]$, so we can make supstitutions $h(x)=x$, $a=T-y$ and $b=T+y$ in (4.7) and get:

$$\frac{\int_{T-y}^{T+y} f(x)g(x)dx}{\int_{T-y}^{T+y} g(x)dx} \leqslant \frac{(T+y)f(T-y)-(T-y)f(T+y)}{2y} + \frac{f(T+y)-f(T-y)}{2y}\frac{\int_{T-y}^{T+y} xg(x)dx}{\int_{T-y}^{T+y} g(x)dx} - \left(\frac{1}{2}-\frac{\tilde{G}}{2y}\right)\delta_f, \tag{4.8}$$

where $\tilde{G} = \frac{1}{\int_{T-y}^{T+y} g(x)dx}\int_{T-y}^{T+y}|x-T|g(x)dx$.

If f is increasing, then $\frac{f(T+y)-f(T-y)}{2y} \geqslant 0$ holds. In that case function g is decreasing, so from Lemma 2.5 we have

$$\frac{1}{\int_{T-y}^{T+y} g(x)dx}\int_{T-y}^{T+y} xg(x)dx \leqslant T. \tag{4.9}$$

Now by combining these inequalities with (4.8) we obtain

$$\begin{aligned}\frac{\int_{T-y}^{T+y} f(x)g(x)dx}{\int_{T-y}^{T+y} g(x)dx} &\leqslant \frac{f(T+y)+f(T-y)}{2} - \left(\frac{1}{2}-\frac{\tilde{G}}{2y}\right)\delta_f \\ &\leqslant \frac{1}{2}\left(\frac{b-T}{b-a}f(a)+\frac{T-a}{b-a}f(b)\right) - \left(\frac{1}{2}-\frac{\tilde{G}}{2y}\right)\delta_f \\ &= \frac{pf(a)+qf(b)}{p+q} - \left(\frac{1}{2}-\frac{\tilde{G}}{2y}\right)\delta_f \end{aligned} \tag{4.10}$$

due to convexity of f. If f is decreasing, then $\frac{f(T+y)-f(T-y)}{2y} \leqslant 0$ holds. In that case function g is increasing, so inequality (4.9) is reversed and we get (4.6) in the same way as above.

If the function f is concave and f and g are monotonic in the same direction, then the inequality in (4.8) is reversed, and the rest of the proof follows in the same way as in the convex case. □

Our following result is a simple consenquence of the previous theorem obtained by choosing $g \equiv 1$ and it provides a refinement of the second inequality in (3.1).

COROLLARY 4.3. *Let p,q be positive real numbers. If $f\colon [a,b] \to \mathbb{R}$ is a convex function, then for all $y \in \mathbb{R}$ such that $0 < y \leqslant \frac{b-a}{p+q}\min\{p,q\}$ we have*

$$\frac{1}{2y}\int_{T-y}^{T+y} f(x)dx \leqslant \frac{f(T+y)+f(T-y)}{2} - \frac{1}{4}\delta_f \leqslant \frac{pf(a)+qf(b)}{p+q} - \frac{1}{4}\delta_f \tag{4.11}$$

where $\delta_f = f(a) + f(b) - 2f\left(\frac{a+b}{2}\right)$. *Inequality (4.11) is reversed if* f *is concave.*

Our next result is a refinement of Theorem 3.9.

THEOREM 4.4. *Let* f *be a continuous convex function on an interval* $I \supset [m,M]$. *Suppose that* $g\colon E \to \mathbb{R}$ *satisfies* $m \leqslant g(t) \leqslant M$ *for every* $t \in E$, $g \in L$ *and* $f(g) \in L$. *Let* $A\colon L \to \mathbb{R}$ *be a positive linear functional with* $A(\mathbf{1}) = 1$, *and let* $p = p_g$, $q = q_g$ *be nonnegative real numbers. If the function* f *is increasing and* $A(g) \leqslant (pm+qM)/(p+q)$ *is valid, then we have*

$$A(f(g)) \leqslant \frac{pf(m)+qf(M)}{p+q} - A(\tilde{g})\delta_f, \tag{4.12}$$

where $\tilde{g}$ *and* δ_f *are defined by (4.2).*

Inequality (4.12) also holds if f *is decreasing and* $A(g) \geqslant (pm+qM)/(p+q)$. *If* f *is an increasing concave function and* $A(g) \geqslant (pm+qM)/(p+q)$, *or if* f *is a decreasing concave function and* $A(g) \leqslant (pm+qM)/(p+q)$, *the inequality sign in (4.12) is reversed.*

Proof. Let f be a convex function. When we rearrange the inequality (4.1) we get

$$\begin{aligned} A(f(g)) &\leqslant \frac{M-A(g)}{M-m}f(m) + \frac{A(g)-m}{M-m}f(M) - A(\tilde{g})\delta_f \\ &= \frac{Mf(m)-mf(M)}{M-m} + \frac{f(M)-f(m)}{M-m}A(g) - A(\tilde{g})\delta_f \end{aligned} \tag{4.13}$$

If f is increasing, then $\frac{f(M)-f(m)}{M-m} \geqslant 0$ holds, so if $A(g) \leqslant \frac{pm+qM}{p+q}$, from (4.13) we easily get (4.12). If f is decreasing, then $\frac{f(M)-f(m)}{M-m} \leqslant 0$ holds, so if $A(g) \geqslant \frac{pm+qM}{p+q}$, from (4.13) we again easily get (4.12).

If f is a concave function, then the inequality sign in (4.13) is reversed. The rest of the proof follows in a similar way. □

If we choose $p = q = 1$ in the previous theorem, we easily obtain the following result.

COROLLARY 4.5. *Let* f *be a continuous convex function on an interval* $I \supset [m,M]$. *Suppose that* $g\colon E \to \mathbb{R}$ *satisfies* $m \leqslant g(t) \leqslant M$ *for every* $t \in E$, $g \in L$ *and* $f(g) \in L$. *Let* $A\colon L \to \mathbb{R}$ *be a positive linear functional with* $A(\mathbf{1}) = 1$. *If the function* f *is increasing and* $A(g) \leqslant (m+M)/2$ *is valid, then we have*

$$A(f(g)) \leqslant \frac{f(m)+f(M)}{2} - A(\tilde{g})\delta_f, \tag{4.14}$$

where $\tilde{g}$ *and* δ_f *are defined by (4.2).*

Inequality (4.14) also holds if f *is decreasing and* $A(g) \geqslant (m+M)/2$. *If* f *is an increasing concave function and* $A(g) \geqslant (m+M)/2$, *or if* f *is a decreasing concave function and* $A(g) \leqslant (m+M)/2$, *the inequality sign in (4.14) is reversed.*

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