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SOME GEOMETRIC PROPERTIES OF MUSIELAK–ORLICZ SEQUENCE SPACES GENERATED BY DE LA VALLÉE–POUSSIN MEANS

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Abstract. Necessary and sufficient conditions for the non-triviality of a Musielak-Orlicz sequence space $V_{\Phi}(\lambda)$ generated by the de la Vallée-Poussin means are obtained. Topological properties such as completeness, separability, order continuity are characterized for the space $V_{\Phi}(\lambda)$. Finally, criteria for the coordinatewise uniformly Kadec-Klee property and the Uniform Opial property are obtained.

1. Introduction

In metric fixed point theory, geometric properties of Banach spaces such as the Kadec-Klee property, the Opial property and their several generalizations play fundamental role. In particular, the Opial property of a Banach space has a great importance in the fixed point theory, differential equation and Integral equations. On the other hand the Kadec-Klee property has several applications in Ergodic theory and many other branches([19]).

In recent days, the theory of Cesàro-Orlicz sequence spaces and Musielak-Orlicz sequence spaces have been studied extensively. Some topological properties like non-triviality, order continuity, separability, completeness and relations between norm and modular as well as some geometric properties like monotonicity, Kadec-Klee property, uniform Opial property, rotundity, local rotundity etc. are discussed in ([1], [9], [10], [13], [15], [23], [24]). The present paper is a continuation of the study of some of the geometric properties for the newly introduced Musielak-Orlicz sequence spaces generated by *de la Vallée-Poussin means*.

Throughout the paper we shall denote $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^+$ as the set of natural numbers, of reals, and of nonnegative reals respectively. Let $(X,\|.\|)$ be a Banach space and being a subspace of l^0, where l^0 be the space of all real sequences $x=(x(i))_{i=1}^{\infty}$. Let $S(X)$ and $B(X)$ denotes the unit sphere and closed unit ball respectively. A sequence $(x_l)\subset X$ is said to be ε*-separated* sequence if separation of sequence (x_l) denoted by $sep(x_l)=\inf\{\|x_l-x_m\|: l\neq m\}>\varepsilon$ for some $\varepsilon>0$ [16].

A Banach space $X\subset l^0$ is said to have the *Kadec-Klee property*, denoted by (H), if weakly convergent sequence on the unit sphere is strongly convergent, i.e., convergent in norm ([7]). A Banach sequence space X is said to possess *coordinatewise Kadec-Klee property*, denoted by *(* H_c *)*([14]), if $x\in X$ and every sequence $(x_l)\subset X$ such that

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$$\|x_l\| \to \|x\| \text{ and } x_l(i) \to x(i) \text{ for each } i, \text{ then } \|x_l - x\| \to 0.$$

It is well known that $X \in (H_c)$ implies $X \in (H)$, because weak convergence in X implies the coordinatewise convergence. A Banach space $X \subset l^0$ has the *coordinatewise uniformly Kadec-Klee property*, denoted by (UKK_c)([21]), if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$(x_l) \subset B(X),\ sep(x_l) \geqslant \varepsilon,\ \|x_l\| \to \|x\| \text{ and } x_l(i) \to x(i) \text{ for each } i \text{ implies } \|x\| \leqslant 1 - \delta.$$

It is well known that the property (UKK_c) implies property (H_c).

A Banach space X is said to have the *Opial property* ([20]) if for every weakly null sequence $(x_l) \subset X$ and every non zero $x \in X$, we have

$$\liminf_{l\to\infty} \|x_l\| < \liminf_{l\to\infty} \|x_l + x\|.$$

A Banach space X is said to have the *uniform Opial property* ([20]) if for each $\varepsilon > 0$ there exists $\mu > 0$ such that for any weakly null sequence (x_l) in $S(X)$ and $x \in X$ with $\|x\| \geqslant \varepsilon$ the following inequality hold:

$$1 + \mu \leqslant \liminf_{l\to\infty} \|x_l + x\|.$$

In any Banach space X an *Opial property* is important because it ensures that X has a weak fixed point property ([4]). Opial in ([26]) has shown that the space $L_p[0, 2\pi]$ $(p \neq 2, 1 < p < \infty)$ does not have this property but the Lebesgue sequence space $l_p(1 < p < \infty)$ has.

A map $\varphi : \mathbb{R} \to [0, \infty]$ is said to be an Orlicz function if it is an even, convex, left continuous on $[0, \infty)$, $\varphi(0) = 0$ and $\varphi(u) \to \infty$ as $u \to \infty$. A sequence $\Phi = (\varphi_n)_{n=1}^{\infty}$ of Orlicz functions φ_n is called Musielak-Orlicz function ([3]). A Musielak-Orlicz function $\Phi = (\varphi_n)_{n=1}^{\infty}$ is said to satisfy condition (∞_1) if

$$\lim_{u\to+\infty} \frac{\varphi_n(u)}{u} = +\infty \text{ for each } n \in \mathbb{N}. \qquad (\infty_1)$$

For a Musielak-Orlicz function Φ, the complementary function $\Psi = (\psi_n)_{n=1}^{\infty}$ of Φ is defined in the sense of Young as

$$\psi_n(u) = \sup_{v\geqslant 0}\{\, |u|v - \varphi_n(v)\} \quad \text{for all } u \in \mathbb{R} \text{ and } n \in \mathbb{N}.$$

Given any Musielak-Orlicz function Φ and $x = (x(n))_{n=1}^{\infty} \in l^0$, a convex modular $I_\Phi : l^0 \to [0, \infty]$ is defined by $I_\Phi(x) = \sum_{n=1}^{\infty} \varphi_n\Big(|x(n)|\Big)$ and the linear space $l_\Phi = \{x \in l^0 : I_\Phi(rx) < \infty \text{ for some } r > 0\}$ is called Musielak-Orlicz sequence space. Let us consider l_Φ with two functionals $|||x|||_\Phi^L$ and $|||x|||_\Phi^A$ defined by

$$|||x|||_\Phi^L = \inf\Big\{r > 0 : I_\Phi\Big(\tfrac{x}{r}\Big) \leqslant 1\Big\} \text{ and}$$

$$|||x|||_\Phi^A = \inf_{k>0}\Big\{\frac{1}{k}\Big(1 + I_\Phi(kx)\Big)\Big\} = \sup\Big\{\sum_{n=1}^{\infty} x(n)y(n) : I_\Psi(y) \leqslant 1\Big\}.$$

These two norms are equivalent. Indeed $|||x|||_\Phi^L \leqslant |||x|||_\Phi^A \leqslant 2|||x|||_\Phi^L$ (see [3]). These functionals, i.e., $|||x|||_\Phi^L$ and $|||x|||_\Phi^A$ are known as the Luxemberg norm and the Amemiya norm (Orlicz norm), respectively, and the corresponding Musielak-Orlicz sequence spaces are denoted by l_Φ^L and l_Φ^A respectively. The set of all $k>0$ such that $|||x|||_\Phi^A = \frac{1}{k}\Big(1+I_\Phi(kx)\Big)$ is attained for a fixed $x \in l_\Phi^A$ is denoted by $K(x)$. Moreover, it is well known that for any $x \in l_\Phi^A$ there exists a $k>0$ such that $|||x|||_\Phi^A = \frac{1}{k}\Big(1+I_\Phi(kx)\Big)$ whenever $\frac{\varphi_n(u)}{u} \to \infty$ as $u \to \infty$ for each $n \in \mathbb{N}$. For the details about Musielak-Orlicz sequence spaces and their geometric properties we refer to ([1]-[3], [18], [24]).

The subspaces

$$\Big\{x = (x(n))_{n=1}^\infty \in l^0 : \forall r>0 \ \exists n_r \in \mathbb{N} \text{ such that } \sum_{n=n_r}^\infty \varphi_n\Big(r|x(n)|\Big) < \infty\Big\},$$

equipped with the Luxemberg norm and the Amemiya norm induced from l_Φ are denoted by h_Φ^L and h_Φ^A are the subspaces of all order continuous elements of l_Φ^L and l_Φ^A respectively.

A Musielak-Orlicz function $\Phi = (\varphi_n)_{n=1}^\infty$ satisfies the δ_2^0-condition denoted by $\Phi \in \delta_2^0$ if there are positive constants a, K, a natural m and a sequence $(c_n)_{n=1}^\infty$ of positive numbers such that $(c_n)_{n=m}^\infty \in l_1$ and the inequality

$$\varphi_n(2u) \leqslant K\varphi_n(u) + c_n \tag{1.1}$$

holds for every $n \in \mathbb{N}$ and $u \in \mathbb{R}$ whenever $\varphi_n(u) \leqslant a$. If a Musielak-Orlicz function Φ satisfies the δ_2^0-condition with $m=1$, then Φ is said to satisfy the δ_2-condition([2], [3]).

For any Musielak-Orlicz function Φ, h_Φ coincides with l_Φ if and only if Φ satisfies δ_2^0-condition([2]).

A Musielak-Orlicz function $\Phi = (\varphi_n)_{n=1}^\infty$ satisfies the condition $(*)$ ([1]) if for any $\varepsilon \in (0,1)$ there is a $\delta > 0$ such that

$$\varphi_n(u) < 1-\varepsilon \text{ implies } \varphi_n((1+\delta)u) \leqslant 1 \text{ for all } n \in \mathbb{N} \text{ and } u \geqslant 0. \tag{1.2}$$

A Musielak-Orlicz function Φ is to said to vanishes only at zero which is denoted by $\Phi > 0$ if $\varphi_n(u) > 0$ for any $n \in \mathbb{N}$ and $u > 0$.

Let $(E, \|.\|_E)$ be a real normed linear subspace of l^0. E is said to be a *normed sequence lattice* ([6]) if it satisfies the following two conditions:

(i) for any $x \in E$ and $y \in l^0$ such that $|y(k)| \leqslant |x(k)|$ for every $k \in \mathbb{N}$, then $y \in E$ and $\|y\|_E \leqslant \|x\|_E$,

(ii) there exists a sequence $x = (x(k))_{k=1}^\infty \in E$ such that $x(k) > 0$ for all $k \in \mathbb{N}$.

A *normed sequence lattice* $(E, \|.\|_E)$ with complete norm $\|.\|_E$ is called *Banach sequence lattice* (see [6]).

