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2–HILBERT C^*–MODULES AND SOME GRÜSS' TYPE INEQUALITIES IN A–2–INNER PRODUCT SPACES

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Abstract. In this paper we generalize the concept of complex valued 2-inner product to A-valued 2-inner product, where A is a C^*-algebra. We define and study the notion of A-valued 2-inner product on right A-module E (E is a linear space) and by new definition, some Grüss type inequalities are given.

1. Preliminaries

In this section, we Introduce 2-inner product spaces and Hilbert C^*-modules. The concepts of 2-inner product and 2-inner product spaces have been studied by many authors, see [2] and also, for Hilbert C^*-modules you can see [10].

Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let X be a linear space of dimension greater than 1 over the field $\mathbb{K} = \mathbb{R}$ of real numbers or the field $\mathbb{K} = \mathbb{C}$ of complex numbers. Suppose that $\langle .,.|.\rangle$ is a $\mathbb{K}$-valued function defined on $X \times X \times X$ satisfying the following conditions:

(a) $\langle x,x|z\rangle \geqslant 0$ and $\langle x,x|z\rangle = 0$ if and only if $x.z$ are linearly dependent,

(b) $\langle x,x|z\rangle = \langle z,z|x\rangle$,

(c) $\langle y,x|z\rangle = \overline{\langle x,y|z\rangle}$,

(d) $\langle \alpha x,y|z\rangle = \alpha\langle x,y|z\rangle$ for any scalar $\alpha \in \mathbb{K}$,

(e) $\langle x+x',y|z\rangle = \langle x,y|z\rangle + \langle x',y|z\rangle$.

$\langle .,.|.\rangle$ is called a *2-inner product* on X and $(X,\langle .,.|.\rangle)$ is called a *2-inner product space* (or *2-pre-Hilbert space*). Some basic properties of 2-inner product $\langle .,.|.\rangle$ can be immediately obtained as follows [3]:

For any $x,y,z \in X$ and $\alpha \in \mathbb{K}$;

$$\langle 0,y|z\rangle = 0, \quad \langle x,0|z\rangle = 0, \tag{1}$$

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$$\langle x, \alpha y|z\rangle = \overline{\alpha}\langle x, y|z\rangle, \tag{2}$$

$$\langle z, z|x \mp y\rangle = \langle x \mp y, x \mp y|z\rangle = \langle x, x|z\rangle + \langle y, y|z\rangle \mp 2Re\langle x, y|z\rangle, \tag{3}$$

$$\langle x, y|z\rangle = \frac{1}{4}[\langle z, z|x+y\rangle - \langle z, z|x-y\rangle] + \frac{i}{4}[\langle z, z|x+iy\rangle - \langle z, z|x-iy\rangle], \tag{4}$$

$$\langle x, y|\alpha z\rangle = |\alpha|^2\langle x, y|z\rangle, \tag{5}$$

$$|\langle x, y|z\rangle|^2 \leqslant \langle x, x|z\rangle\langle y, y|z\rangle \quad \text{(Cauchy-Schwarz inequality)}, \tag{6}$$

$$\langle z, y|z\rangle = \langle y, z|z\rangle = 0. \tag{7}$$

In any given 2-inner product space $(X, \langle ., .|.\rangle)$, we can define a function $\|.|.\|$ on $X \times X$ by

$$\|x|z\| = \sqrt{\langle x, x|z\rangle} \tag{8}$$

for all $x, z \in X$.

It is easy to see that this function satisfies the following conditions:

(i) $\|x|z\| \geqslant 0$ and $\|x|z\| = 0$ if and only if x and z are linearly dependent,

(ii) $\|z|x\| = \|x|z\|$,

(iii) $\|\alpha x|z\| = |\alpha|\|x|z\|$ for any scalar $\alpha \in \mathbb{K}$,

(iv) $\|x + x'|z\| \leqslant \|x|z\| + \|x'|z\|$.

Any function $\|.|.\|$ defined on $X \times X$ and satisfying the above conditions is called a *2-norm* on X and $(X, \|.|.\|)$ is called a *linear 2-normed space* [8]. Whenever a 2-inner product space $(X, \langle ., .|.\rangle)$ is given, we consider it as a linear 2-normed space $(X, \|.|.\|)$ with the 2-norm defined by (8).

Now, we give briefly definition of Hilbert C^*-modules.