NOTE. In many literatures a *Banach sequence lattice* E is also called as *Köthe sequence space* ([9], [23]).

A *Banach sequence lattice* E is said to have the *Fatou property* if for any $x \in l^0$ and sequence $(x_l) \subset E_+$ (where $E_+ = \{x \in E : x \geqslant 0\}$) satisfying $0 \leqslant x_l(i) \nearrow x(i)$, i.e.,

$x_l(i)$ increases to $x(i)$ as $l \to \infty$ for each $i \in \mathbb{N}$ and $\sup_l \|x_l\|_E < \infty$, then $x \in E$ and $\|x\|_E = \lim_{l\to\infty} \|x_l\|_E$ (see [23]).

An element $x \in E$ is said to be *order continuous* if for any sequence $(x_l) \subset E_+$ such that $|x(i)| \geqslant x_l(i) \searrow 0$, i.e., $x_l(i)$ decreases to zero as $l \to \infty$ for each i implies that $\|x_l\|_E \to 0$. The set of all order continuous element in E is denoted by E_a. A *Banach sequence lattice* E is said to be *order continuous* if $E_a = E$. It is known that E is *order continuous* if and only if $\|(0,0,\ldots,x(i+1),x(i+2),\ldots)\|_E \to 0$ as $i \to \infty$ for any $x \in E$ ([23]).

2. Class $V_\Phi(\lambda)$

Let $\lambda = (\lambda_n)_{n=1}^\infty$ be a non decreasing sequence of natural numbers satisfying $\lambda_1 = 1, \lambda_{n+1} - \lambda_n \leqslant 1$, $\forall n \in \mathbb{N}$. For any sequence $x = (x(n))_{n=1}^\infty \in l^0$ the *de la Vallée-Poussin means* is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k\in I_n} x(k),$$

where $I_n = [n - \lambda_n + 1, n], n = 1,2,\ldots$ ([5], [8]).

We introduce the *de la Vallée-Poussin means* map V_λ on l^0 as $V_\lambda : l^0 \to [0,\infty)$ such that $x \to V_\lambda x$, where

$$V_\lambda x = (V_\lambda x(n))_{n=1}^\infty, \text{ with } V_\lambda x(n) = \frac{1}{\lambda_n} \sum_{k\in I_n} |x(k)| \text{ for each } n = 1,2,\ldots \text{ and } x \in l^0.$$

Using this *de la Vallée-Poussin means* map and a Musielak-Orlicz function $\Phi = (\varphi_n)_{n=1}^\infty$, we define on l^0 a functional $\sigma_\Phi(x)$ by

$$\sigma_\Phi(x) = I_\Phi(V_\lambda x) = \sum_{n=1}^\infty \varphi_n\Big(\frac{1}{\lambda_n} \sum_{k\in I_n} |x(k)|\Big).$$

Since Φ is convex, so it is easy to verify that $\sigma_\Phi(x)$ is a convex modular (for definition see [3]) on l^0.

We now introduce the space $V_\Phi(\lambda)$ as follows

$$V_\Phi(\lambda) = \{x \in l^0 : V_\lambda x \in l_\Phi\} = \{x \in l^0 : \sigma_\Phi(rx) < \infty \text{ for some } r > 0\}.$$

Clearly, it is a linear space and also forms a normed linear space under the norms $\|x\|_\Phi^L = |||V_\lambda x|||_\Phi^L$ and $\|x\|_\Phi^A = |||V_\lambda x|||_\Phi^A$ introduced with the help of the norms on l_Φ. We call $V_\Phi(\lambda)$ as the Musielak-Orlicz sequence space generated by *de la Vallée-Poussin means*.

Our class $V_\Phi(\lambda)$ includes the following classes as particular cases:

(i) When $\lambda_n = n$, $n = 1,2,\ldots$, the $V_\Phi(\lambda)$ reduces to the Cesàro-Musielak-Orlicz sequence space ces_Φ studied by Wangkeeree ([17]), where

$$ces_\Phi = \Big\{x \in l^0 : \sum_{n=1}^\infty \varphi_n\Big(\frac{r}{n} \sum_{k=1}^n |x(k)|\Big) < \infty \text{ for some } r > 0\Big\},$$

(ii) For $\varphi_n = \varphi$, $\forall n$ the ces_Φ becomes well-known Cesàro-Orlicz sequence space ces_φ studied recently by Petrot and Suantai [9], Foralewski et al.[13], Cui et al. [23],

(iii) For $\lambda_n = 1$, $\forall n$ the $V_\Phi(\lambda)$ reduces to the Musielak-Orlicz sequence space l_Φ.

More recently, some geometric properties of sequence spaces involving *de la Vallée-Poussin means* has been studied by Şimşek et al. [11], [12]. In this paper, we introduce and study a new sequence space defined with the help of a Musielak-Orlicz function and *de la Vallée-Poussin means*. Some basic topological properties like non-triviality, order continuity, separability, completeness as well as criteria for some geometric properties like coordinatewise Uniform Kadec-Klee property, Uniform Opial property with respect to both, the Luxemberg norm and Amemiya norm, are obtained for this class.

NOTATIONS. For any $x \in l^0$ and $i \in \mathbb{N}$, we use the following notations throughout the paper:

$x|_i = (x(1), x(2), x(3), \ldots, x(i), 0, 0, \ldots)$, called the truncation of x at i,

$x|_{\mathbb{N}-i} = (0, 0, 0, \ldots, 0, x(i+1), x(i+2), \ldots)$,

$x|_I = \{x = (x(i))_{i=1}^{\infty} \in l^0 : x(i) \neq 0 \text{ for all } i \in I \subseteq \mathbb{N} \text{ and } x(i) = 0 \text{ for all } i \in \mathbb{N} \setminus I\}$ and

$supp\, x = \{i \in \mathbb{N} : x(i) \neq 0\}$ and clA denotes the closure of a set A.

For simplifying notations, we write $V_\Phi^L(\lambda) = (V_\Phi(\lambda), \|.\|_\Phi^L)$ and $V_\Phi^A(\lambda) = (V_\Phi(\lambda), \|.\|_\Phi^A)$. If norm is not specified we write simply $(V_\Phi(\lambda), \|.\|)$, where $\|.\|$ may be $\|.\|_\Phi^L$ or $\|.\|_\Phi^A$.

3. Main Results

First, we give the criteria for non-triviality of the space $V_\Phi(\lambda)$:

PROPOSITION 1. *Let $\Phi = (\varphi_n)_{n=1}^{\infty}$ be a Musielak-Orlicz function such that $\varphi_i(u) \geqslant \varphi_j(u)$ if $i \leqslant j$, $i, j \in \mathbb{N}$, $u \in [0, \infty)$ and a non decreasing sequence $\lambda = (\lambda_n)_{n=1}^{\infty}$ of natural numbers tending to ∞, satisfies $\lambda_1 = 1, \lambda_{n+1} - \lambda_n \leqslant 1$ $\forall\, n \in \mathbb{N}$, $\lambda_{mn} \geqslant \lambda_m \lambda_n$ $\forall\, m, n \in \mathbb{N}$. Then the followings are equivalent:*

(a) $V_\Phi(\lambda) \neq \{0\}$

(b) *there exists* $n_1 \in \mathbb{N}$ *such that* $\displaystyle\sum_{n=n_1}^{\infty} \varphi_n\left(\frac{1}{\lambda_n}\right) < \infty$

(c) $\forall r > 0$, $\exists n_r$ *such that* $\displaystyle\sum_{n=n_r}^{\infty} \varphi_n\left(\frac{r}{\lambda_n}\right) < \infty.$

Proof. $(a) \Rightarrow (b)$ Let $y = (y(k))_{k=1}^{\infty}$ be a nonzero element of $V_\Phi(\lambda)$. Since $y \neq 0$, so there exists $t \in \mathbb{N}$ such that $y(t) \neq 0$. Suppose $z = (0, 0, \ldots, 0, y(t), 0, \ldots) \in V_\Phi(\lambda)$. In particular, we choose $x = (0, 0, \ldots, 0, 1, 0, \ldots) \in V_\Phi(\lambda)$, where 1 is at the t-th position. Then we have

$$\sigma_\Phi(rx) = \sum_{n=t}^{\infty} \varphi_n\left(\frac{r}{\lambda_n}\right) < \infty.$$

We consider the following two cases:

(i) If $r>1$, then for each $n\in\mathbb{N}$, we have $\frac{1}{\lambda_n}<\frac{r}{\lambda_n}$ and by the monotonicity of $\Phi=(\varphi_n)_{n=1}^{\infty}$, we have $\varphi_n\left(\frac{1}{\lambda_n}\right)<\varphi_n\left(\frac{r}{\lambda_n}\right)$ for each $n\in\mathbb{N}$. Therefore

$$\sum_{n=t}^{\infty}\varphi_n\left(\frac{1}{\lambda_n}\right)<\sum_{n=t}^{\infty}\varphi_n\left(\frac{r}{\lambda_n}\right)<\infty.$$

Hence (b) follows easily from above by choosing $t=n_1$.

(ii) If $0<r<1$, then $\exists\, m\in\mathbb{N}$ such that $r\geqslant\frac{1}{\lambda_m}$. Hence $\frac{1}{\lambda_m\lambda_n}\leqslant\frac{r}{\lambda_n}$ for all $n\in\mathbb{N}$. So $\sum_{n=t}^{\infty}\varphi_n\left(\frac{1}{\lambda_m\lambda_n}\right)\leqslant\sum_{n=t}^{\infty}\varphi_n\left(\frac{r}{\lambda_n}\right)$.

Now, since $\lambda=(\lambda_n)_{n=1}^{\infty}$ is a non decreasing sequence of natural numbers and $\Phi=(\varphi_n)_{n=1}^{\infty}$ be a Musielak-Orlicz function such that $\varphi_i(u)\geqslant\varphi_j(u)$ if $i\leqslant j$, $i,j\in\mathbb{N}$, $u\in[0,\infty)$, we have

$$\begin{aligned}
&\sum_{n=mt}^{\infty}\varphi_n\left(\frac{1}{\lambda_n}\right)\\
&=\varphi_{mt}\left(\frac{1}{\lambda_{mt}}\right)+\varphi_{mt+1}\left(\frac{1}{\lambda_{mt+1}}\right)+\ldots+\varphi_{mt+(m-1)}\left(\frac{1}{\lambda_{mt+(m-1)}}\right)\\
&\quad+\varphi_{mt+m}\left(\frac{1}{\lambda_{mt+m}}\right)+\varphi_{m(t+1)+1}\left(\frac{1}{\lambda_{m(t+1)+1}}\right)+\ldots+\varphi_{m(t+1)+(m-1)}\left(\frac{1}{\lambda_{m(t+1)+(m-1)}}\right)\\
&\quad+\varphi_{m(t+2)}\left(\frac{1}{\lambda_{m(t+2)}}\right)+\varphi_{m(t+2)+1}\left(\frac{1}{\lambda_{m(t+2)+1}}\right)+\ldots\\
&\leqslant\varphi_{mt}\left(\frac{1}{\lambda_{mt}}\right)+\varphi_{mt+1}\left(\frac{1}{\lambda_{mt}}\right)+\ldots+\varphi_{mt+(m-1)}\left(\frac{1}{\lambda_{mt}}\right)\\
&\quad+\varphi_{m(t+1)}\left(\frac{1}{\lambda_{m(t+1)}}\right)+\varphi_{m(t+1)+1}\left(\frac{1}{\lambda_{m(t+1)}}\right)+\ldots+\varphi_{m(t+1)+(m-1)}\left(\frac{1}{\lambda_{m(t+1)}}\right)+\ldots\\
&\leqslant\varphi_{mt}\left(\frac{1}{\lambda_{mt}}\right)+\varphi_{mt}\left(\frac{1}{\lambda_{mt}}\right)+\ldots+\varphi_{mt}\left(\frac{1}{\lambda_{mt}}\right)\\
&\quad+\varphi_{m(t+1)}\left(\frac{1}{\lambda_{m(t+1)}}\right)+\ldots+\varphi_{m(t+1)}\left(\frac{1}{\lambda_{m(t+1)}}\right)+\ldots\\
&\leqslant m\varphi_{mt}\left(\frac{1}{\lambda_{mt}}\right)+m\varphi_{m(t+1)}\left(\frac{1}{\lambda_{m(t+1)}}\right)+\ldots\\
&=m\sum_{n=t}^{\infty}\varphi_{mn}\left(\frac{1}{\lambda_{mn}}\right)\leqslant m\sum_{n=t}^{\infty}\varphi_n\left(\frac{1}{\lambda_{mn}}\right)\leqslant m\sum_{n=t}^{\infty}\varphi_n\left(\frac{1}{\lambda_m\lambda_n}\right)\leqslant m\sum_{n=t}^{\infty}\varphi_n\left(\frac{r}{\lambda_n}\right)<\infty,
\end{aligned}$$

By choosing $mt=n_1$, we get $(a)\Rightarrow(b)$.

$(b)\Rightarrow(c)$ Suppose (b) is true. So, there exists n_1 such that $\sum_{n=n_1}^{\infty}\varphi_n\left(\frac{1}{\lambda_n}\right)<\infty$.

We consider the following two cases:

(i) If $0<r<1$, then for all n, $\frac{r}{\lambda_n}<\frac{1}{\lambda_n}$ and $\sum_{n=n_1}^{\infty}\varphi_n\left(\frac{r}{\lambda_n}\right)<\sum_{n=n_1}^{\infty}\varphi_n\left(\frac{1}{\lambda_n}\right)<\infty$. So by choosing $n_r:=n_1$, we obtain $\sum_{n=n_r}^{\infty}\varphi_n\left(\frac{r}{\lambda_n}\right)<\infty$.

(ii) If $r>1$, then we can find an $m\in\mathbb{N}$ such that $r\leqslant\lambda_m$. Defining $n_r=n_1m$ and using the same steps as above with the fact that $\lambda_{mn}\geqslant\lambda_m\lambda_n$, $\forall m,n\in\mathbb{N}$, we have

$$\sum_{n=n_r}^{\infty}\varphi_n\left(\frac{r}{\lambda_n}\right)\leqslant\sum_{n=n_r}^{\infty}\varphi_n\left(\frac{\lambda_m}{\lambda_n}\right)=\sum_{n=n_1m}^{\infty}\varphi_n\left(\frac{\lambda_m}{\lambda_n}\right)\leqslant m\sum_{n=n_1}^{\infty}\varphi_n\left(\frac{1}{\lambda_n}\right)<\infty.$$

Therefore $(b)\Rightarrow(c)$.

$(c)\Rightarrow(a)$ Take $r=1$. Since (c) holds, there exists $n_1\in\mathbb{N}$ such that $\sum_{n=n_1}^{\infty}\varphi_n\left(\frac{1}{\lambda_n}\right)<\infty$. Let us consider a sequence $x=(0,0,\dots,0,1,0,\dots)$, 1 is at the n_1-th term. Then $x\in l^0$ and

$$\sigma_\Phi(rx)=\sigma_\Phi(x)=\sum_{n=n_1}^{\infty}\varphi_n\left(\frac{1}{\lambda_n}\right)<\infty,$$

which implies that $x\in V_\Phi(\lambda)$. $\square$

It is easy to observe that any nontrivial Musielak-Orlicz sequence space $V_\Phi(\lambda)$ generated by *de la Vallée-Poussin means* belongs to the class of normed sequence lattice.

LEMMA 1. (Fatou property) *Let* $x=(x(i))_{i=1}^{\infty}\in l^0$, $(x_l)\subset V_\Phi^A(\lambda)$, $x_l=(x_l(i))_{i=1}^{\infty}$, $l\in\mathbb{N}$ *(or* $(x_l)\subset V_\Phi^L(\lambda)$*) are such that* $0\leqslant x_l(i)\nearrow x(i)$ *as* $l\to\infty$ *for each* $i\in\mathbb{N}$ *and* $\sup_l\|x_l\|_\Phi^A<\infty$ *(or* $\sup_l\|x_l\|_\Phi^L<\infty$*), then* $x\in V_\Phi^A(\lambda)$ *(or* $x\in V_\Phi^L(\lambda)$*) and* $\|x\|_\Phi^A=\lim_{l\to\infty}\|x_l\|_\Phi^A$ *(or* $\|x\|_\Phi^L=\lim_{l\to\infty}\|x_l\|_\Phi^L$*).*

Proof. Assume that $x_l\subset V_\Phi^A(\lambda)$ for all $l\in\mathbb{N}$, $\sup_l\|x_l\|_\Phi^A<\infty$ and $0\leqslant x_l(i)\nearrow x(i)$ as $l\to\infty$ for any $x\in l^0$. Let $\|x_l\|_\Phi^A=\alpha_l$. Since $\sup_l\alpha_l<\infty$ and (α_l) is non decreasing, there exists a finite α such that $\alpha_l\nearrow\alpha=\sup_l\alpha_l$. By the definition of norm $\|x_l\|_\Phi^A$, we have

$$\alpha_l=\|x_l\|_\Phi^A=\inf_{k>0}\frac{1}{k}\{\sigma_\Phi(kx_l)+1\}\leqslant\inf_{k>0}\frac{1}{k}\{\sigma_\Phi(kx)+1\}=\|x\|_\Phi^A,$$

Therefore $\alpha=\sup_l\alpha_l\leqslant\|x\|_\Phi^A$. We want to prove that $\|x\|_\Phi^A\leqslant\alpha$.

From the definition of norm $\|x_l\|_\Phi^A$, for every $\varepsilon>0$, there exists $k_l>0$ such that

$$\|x_l\|_\Phi^A+\varepsilon>\tfrac{1}{k_l}\{\sigma_\Phi(k_lx_l)+1\}>\tfrac{1}{k_l}\Rightarrow k_l>\tfrac{1}{\|x_l\|_\Phi^A+\varepsilon}\geqslant\tfrac{1}{\alpha+\varepsilon}>0 \text{ for each } l.$$

Therefore $(k_l)_{l=1}^{\infty}$ is bounded from below. Now we have two cases:

Case (i): when $(k_l)_{l=1}^{\infty}$ is bounded from above. In this case $(k_l)_{l=1}^{\infty}$ is a bounded sequence so it will have a convergent subsequence $(k_{l_m})_{m=1}^{\infty}$ (say). So, we may assume that $k_{l_m} \to k$, $k \in (0,\infty)$. Since $x_l(i) \nearrow x(i)$ for each $i \in \mathbb{N}$, so by the Fatou's lemma it follows that

$$\tfrac{1}{k}\{\sigma_\Phi(kx)+1\} \leqslant \liminf_{m\to\infty} \frac{1}{k_{l_m}}(\sigma_\Phi(k_{l_m}x_{l_m})+1) \leqslant \lim_{m\to\infty}(\|x_{l_m}\|_\Phi^A+\varepsilon)=\alpha+\varepsilon,$$

for some subsequence $(x_{l_m})_{m=1}^{\infty}$ of $(x_l)_{l=1}^{\infty}$. Since ε is arbitrary, we obtain $\|x\|_\Phi^A \leqslant \alpha$. Also for some $k_0>0$, we have $\sigma_\Phi(k_0x)<\infty$, i.e, $x \in V_\Phi^A(\lambda)$.

Case (ii): when $(k_l)_{l=1}^{\infty}$ is not bounded from above, so we assume $k_l \to \infty$ as $l \to \infty$. For a given $\varepsilon>0$, we choose $L=\frac{1}{\varepsilon}$. Then for $l \geqslant L$ we have $k_l \geqslant L$. Using the convexity of σ_Φ, we have

$$\frac{\sigma_\Phi(Lx_l)}{L}+\frac{1}{L}=\frac{1}{L}\sigma_\Phi\left(\frac{L}{k_l}k_lx_l\right)+\frac{1}{k_l}+\frac{1}{L}-\frac{1}{k_l} \leqslant \frac{\sigma_\Phi(k_lx_l)}{k_l}+\frac{1}{k_l}+\varepsilon-\frac{1}{k_l} \leqslant \|x_l\|_\Phi^A+2\varepsilon-\frac{1}{k_l}.$$

Consequently, using Fatou's lemma $\|x\|_\Phi^A \leqslant \frac{1}{L}\{\sigma_\Phi(Lx)+1\} \leqslant \liminf_{l\to\infty}\frac{1}{L}\{\sigma_\Phi(Lx_l)+1\} \leqslant \lim_{l\to\infty}(\|x_l\|_\Phi^A+2\varepsilon-\frac{1}{k_l})=\alpha+2\varepsilon$. This implies that $\|x\|_\Phi^A \leqslant \alpha$, since ε is arbitrary. Thus the proof is completed.

In case of Luxemberg norm $\|.\|_\Phi^L$, the proof runs on the parallel lines as given in [23]. □

It is known that any *normed sequence lattice* with the *Fatou property* is complete in its norm ([22], p.4). Hence the spaces $V_\Phi^A(\lambda)$ and $V_\Phi^L(\lambda)$ are Banach spaces.

THEOREM 1. *Let*

$$h_\Phi^A(\lambda)=\left\{x \in V_\Phi^A(\lambda): \forall r>0\ \exists n_r \text{ such that } \sum_{n=n_r}^{\infty}\varphi_n\left(\frac{r}{\lambda_n}\sum_{k\in I_n}|x(k)|\right)<\infty\right\}$$

and suppose $\Phi=(\varphi_n)_{n=1}^{\infty}$, $\lambda=(\lambda_n)_{n=1}^{\infty}$ *satisfies the assumptions in Proposition 1.*

Then followings are true:

(i) $h_\Phi^A(\lambda)$ *is a closed subspaces of* $V_\Phi^A(\lambda)$,

(ii) $h_\Phi^A(\lambda)=clD_\Phi$, *where* $D_\Phi=\{x \in V_\Phi^A(\lambda): x(k)\neq 0 \text{ for finite } k \in \mathbb{N}\}$,

(iii) $h_\Phi^A(\lambda)$ *is order continuous,*

(iv) $h_\Phi^A(\lambda)$ *is a separable subspaces of* $V_\Phi^A(\lambda)$.

Proof. (i) Clearly $h_\Phi^A(\lambda)$ is a subspace of $V_\Phi^A(\lambda)$. We need to show only that the closedness of $h_\Phi^A(\lambda)$. For this, let $x_i=(x_i(k))_{k=1}^{\infty} \in h_\Phi^A(\lambda)$, $i \in \mathbb{N}$ and $\|x-x_i\|_\Phi^A \to 0$ as $i\to\infty$ and $x \in V_\Phi^A(\lambda)$. We show that $x \in h_\Phi^A(\lambda)$. By the equivalent definition of norm and modular convergence, we have $\sigma_\Phi(r(x-x_i)) \to 0$ as $i \to \infty$ for all $r>0$. So for all $r>0$ there exists $J \in \mathbb{N}$ such that $\sigma_\Phi(2r(x-x_J))<1$. Since $x_J \in h_\Phi^A(\lambda)$

so there exists n_J such that $\sum_{n=n_J}^{\infty} \varphi_n\Big(\frac{2r}{\lambda_n}\sum_{k\in I_n}|x_J(k)|\Big) < \infty \ \forall r > 0$. We choose $n_r = n_J$, then we have

$$\begin{aligned}\sum_{n=n_J}^{\infty} \varphi_n\Big(\frac{r}{\lambda_n}\sum_{k\in I_n}|x(k)|\Big) &\leqslant \sum_{n=n_J}^{\infty} \varphi_n\Big(\frac{r}{2\lambda_n}\sum_{k\in I_n}(2|x(k)-x_J(k)|) + \frac{r}{2\lambda_n}\sum_{k\in I_n}2|x_J(k)|\Big)\\ &\leqslant \frac{1}{2}\sum_{n=n_J}^{\infty} \varphi_n\Big(\frac{2r}{\lambda_n}\sum_{k\in I_n}|x(k)-x_J(k)|\Big) + \frac{1}{2}\sum_{n=n_J}^{\infty} \varphi_n\Big(\frac{2r}{\lambda_n}\sum_{k\in I_n}|x_J(k)|\Big)\\ &\leqslant \frac{1}{2}\sigma_\Phi(2r(x-x_J)) + \frac{1}{2}\sum_{n=n_J}^{\infty} \varphi_n\Big(\frac{2r}{\lambda_n}\sum_{k\in I_n}|x_J(k)|\Big) < \infty.\end{aligned}$$

Since r is arbitrary, so we have $x \in h_\Phi^A(\lambda)$. This completes the proof.

(ii) We first show the inclusion $clD_\Phi \subset h_\Phi^A(\lambda)$. If $clD_\Phi = \emptyset$, then the inclusion is true. So, let $clD_\Phi \neq \emptyset$. Since $h_\Phi^A(\lambda)$ is a closed linear subspace of $V_\Phi^A(\lambda)$, it is enough to prove that $x = e_t \in h_\Phi^A(\lambda)$, because any element of the form $y = (y(1), y(2), \ldots, y(k), 0, 0, \ \ldots)$ can be written as the linear combination of $x = e_t, t = 1, 2, \ldots$. Then by Proposition 1, there exists n_r such that

$$\sum_{n=n_r}^{\infty} \varphi_n\Big(\frac{r}{\lambda_n}\Big) < \infty.$$

We may assume that $n_r \geqslant t$ for each r. Hence $x \in h_\Phi^A(\lambda)$.

For reverse inclusion, let $x = (x(1), x(2), \ldots, x(p), x(p+1), \ldots) \in h_\Phi^A(\lambda)$. For any $p \in \mathbb{N}$, we denote $x^p = (x(1), x(2), \ldots, x(p), 0, 0, \ldots)$. Then $x^p \in clD_\Phi$. We prove that for every $x \in h_\Phi^A(\lambda)$, $\|x - x^p\|_\Phi^A \to 0$ as $p \to \infty$. Since $x \in h_\Phi^A(\lambda)$, so for a given $\varepsilon > 0$ there exists a p_0, we have

$$\sum_{n=p_0+1}^{\infty} \varphi_n\Big(\frac{r}{\lambda_n}\sum_{k\in I_n}|x(k)|\Big) < \frac{\varepsilon}{2} \ \ \forall r > 0.$$

Then for any $r > 0$ and $p \geqslant p_0$, we have

$$\begin{aligned}\sigma_\Phi(r(x-x^p)) \leqslant \sigma_\Phi(r(x-x^{p_0})) &= \sigma_\Phi(r(0,0,\ldots,0,x(p_0+1),x(p_0+2),\ldots))\\ &= \sum_{n=p_0+1}^{\infty} \varphi_n\Big(\frac{r}{\lambda_n}\sum_{k\in I_n}|x(k)|\Big) < \frac{\varepsilon}{2},\end{aligned}$$

i.e., $\|x - x^p\|_\Phi^A \to 0$ as $p \to \infty$ and so $x \in clD_\Phi$. Hence the result (ii) is proved.

(iii) Let $x \in h_\Phi^A(\lambda)$, we show that x is an order continuous element. Let $x_l(i) \searrow 0$ for each i and $x_l(i) \leqslant |x(i)|$ for all $l \in \mathbb{N}$. Since $x \in h_\Phi^A(\lambda)$, so for any $r > 0$, $\varepsilon > 0$ there exists $n_r \in \mathbb{N}$ such that

$$\sum_{n=n_r}^{\infty} \varphi_n\Big(\frac{r}{\lambda_n}\sum_{k\in I_n}|x(k)|\Big) < \frac{\varepsilon}{2}.$$

We denote for every $n \in \mathbb{N}$,

$$\beta(n) = \varphi_n\Big(\tfrac{r}{\lambda_n}\sum_{k\in I_n}|x(k)|\Big) \text{ and } \beta_l(n) = \varphi_n\Big(\tfrac{r}{\lambda_n}\sum_{k\in I_n}|x_l(k)|\Big).$$

Since $x_l(i)\searrow 0$ for each i, so $\beta_l(n)\to 0$ as $l\to\infty$ for each $n\in\mathbb{N}$. Hence there is a $l_\varepsilon\in\mathbb{N}$ such that $\sum_{n=1}^{n_{r-1}}\beta_l(n) < \frac{\varepsilon}{2}$ for all $l \geqslant l_\varepsilon$. Also since $\sum_{n=n_r}^{\infty}\beta_l(n) \leqslant \sum_{n=n_r}^{\infty}\beta(n) < \frac{\varepsilon}{2}$ for all $n\geqslant n_r$ and $l\in\mathbb{N}$. Therefore for arbitrary $r>0$, we have $\sigma_\Phi(rx_l)<\varepsilon$ for all $l\geqslant l_\varepsilon$. This means that $\|x_l\|_\Phi^A\to 0$ as $l\to\infty$. So x is an order continuous element.

Now let $x\in V_\Phi^A(\lambda)$ be an order continuous element. Then by definition

$$\|(0,0,\ldots,x(i+1),x(i+2),\ldots)\|_\Phi^A \to 0 \text{ as } i\to\infty.$$

This means that $x\in cl\{x\in V_\Phi^A(\lambda): x(k)\neq 0 \text{ for finite } k\in\mathbb{N}\} = h_\Phi^A(\lambda)$.

(iv) Define a countable set $S_\Phi = cl\{x\in V_\Phi^A(\lambda): x(k)\neq 0$ for finite $k\in\mathbb{N}$ and $x(k)\in\mathbb{Q}\}$. It is easy to show that S_Φ is separable. We show that $clD_\Phi = S_\Phi$. The inclusion $S_\Phi\subset clD_\Phi$ is obvious. We show only the inclusion $clD_\Phi\subset S_\Phi$. Let $x = (x(1),x(2),\ldots,x(p),0,0,\ldots)\in clD_\Phi$ and $x_j = (x_j(1),x_j(2),\ldots,x_j(p),0,0,\ldots)\in S_\Phi$ is such that $x_j(k)\to x(k)$ as $j\to\infty$ for each $k\in\mathbb{N}$. We prove that for each $x\in clD_\Phi$ there exists $x_j\in S_\Phi$ such that $\|x_j - x\|_\Phi^A\to 0$ as $j\to\infty$. We choose for any $r>0$, $\varepsilon\in(0,M)$, where $M>0$, a constant. Then for any $r>0$ there exists a $j_0\in\mathbb{N}$ such that

$$r\sum_{k\in I_n}|x(k)-x_j(k)| < \frac{\varepsilon}{M} \text{ for } j\geqslant j_0 \text{ and } n=1,2,\ldots,p.$$

By the condition (b) of Proposition 1, we have there exists a constant $M>0$ such that $\sum_{n=1}^{\infty}\varphi_n\Big(\frac{1}{\lambda_n}\Big)\leqslant M$. Therefore for $j\geqslant j_0$ and convexity of $\Phi=(\varphi_n)_{n=1}^{\infty}$, we have

$$\begin{aligned}
\sigma_\Phi(r(x-x_j)) &= \sum_{n=1}^{p}\varphi_n\Big(\frac{r}{\lambda_n}\sum_{k\in I_n}|x(k)-x_j(k)|\Big) + \sum_{n=p+1}^{\infty}\varphi_n\Big(\frac{r}{\lambda_n}\sum_{k=n-\lambda_n+1}^{p}|x(k)-x_j(k)|\Big)\\
&\leqslant \sum_{n=1}^{p}\varphi_n\Big(\frac{r}{\lambda_n}\sum_{k\in I_n}|x(k)-x_j(k)|\Big) + \sum_{n=p+1}^{\infty}\varphi_n\Big(\frac{r}{\lambda_n}\sum_{k=p-\lambda_p+1}^{p}|x(k)-x_j(k)|\Big)\\
&< \frac{\varepsilon}{M}\sum_{n=1}^{\infty}\varphi_n\Big(\frac{1}{\lambda_n}\Big)\leqslant\varepsilon.
\end{aligned}$$

Since r is arbitrary, so we have $\|x_j - x\|_\Phi^A\to 0$ as $j\to\infty$. Hence $clD_\Phi = S_\Phi$. Since S_Φ is separable and $h_\Phi^A(\lambda) = clD_\Phi = S_\Phi$, so $h_\Phi^A(\lambda)$ is separable. □

REMARK 1. Since the Luxemberg norm $\|.\|_\Phi^L$ and the Amemiya norm $\|.\|_\Phi^A$ are equivalent, so all the results of Theorem 1 are also true for the space $h_\Phi^L(\lambda)$.

THEOREM 2. *Let $\Phi=(\varphi_n)_{n=1}^{\infty}$ be a Musielak-Orlicz function satisfying the δ_2-condition. Then $h_\Phi(\lambda) = V_\Phi(\lambda)$.*

Proof. We need to show here only the inclusion $V_\Phi(\lambda) \subset h_\Phi(\lambda)$. Let $x \in V_\Phi(\lambda)$. Then for some $t > 0$, $\sigma_\Phi(tx) < \infty$, i.e., $\sum_{n=1}^{\infty} \varphi_n\Big(\frac{t}{\lambda_n}\sum_{k\in I_n}|x(k)|\Big) < \infty$. We show that for any $r > 0$ there exists a $n_r \in \mathbb{N}$ such that $\sum_{n=n_r}^{\infty} \varphi_n\Big(\frac{r}{\lambda_n}\sum_{k\in I_n}|x(k)|\Big) < \infty$. If $r \in [0,t]$ then it is easily follows because $\sum_{n=n_r}^{\infty} \varphi_n\Big(\frac{r}{\lambda_n}\sum_{k\in I_n}|x(k)|\Big) \leqslant \sum_{n=n_r}^{\infty} \varphi_n\Big(\frac{t}{\lambda_n}\sum_{k\in I_n}|x(k)|\Big) < \infty$. Now, we fix t and choose $r > t$. Since $x \in V_\Phi(\lambda)$, i.e., for some $t > 0$, $\sigma_\Phi(tx) < \infty$, so there exists n_r and a constant a such that $\sum_{n=n_r}^{\infty} \varphi_n\Big(\frac{t}{\lambda_n}\sum_{k\in I_n}|x(k)|\Big) < \frac{a}{2}$. Therefore for each $n \geqslant n_r$, we have $\varphi_n\Big(\frac{t}{\lambda_n}\sum_{k\in I_n}|x(k)|\Big) < \frac{a}{2}$. Choose a sequence $(c_n)_{n=1}^{\infty}$ of positive real numbers such that $\sum_{n=1}^{\infty} c_n < \infty$. So for a given $\varepsilon > 0$, there exists a n_r such that $\sum_{n=n_r}^{\infty} c_n < \frac{\varepsilon}{2}$. Let $u = \frac{t}{\lambda_n}\sum_{k\in I_n}|x(k)|$, $K > 0$ be a constant and a is chosen above. Since $r > t$ so there is a $l \in \mathbb{N}$ such that $r \leqslant 2^l t$. Now applying δ_2-condition for all $n \geqslant n_r$, we have

$$\varphi_n\Big(\frac{r}{\lambda_n}\sum_{k\in I_n}|x(k)|\Big) \leqslant \varphi_n\Big(\frac{2^l t}{\lambda_n}\sum_{k\in I_n}|x(k)|\Big) + c_n \leqslant K^l \varphi_n\Big(\frac{t}{\lambda_n}\sum_{k\in I_n}|x(k)|\Big) + \Big(\sum_{i=0}^{l-1} K^i\Big) c_n$$

Taking summation both sides over $n \geqslant n_r$, we obtain

$$\sum_{n=n_r}^{\infty} \varphi_n\Big(\frac{r}{\lambda_n}\sum_{k\in I_n}|x(k)|\Big) \leqslant K^l \sum_{n=n_r}^{\infty} \varphi_n\Big(\frac{t}{\lambda_n}\sum_{k\in I_n}|x(k)|\Big) + \Big(\sum_{i=0}^{l-1} K^i\Big) \sum_{n=n_r}^{\infty} c_n < \infty.$$

Hence $x \in h_\Phi(\lambda)$. □

COROLLARY 1. *Let $\Phi \in \delta_2$, i.e., (1.1) holds. Then*
(i) $V_\Phi(\lambda)$ is separable,
(ii) $V_\Phi(\lambda)$ is order continuous.

We will assume in the rest of the paper that $\Phi = (\varphi_n)_{n=1}^{\infty}$ is a Musielak-Orlicz function with all φ_n being finitely valued. The following known Lemmas are useful in the sequel:

LEMMA 2. *Let $x \in h_\Phi^L(\lambda)$ be an arbitrary element. Then $\|x\|_\Phi^L = 1$ if and only if $\sigma_\Phi(x) = 1$.*

Proof. The proof will run on the parallel lines of the proof of Lemma 2.1 in [23]. □

LEMMA 3. *Suppose $\Phi \in \delta_2$ and $\Phi > 0$. Then for any $(x_l) \subset V_\Phi^L(\lambda)$, $\|x_l\|_\Phi^L \to 0$ ($\|x_l\|_\Phi^A \to 0$) if and only if $\sigma_\Phi(x_l) \to 0$.*

Proof. For the proof this lemma see [1], [14]. □

LEMMA 4. *If* $\Phi \in \delta_2$*, i.e.,* (1.1) *holds, then for any* $x \in V_\Phi^L(\lambda)$*,*

$$\|x\|_\Phi^L = 1 \text{ if and only if } \sigma_\Phi(x) = 1.$$

Proof. Since $\Phi \in \delta_2$ implies $V_\Phi^L(\lambda) = h_\Phi^L(\lambda)$. The proof will be follows from Lemma 2. □

LEMMA 5. *Let* $\Phi \in \delta_2$*, i.e.,* (1.1) *holds and satisfy the condition* $(*)$*, i.e.,* (1.2) *holds. Then for any* $x \in V_\Phi^L(\lambda)$ *and every* $\varepsilon \in (0,1)$ *there exists* $\delta(\varepsilon) \in (0,1)$ *such that* $\sigma_\Phi(x) \leqslant 1-\varepsilon$ *implies* $\|x\|_\Phi^L \leqslant 1-\delta$*.*

Proof. The proof of this lemma will be given in a similar way as the proof of Lemma 9 in [1]. □

LEMMA 6. [1] *Let* $(X, \|.\|)$ *be a normed space. If* $f : X \to \mathbb{R}$ *is a convex function in the set* $K(0,1) = \{x \in X : \|x\| \leqslant 1\}$ *and* $|f(x)| \leqslant M$ *for all* $x \in K(0,1)$ *and some* $M > 0$*. Then* f *is almost uniformly continuous in* $K(0,1)$*; i.e., for all* $d \in (0,1)$ *and* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $\|y\| \leqslant d$ *and* $\|x-y\| < \delta$ *implies* $|f(x)-f(y)| < \varepsilon$ *for all* $x, y \in K(0,1)$*.*

LEMMA 7. *Let* $\Phi \in \delta_2$*, i.e.,* (1.1) *holds,* $\Phi > 0$ *and satisfies the condition* $(*)$*, i.e.,* (1.2) *holds. Then for each* $d \in (0,1)$ *and* $\varepsilon > 0$ *there exists* $\delta = \delta(d,\varepsilon) > 0$ *such that* $\sigma_\Phi(x) \leqslant d$*,* $\sigma_\Phi(y) \leqslant \delta$ *imply*

$$|\sigma_\Phi(x+y) - \sigma_\Phi(x)| < \varepsilon \text{ for any } x, y \in V_\Phi^L(\lambda). \tag{3.1}$$

Proof. Since $\Phi \in \delta_2$ and satisfies condition $(*)$, so by Lemma 5, there exists $d_1 \in (0,1)$ such that $\|x\|_\Phi^L \leqslant d_1$. Also, by Lemma 3, we find a $\delta > 0$ such that for every $\delta_1 > 0$, $\sigma_\Phi(y) \leqslant \delta$ implies $\|y\|_\Phi^L \leqslant \delta_1$ for any $y \in V_\Phi^L(\lambda)$. So, if $\sigma_\Phi(x) \leqslant d$ and $\sigma_\Phi(y) \leqslant \delta$ then $\|x\|_\Phi^L \leqslant d_1$ and $\|y\|_\Phi^L \leqslant \delta_1$. Hence, by Lemma 6, we have $|\sigma_\Phi(x+y) - \sigma_\Phi(x)| < \varepsilon$ because the functional σ_Φ satisfies all the assumptions of f defined in Lemma 6. □

LEMMA 8. *Let* $\Phi \in \delta_2$*, i.e.,* (1.1) *holds and satisfies the condition* $(*)$*, i.e.,* (1.2) *holds and* $\Phi > 0$*. Then for any* $x \in V_\Phi^L(\lambda)$ *and any* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon) > 0$ *such that* $\sigma_\Phi(x) \geqslant 1+\varepsilon$ *implies* $\|x\|_\Phi^L \geqslant 1+\delta$*.*

Proof. The proof of this lemma goes on the parallel lines of the proof of the Lemma 4 in [24]. □

It is to be noted that, for a fixed $x \in V_\Phi^A(\lambda)$ the set $K(x)$ defined earlier (see Section 1) will be of the form $K(x) = \left\{k > 0 : \|x\|_\Phi^A = \frac{1}{k}(1+\sigma_\Phi(kx))\right\}$.

LEMMA 9. *Let $x \in V_{\Phi}^{A}(\lambda)$ be given and $x \neq 0$. If $K(x) = \emptyset$, then $\|x\|_{\Phi}^{A} = \sum_{n=1}^{\infty} \alpha_n V_{\lambda} x(n)$, where $\alpha_n = \lim_{u\to\infty} \frac{\varphi_n(u)}{u}$ and $V_{\lambda}x(n) = \frac{1}{\lambda_n} \sum_{i\in I_n} |x(i)|$, $n \in \mathbb{N}$.*

Proof. Let $f(k) = \frac{1}{k}(1+\sigma_{\Phi}(kx))$, where

$$\sigma_{\Phi}(x) = \sum_{n=1}^{\infty} \varphi_n\Big(\frac{1}{\lambda_n}\sum_{i\in I_n}|x(i)|\Big) = \sum_{n=1}^{\infty} \varphi_n\Big(V_{\lambda}x(n)\Big).$$

Since $f(k)$ is continuous and $K(x) = \emptyset$, so we have $\|x\|_{\Phi}^{A} = \lim_{k\to\infty} f(k) = \lim_{k\to\infty} \frac{\sigma_{\Phi}(kx)}{k}$. Then $\alpha_n = \lim_{u\to\infty} \frac{\varphi_n(u)}{u}$ is finite for all $n \in supp\ x$. If not, there exists a $n_0 \in supp\ x$ such that $\|x\|_{\Phi}^{A} = \lim_{k\to\infty} \frac{\sigma_{\Phi}(kx)}{k} \geqslant \lim_{k\to\infty} \frac{\varphi_{n_0}(kV_{\lambda}x(n_0))}{kV_{\lambda}x(n_0)} V_{\lambda}x(n_0) = \infty$. So we have

$$\|x\|_{\Phi}^{A} = \lim_{k\to\infty} \frac{\sigma_{\Phi}(kx)}{k} = \lim_{k\to\infty} \sum_{n=1}^{\infty} \frac{\varphi_n(kV_{\lambda}x(n))}{kV_{\lambda}x(n)} V_{\lambda}x(n) = \sum_{n=1}^{\infty} \alpha_n V_{\lambda}x(n). \quad \square$$

LEMMA 10. *Let $x \in V_{\Phi}^{A}(\lambda)$ be given and $x \neq 0$. If $\Phi = (\varphi_n)_{n=1}^{\infty}$ is a Musielak-Orlicz function satisfying condition (∞_1), then $K(x) \neq \emptyset$.*

Proof. Suppose in contrary that $K(x) = \emptyset$, then by the Lemma 9, we obtain $\lim_{u\to\infty} \frac{\varphi_n(u)}{u} < \infty$ for each $n \in supp\ x$, a contradiction to the assumption that Φ satisfying condition (∞_1). $\square$

THEOREM 3. *Let $\Phi = (\varphi_n)_{n=1}^{\infty}$ be a Musielak-Orlicz function satisfying condition $(*)$, i.e., (1.2) holds, $\Phi \in \delta_2$, i.e., (1.1) holds and $\Phi > 0$. Then the space $V_{\Phi}^{L}(\lambda)$ has the UKK_c-property.*

Proof. Since $\Phi \in \delta_2$ and $\Phi > 0$, so for a given $\varepsilon > 0$, by Lemma 3, there exists a $\eta > 0$ such that

$$\|x\|_{\Phi}^{L} \geqslant \frac{\varepsilon}{4} \Rightarrow \sigma_{\Phi}(x) \geqslant \eta. \tag{3.2}$$

Now with this $\eta > 0$, from Lemma 5, we can find a $\delta_1 \in (0,1)$ such that

$$\|x\|_{\Phi}^{L} > 1-\delta_1 \Rightarrow \sigma_{\Phi}(x) > 1-\eta. \tag{3.3}$$

Suppose $(x_l) \subset B(V_{\Phi}^{L}(\lambda))$, $\|x_l\|_{\Phi}^{L} \to \|x\|_{\Phi}^{L}$, $x_l(i) \to x(i)$ for all $i \in \mathbb{N}$ and $sep(x_l) \geqslant \varepsilon$. We will show that there exists a $\delta > 0$ such that $\|x\|_{\Phi}^{L} \leqslant 1-\delta$. If possible, suppose that $\|x\|_{\Phi}^{L} > 1-\delta$. Then we can select a finite set $I = \{1,2,\dots,N-1\}$ on which $\|x|_I\|_{\Phi}^{L} > 1-\delta$. Since $x_l(i) \to x(i)$ for each $i \in \mathbb{N}$, so we get that $x_l \to x$ uniformly on I. Consequently, since $\|x_l\|_{\Phi}^{L} \to \|x\|_{\Phi}^{L}$, there exists $l_N \in \mathbb{N}$ such that

$$\|x_l|_I\|_{\Phi}^{L} > 1-\delta \text{ and } \|(x_l - x_m)|_I\|_{\Phi}^{L} \leqslant \tfrac{\varepsilon}{2} \text{ for all } l,m \geqslant l_N.$$

Keeping in mind (3.3), the first inequality implies that $\sigma_\Phi(x_l|_I) > 1-\eta$ for $l \geqslant l_N$. Since $sep(x_l) \geqslant \varepsilon$, i.e., $\|x_l - x_m\|_\Phi^L \geqslant \varepsilon$, so the second inequality implies that $\|(x_l - x_m)|_{\mathbb{N}-I}\|_\Phi^L \geqslant \frac{\varepsilon}{2}$ for $l,m \geqslant l_N, l \neq m$. Hence for $N \in \mathbb{N}$ there exists a l_N such that $\|x_{l_N}|_{\mathbb{N}-I}\|_\Phi^L \geqslant \frac{\varepsilon}{4}$. Without loss of generality, we may assume that $\|x_l|_{\mathbb{N}-I}\|_\Phi^L \geqslant \frac{\varepsilon}{4}$ for all $l,N \in \mathbb{N}$. Therefore, by (3.2), we have $\sigma_\Phi(x_l|_{\mathbb{N}-I}) \geqslant \eta$.

By convexity of φ_n for each $n \in \mathbb{N}$, we have for any $\alpha \in [0,1]$ and $u \in \mathbb{R}$ that $\varphi_n(\alpha u) = \varphi_n(\alpha u + (1-\alpha)0) \leqslant \alpha\varphi_n(u)$. Therefore, if $0 \leqslant u < v < \infty$, then $\varphi_n(u) = \varphi_n(\frac{u}{v}v) \leqslant \frac{u}{v}\varphi_n(v)$, which means that $\frac{\varphi_n(u)}{u} \leqslant \frac{\varphi_n(v)}{v}$ for each $n \in \mathbb{N}$. Assuming now that $0 \leqslant u,v < \infty$, $u+v > 0$, we get

$$\varphi_n(u+v) = u\frac{\varphi_n(u+v)}{u+v} + v\frac{\varphi_n(u+v)}{u+v} \geqslant u\frac{\varphi_n(u)}{u} + v\frac{\varphi_n(v)}{v} = \varphi_n(u) + \varphi_n(v)$$

for each $n \in \mathbb{N}$. Using this fact, we obtain $\sigma_\Phi(x_l|_I) + \sigma_\Phi(x_l|_{\mathbb{N}-I}) \leqslant \sigma_\Phi(x_l) \leqslant 1$. This implies that $\sigma_\Phi(x_l|_{\mathbb{N}-I}) \leqslant 1 - \sigma_\Phi(x_l|_I) < 1-(1-\eta) = \eta$, i.e., $\sigma_\Phi(x_l|_{\mathbb{N}-I}) < \eta$. This contradicts to the fact that $\sigma_\Phi(x_l|_{\mathbb{N}-I}) \geqslant \eta$. This finishes the proof. $\square$

THEOREM 4. *The space $V_\Phi^A(\lambda)$ has the UKK_c-property whenever Φ satisfies condition (∞_1), $\Phi \in \delta_2$, i.e., (1.1) holds and $\Phi > 0$.*

Proof. For a given $\varepsilon > 0$, let $(x_l) \subset B(V_\Phi^A(\lambda))$, $\|x_l\|_\Phi^A \to \|x\|_\Phi^A$, $x_l(i) \to x(i)$ for each $i \in \mathbb{N}$ and $sep(x_l) \geqslant \varepsilon$. We shall prove that $\|x\|_\Phi^A \leqslant 1-\delta$. It is trivial when $x = 0$. So, we assume that $x \neq 0$. Then by Lemma 10, we have $K(x) \neq \emptyset$, i.e., for each $x \in V_\Phi^A(\lambda)$ there exists a $k_l \in \mathbb{R}_+$ such that $\|x\|_\Phi^A = \frac{1}{k_l}\big(1+\sigma_\Phi(k_l x)\big)$. Since $x_l \to x$ weakly implies $x_l(i) \to x(i)$ for each $i \in \mathbb{N}$, so we can select a finite set $I = \{1,2,3,\ldots,N-1\}$ on which $x_l \to x$ uniformly. So, there exists $l_N \in \mathbb{N}$ such that

$$\|(x_l - x_m)|_I\|_\Phi^A \leqslant \frac{\varepsilon}{2} \text{ for all } l,m \geqslant l_N. \tag{3.4}$$

Since $sep(x_l) \geqslant \varepsilon$, so we have by definition that $\|x_l - x_m\|_\Phi^A \geqslant \varepsilon$ for $l \neq m$. This inequality together with (3.4) implies that $\|(x_l - x_m)|_{\mathbb{N}-I}\|_\Phi^A \geqslant \frac{\varepsilon}{2}$ for $l \neq m$ and $l,m \geqslant l_N$. Hence for each $N \in \mathbb{N}$ there exists a l_N such that $\|x_{l_N}|_{\mathbb{N}-I}\|_\Phi^A \geqslant \frac{\varepsilon}{4}$. Without loss of generality, we may assume that $\|x_l|_{\mathbb{N}-I}\|_\Phi^A \geqslant \frac{\varepsilon}{4}$ for all $l,N \in \mathbb{N}$. Therefore, by Lemma 3, there exists $\delta_1 \in (0,\varepsilon)$ such that $\sigma_\Phi(x_l|_{\mathbb{N}-I}) \geqslant \delta_1$.

Since $V_\Phi^A(\lambda)$ is order continuous, i.e., $\|x - x|_I\|_\Phi^A \to 0$ for sufficiently large N and hence there exists a $\frac{\delta_1}{4} > 0$ such that $\|x|_I\|_\Phi^A > \|x\|_\Phi^A - \frac{\delta_1}{4}$. Also, since $x_l(i) \to x(i)$ for each i and $\|x_l\|_\Phi^A \to \|x\|_\Phi^A$, so there exists $N_0 \in \mathbb{N}$ such that

$$\|x_l|_I\|_\Phi^A > \|x\|_\Phi^A - \tfrac{\delta_1}{4} \text{ for } l > N_0.$$

Since $\|x_l\|_\Phi^A \leqslant 1$ implies $k_l \geqslant 1$ for all $l \in \mathbb{N}$, so for each $n \in \mathbb{N}$ the convexity of φ_n and the inequality $\varphi_n(u+v) \geqslant \varphi_n(u) + \varphi_n(v)$ for all $u,v \in \mathbb{R}^+$ implies that

$$\begin{aligned} 1 &\geqslant \|x_l\|_\Phi^A \\ &= \frac{1}{k_l}\Big(1 + \sum_{n=1}^{N-1}\varphi_n\Big(\frac{k_l}{\lambda_n}\sum_{j\in I_n}|x_l(j)|\Big) + \sum_{n=N}^{\infty}\varphi_n\Big(\frac{k_l}{\lambda_n}\sum_{j\in I_n}|x_l(j)|\Big)\Big) \end{aligned}$$

$$
\begin{aligned}
&=\frac{1}{k_l}\Big(1+\sum_{n=1}^{N-1}\varphi_n\Big(\frac{k_l}{\lambda_n}\sum_{j\in I_n}|x_l(j)|\Big)\Big)+\frac{1}{k_l}\Big(\sum_{n=N}^{\infty}\varphi_n\Big(\frac{k_l}{\lambda_n}\sum_{j=n-\lambda_n+1}^{N-1}|x_l(j)|+\frac{k_l}{\lambda_n}\sum_{j=N}^{n}|x_l(j)|\Big)\Big)\\
&\geqslant\frac{1}{k_l}\Big(1+\sum_{n=1}^{N-1}\varphi_n\Big(\frac{k_l}{\lambda_n}\sum_{j\in I_n}|x_l(j)|\Big)\Big)+\frac{1}{k_l}\sum_{n=N}^{\infty}\varphi_n\Big(\frac{k_l}{\lambda_n}\sum_{j=n-\lambda_n+1}^{N-1}|x_l(j)|\Big)\\
&+\frac{1}{k_l}\sum_{n=N}^{\infty}\varphi_n\Big(\frac{k_l}{\lambda_n}\sum_{j=N}^{n}|x_l(j)|\Big)\\
&=\frac{1}{k_l}\Big(1+\sigma_\Phi\Big(k_l x_l|_I\Big)\Big)+\frac{1}{k_l}\sum_{n=N}^{\infty}\varphi_n\Big(\frac{k_l}{\lambda_n}\sum_{j=N}^{n}|x_l(j)|\Big)\\
&\geqslant\frac{1}{k_l}\Big(1+\sigma_\Phi\Big(k_l x_l|_I\Big)\Big)+\sum_{n=N}^{\infty}\varphi_n\Big(\frac{1}{\lambda_n}\sum_{j=N}^{n}|x_l(j)|\Big)\ \ [\because k_l\geqslant 1.]\\
&\geqslant\|x_l|_I\|_\Phi^A+\sigma_\Phi\Big(x_l|_{\mathbb{N}-I}\Big)\ \ \Big[\because\sum_{n=N}^{\infty}\varphi_n\Big(\frac{1}{\lambda_n}\sum_{j=N}^{n}|x_l(j)|\Big)\geqslant\sum_{n=N}^{\infty}\varphi_n\Big(\frac{1}{\lambda_n}\sum_{j=n-\lambda_n+1}^{n}|x_l(j)|\Big)\Big]\\
&>\|x\|_\Phi^A-\frac{\delta_1}{4}+\delta_1=\|x\|_\Phi^A+\frac{3\delta_1}{4}\ \text{for } l>N_0.
\end{aligned}
$$

Therefore $\|x\|_\Phi^A \leqslant 1-\frac{3\delta_1}{4}$, which means that the space $V_\Phi^A(\lambda)$ has the coordinatewise Uniform Kadec-Klee property. □

THEOREM 5. *Let $\Phi=(\varphi_n)_{n=1}^{\infty}>0$ be a Musielak-Orlicz function satisfying condition $(*)$, i.e., (1.2) holds and δ_2, i.e., (1.1) holds. Then $V_\Phi^L(\lambda)$ has the uniform Opial property.*

Proof. Let $\varepsilon>0$ be any number and $(x_l)\subset S(V_\Phi^L(\lambda))$ be any weakly null sequence. We show that for any $\varepsilon>0$ there is a $\mu>0$ such that

$$\liminf_{l\to\infty}\|x_l+x\|_\Phi^L\geqslant 1+\mu,$$

for each $x\in V_\Phi^L(\lambda)$ satisfying $\|x\|_\Phi^L\geqslant\varepsilon$. Since $\Phi\in\delta_2$ and $\Phi>0$, so by Lemma 3, for each $\varepsilon>0$ there is a number $\delta\in(0,1)$ such that for each $x\in V_\Phi^L(\lambda)$, we have $\sigma_\Phi(x)\geqslant\delta$. Since $\Phi\in\delta_2$, $\Phi>0$ and Φ satisfies the condition $(*)$, so by Lemma 7 for any $\varepsilon>0$, there exists $\delta_1\in(0,\delta)$ such that $\sigma_\Phi(u)\leqslant 1$, $\sigma_\Phi(v)\leqslant\delta_1$ imply

$$|\sigma_\Phi(u+v)-\sigma_\Phi(u)|<\frac{\delta}{8}\ \text{for any}\ u,v\in V_\Phi^L(\lambda). \tag{3.5}$$

Since $\sigma_\Phi(x)<\infty$, so there is a number $n_0\in\mathbb{N}$ such that

$$\sum_{n=n_0+1}^{\infty}\varphi_n\Big(\frac{1}{\lambda_n}\sum_{j\in I_n}|x(j)|\Big)\leqslant\frac{\delta_1}{8}. \tag{3.6}$$

From (3.6) it follows that

$$\delta \leqslant \sum_{n=1}^{n_0} \varphi_n\Big(\frac{1}{\lambda_n}\sum_{j\in I_n}|x(j)|\Big) + \sum_{n=n_0+1}^{\infty} \varphi_n\Big(\frac{1}{\lambda_n}\sum_{j\in I_n}|x(j)|\Big)$$
$$\leqslant \sum_{n=1}^{n_0} \varphi_n\Big(\frac{1}{\lambda_n}\sum_{j\in I_n}|x(j)|\Big) + \frac{\delta_1}{8},$$

which implies $\sum_{n=1}^{n_0} \varphi_n\Big(\frac{1}{\lambda_n}\sum_{j\in I_n}|x(j)|\Big) \geqslant \delta - \frac{\delta_1}{8} > \delta - \frac{\delta}{8} = \frac{7\delta}{8}$. Since $x_l \to 0$ weakly, i.e., $x_l(i) \to 0$ for each i, so there exists a l_0 such that for all $l \geqslant l_0$, the last inequality yields

$$\sum_{n=1}^{n_0} \varphi_n\Big(\frac{1}{\lambda_n}\sum_{j\in I_n}|x_l(j)+x(j)|\Big) \geqslant \frac{7\delta}{8}. \tag{3.7}$$

Again, by the fact that $x_l \to 0$ weakly, we can choose an n_0 such that $\sigma_\Phi(x_l|_{n_0}) \to 0$ as $l \to \infty$. So there exists a $l_1 > l_0$ such that $\sigma_\Phi(x_l|_{n_0}) \leqslant \delta_1$ for all $l \geqslant l_1$. Since $(x_l) \subset S(V_\Phi^L(\lambda))$, i.e., $\|x_l\|_\Phi^L = 1$, so by Lemma 4, we have $\sigma_\Phi(x_l) = 1$. This implies that there exists n_0 such that $\sigma_\Phi(x_l|_{\mathbb{N}-n_0}) \leqslant 1$. Now choose $u = x_l|_{\mathbb{N}-n_0}$ and $v = x_l|_{n_0}$. Then $u, v \in V_\Phi^L(\lambda)$, $\sigma_\Phi(u) \leqslant 1$, $\sigma_\Phi(v) \leqslant \delta_1$. So from (3.5), for all $l \geqslant l_1$, we have

$$\big|\sigma_\Phi(x_l|_{\mathbb{N}-n_0} + x_l|_{n_0}) - \sigma_\Phi\big(x_l|_{\mathbb{N}-n_0}\big)\big| < \tfrac{\delta}{8},$$

which implies that $\sigma_\Phi(x_l) - \frac{\delta}{8} < \sigma_\Phi\big(x_l|_{\mathbb{N}-n_0}\big)$ for all $l \geqslant l_1$, i.e., $\sum_{n=n_0+1}^{\infty} \varphi_n\Big(\frac{1}{\lambda_n}\sum_{j\in I_n}|x_l(j)|\Big)$ $> 1 - \frac{\delta}{8}$ for all $l \geqslant l_1$. Again, since $\sigma_\Phi\big(x_l|_{\mathbb{N}-n_0}\big) \leqslant 1$ and $\sigma_\Phi\big(x|_{\mathbb{N}-n_0}\big) \leqslant \frac{\delta_1}{8} < \delta_1$, so from the equations (3.5) and (3.7), we obtain

$$\sigma_\Phi(x_l + x) = \sum_{n=1}^{n_0} \varphi_n\Big(\frac{1}{\lambda_n}\sum_{j\in I_n}|x_l(j)+x(j)|\Big) + \sum_{n=n_0+1}^{\infty} \varphi_n\Big(\frac{1}{\lambda_n}\sum_{j\in I_n}|x_l(j)+x(j)|\Big)$$
$$> \sum_{n=1}^{n_0} \varphi_n\Big(\frac{1}{\lambda_n}\sum_{j\in I_n}|x_l(j)+x(j)|\Big) + \sum_{n=n_0+1}^{\infty} \varphi_n\Big(\frac{1}{\lambda_n}\sum_{j\in I_n}|x_l(j)|\Big) - \frac{\delta}{8}$$
$$> \frac{7\delta}{8} + (1 - \frac{\delta}{8}) - \frac{\delta}{8} = 1 + \frac{5\delta}{8}.$$

Since $\Phi \in \delta_2$ and satisfying the condition $(*)$ and $\Phi > 0$, so by Lemma 8, there is a $\mu > 0$ depending only on δ such that $\|x_l + x\|_\Phi^L > 1 + \mu$. Hence $\liminf_{l\to\infty} \|x_l + x\|_\Phi^L \geqslant 1 + \mu$. This completes the proof. □

COROLLARY 2. *(i) If Φ satisfies condition-δ_2 and $(*)$ then l_Φ^L* [24] *has the uniform Opial property.*

(ii) Suppose $\lambda_n = n$, $n = 1, 2, \dots$ and $\varphi_n(u) = |u|^{p_n}$ for all $u \in \mathbb{R}$, $1 < p_n < \infty$ $\forall n$. Then it is easy to verify that $\Phi \in \delta_2$ if and only if $\limsup_{n\to\infty} p_n < \infty$. Therefore $ces_{(p)}^L$ [10] *has the uniform Opial property.*

(iii) If $\varphi_n = \varphi$ $\forall n$ *and* $\Phi \in \delta_2$*, then Cesàro-Orlicz sequence spaces* ces^L_φ [9] *has the uniform Opial property.*

THEOREM 6. *Let* $\Phi > 0$ *be a Musielak-Orlicz function satisfying conditions* (∞_1) *and* δ_2*, i.e.,* (1.1)*. Then* $V^A_\Phi(\lambda)$ *has the uniform Opial property.*

Proof. Take any $\varepsilon > 0$ and $x \in V^A_\Phi(\lambda)$ with $\|x\|^A_\Phi \geqslant \varepsilon$. Let $(x_l) \subset S(V^A_\Phi(\lambda))$ be any weakly null sequence. We will show that for every $\varepsilon > 0$ there is a $\mu > 0$ such that

$$\liminf_{l\to\infty} \|x_l + x\|^A_\Phi \geqslant 1 + \mu,$$

for each $x \in V^A_\Phi(\lambda)$. Since $\Phi \in \delta_2$ and $\Phi > 0$ by Lemma 3, there is a $\delta \in (0, \frac{4}{5})$ independent of x such that $\sigma_\Phi(\frac{x}{2}) \geqslant \delta$. Since $\Phi \in \delta_2$ implies $V^A_\Phi(\lambda) = h^A_\Phi(\lambda)$, which is order continuous by Theorem 1. Hence x is an order continuous element. So, there exists a natural number $n_0 \in \mathbb{N}$ such that

$$\left\|x - x|_{n_0}\right\|^A_\Phi = \left\|x|_{\mathbb{N}-n_0}\right\|^A_\Phi < \tfrac{\delta}{8} \text{ and } \sum_{n=n_0+1}^{\infty} \varphi_n\Big(\frac{1}{\lambda_n}\sum_{j\in I_n}\frac{|x(j)|}{2}\Big) < \frac{\delta}{8}.$$

Since $\sigma_\Phi(\frac{x}{2}) \geqslant \delta$, it follows that

$$\begin{aligned}\delta &\leqslant \sum_{n=1}^{n_0} \varphi_n\Big(\frac{1}{\lambda_n}\sum_{j\in I_n}\frac{|x(j)|}{2}\Big) + \sum_{n=n_0+1}^{\infty} \varphi_n\Big(\frac{1}{\lambda_n}\sum_{j\in I_n}\frac{|x(j)|}{2}\Big)\\ &< \sum_{n=1}^{n_0} \varphi_n\Big(\frac{1}{\lambda_n}\sum_{j\in I_n}\frac{|x(j)|}{2}\Big) + \frac{\delta}{8}.\end{aligned}$$

This gives $\displaystyle\sum_{n=1}^{n_0} \varphi_n\Big(\frac{1}{\lambda_n}\sum_{j\in I_n}\frac{|x(j)|}{2}\Big) > \frac{7\delta}{8}$. Since $x_l \to 0$ weakly implies that $x_l(i) \to 0$ as $l \to \infty$ for each i, so we have $\sigma_\Phi(x_l|_{n_0}) \to 0$ as $l \to \infty$. Hence, by Lemma 3, there exists a natural number l_0 such that

$$\left\|x_l|_{n_0}\right\|^A_\Phi < \frac{\delta}{8} \text{ for all } l > l_0 \text{ which implies } \left\|x_l|_{\mathbb{N}-n_0}\right\|^A_\Phi > 1 - \frac{\delta}{8}, \text{ since } \|x_l\|^A_\Phi = 1. \tag{3.8}$$

Now for all $l \geqslant l_0$, we have

$$\begin{aligned}\|x_l + x\|^A_\Phi &= \left\|(x_l + x)|_{n_0} + (x_l + x)|_{\mathbb{N}-n_0}\right\|^A_\Phi \geqslant \left\|(x_l + x)|_{n_0} + x_l|_{\mathbb{N}-n_0}\right\|^A_\Phi - \frac{\delta}{8}\\ &\geqslant \left\|x|_{n_0} + x_l|_{\mathbb{N}-n_0}\right\|^A_\Phi - \frac{\delta}{8} - \frac{\delta}{8} = \left\|x|_{n_0} + x_l|_{\mathbb{N}-n_0}\right\|^A_\Phi - \frac{\delta}{4}.\end{aligned}$$

Since Φ satisfying condition (∞_1), so by Lemma 10, there exists $k_l > 0$ such that for $l \geqslant l_0$, we have

$$\left\|x|_{n_0} + x_l|_{\mathbb{N}-n_0}\right\|^A_\Phi = \tfrac{1}{k_l}\Big(1 + \sigma_\Phi\Big(k_l\Big(x|_{n_0} + x_l|_{\mathbb{N}-n_0}\Big)\Big)\Big).$$

Now using the fact that $\sigma_\Phi(u+v) \geqslant \sigma_\Phi(u) + \sigma_\Phi(v)$, whenever $supp(u) \cap supp(v) = \emptyset$, then we have

$$\begin{aligned}\|x_l + x\|_\Phi^A &\geqslant \frac{1}{k_l} + \frac{1}{k_l}\sigma_\Phi\left(k_l x|_{n_0}\right) + \frac{1}{k_l}\sigma_\Phi\left(k_l x_l|_{\mathbb{N}-n_0}\right) - \frac{\delta}{4} \\ &\geqslant \left\|x_l|_{\mathbb{N}-n_0}\right\|_\Phi^A + \frac{1}{k_l}\sigma_\Phi\left(k_l x|_{n_0}\right) - \frac{\delta}{4}.\end{aligned}$$

Without loss of generality, we may assume that $k_l \geqslant \frac{1}{2}$ for all l because if $k_l < \frac{1}{2}$ then we have $\|x_l + x\|_\Phi^A > 2 - \frac{\delta}{4} > 1 + \delta$. Using the convexity of Φ, we have $\sigma_\Phi\left(k_l x|_{n_0}\right) \geqslant 2k_l \sigma_\Phi\left(\frac{1}{2}x|_{n_0}\right)$. Therefore (3.8) implies that

$$\begin{aligned}\|x_l + x\|_\Phi^A &\geqslant \left\|x_l|_{\mathbb{N}-n_0}\right\|_\Phi^A + 2\sigma_\Phi\left(\frac{1}{2}x|_{n_0}\right) - \frac{\delta}{4} \\ &> \left\|x_l|_{\mathbb{N}-n_0}\right\|_\Phi^A + 2\sum_{n=1}^{n_0}\varphi_n\left(\frac{1}{\lambda_n}\sum_{j\in I_n}\frac{|x(j)|}{2}\right) - \frac{\delta}{4} \\ &\geqslant 1 - \frac{\delta}{8} + 2.\frac{7\delta}{8} - \frac{\delta}{4} = 1 + \frac{11\delta}{8},\end{aligned}$$

which implies that $\liminf_{l\to\infty} \|x_l + x\|_\Phi^A \geqslant 1 + \mu$, where μ depends upon δ. This completes the proof. □

COROLLARY 3. *(i) Let $\lambda_n = n$, $n = 1, 2, \ldots$ and $\varphi_n(u) = |u|^{p_n}$ for all $u \in \mathbb{R}$, $1 < p_n < \infty$ $\forall n$. Then it is easy to verify that $\Phi \in \delta_2$ if and only if $\limsup_{n\to\infty} p_n < \infty$. Therefore $ces^A_{(p)}$ [10] has the uniform Opial property.*

(ii) Suppose $\varphi_n = \varphi$ and $\lambda_n = 1$ for all $n \in \mathbb{N}$ and $\Phi \in \delta_2$. Then l^A_φ [25] has the uniform Opial property.

(iii) If $\varphi_n = \varphi$ $\forall n$, $\lambda_n = n$, $n = 1, 2, \ldots$ and $\Phi \in \delta_2$, then the Cesàro-Orlicz sequence spaces ces^A_φ [9] has the uniform Opial property.

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