Let A be a C^*-algebra. An *inner product A-module* is a linear space E which is a right A-module (with compatible scalar multiplication: $\lambda(xa) = (\lambda x)a = x(\lambda a)$ for $x \in E, a \in A, \lambda \in \mathbb{C}$), together with a map $(x, y) \mapsto \langle x, y\rangle : E \times E \to A$ such that for all $x, y, z \in E$, $\alpha, \beta \in \mathbb{C}$ and $a \in A$:

(a$'$) $\langle x, x\rangle \geqslant 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$,

(b$'$) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$,

(c$'$) $\langle xa, y\rangle = a\langle x, y\rangle$,

(d$'$) $\langle y, x\rangle = \langle x, y\rangle^*$.

Recall that an element a of C^*-algebra A is positive if a is hermition and $\sigma(a) \subseteq \mathbb{R}^+$. We write $a \geqslant 0$ to mean that a is positive (see [11]).

Note that, it follows from the last condition that the inner product is conjugate linear in its second variable. So that an inner product space is the same thing as an inner product $\mathbb{C}$-module.

If E satisfies all the conditions for an inner product A-module except for the second part of the first condition then we call E a *semi-inner product A-module*. For such modules there is an useful version of the Cauchy-Schwarz inequality [10]:

If E is a semi-inner product A-module and $x,y \in E$ then

$$\langle y,x\rangle\langle x,y\rangle \leqslant \| \langle x,x\rangle \| \langle y,y\rangle. \tag{9}$$

For x in E we write $\|x\| = \|\langle x,x\rangle\|^{\frac{1}{2}}$. It is easy to deduce from this that if E is an inner product A-module then $\|.\|$ is a norm on E and $\|\langle x,y\rangle\| \leqslant \|x\|\|y\|$.

An inner product A-module which is complete with respect to its norm is called a *Hilbert A-module* or a *Hilbert C^*-module* over the C^*-algebra A.

In this paper, as in the case of inner product C^*-modules in [10], we are going to define and study the concept of 2-inner product C^*-modules. Indeed, 2-inner product C^*-modules are generalization of 2-inner product spaces for C^*-algebra A rather than $\mathbb{C}$ (So that a 2-inner product space is the same thing as a 2-inner product $\mathbb{C}$-module).

2. 2-pre-Hilbert C^*-modules

In this section, we define and study the notion of 2-inner product C^*-modules (or 2-pre-Hilbert C^*-modules) and state elementary properties in this case.

Let E be a right A-module where A is a C^*-algebra, an *A-combination* of $x_1, x_2, \ldots, x_n$ in E is written as follows

$$\sum_{i=1}^{n} x_i a_i = x_1 a_1 + x_2 a_2 + \ldots + x_n a_n \qquad (a_i \in A)$$

and $x_1, x_2, \ldots, x_n$ are called *A-independent* if the equation

$$x_1 a_1 + x_2 a_2 + \ldots + x_n a_n = 0$$

has exactly one solution, namely $a_1 = a_2 = \ldots = a_n = 0$, otherwise, we say that $x_1, x_2, \ldots, x_n$ are *A-dependent*.

The maximum number of elements in E that are A-independent, is called *A-rank* of E.

DEFINITION 1. Let A be a C^*-algebra. A *2-inner product A-module* (or *2-pre-Hilbert C^*-modules*) is a linear space E by A-rank greater than 1, which is a right A-module(with compatible scalar multiplication: $\lambda(xa) = (\lambda x)a = x(\lambda a)$ for $x \in E$, $a \in A$, $\lambda \in \mathbb{C}$) together with a map $(x,y,z) \mapsto \langle x,y|z\rangle$: $E \times E \times E \to A$ such that for all $x,x',y,z \in E$, $a \in A$, $\alpha,\beta \in \mathbb{C}$;

(A) $\langle x,x|z\rangle \geqslant 0$; $\langle x,x|z\rangle = 0$ if and only if x,z are A-dependent,

(B) $\langle \alpha x + \beta x', y|z\rangle = \alpha\langle x,y|z\rangle + \beta\langle x',y|z\rangle$,

(C) $\langle xa, y|z\rangle = a\langle x,y|z\rangle$,

(D) $\langle x,y|z\rangle = \langle y,x|z\rangle^{\star}$,

(E) $\langle x,x,|z\rangle = \langle z,z|x\rangle$.

$(E,\langle .,.|.\rangle)$ is called an *A-2-inner product space* or *C^*-2-inner product space*. Note that if $A=\mathbb{C}$ then E with a 2-inner product is a 2-inner product $\mathbb{C}$-module. If E satisfies all the condition for a 2-inner product A-module except for the second part of condition (A), then we call E a *2-semi-inner product A-module*.

Recall that in a C^*-algebra A, for each $a\in A$ there exist a unique pair of hermition elements $b,c\in A$ such that $a=b+ic$ $(b=\frac{1}{2}(a+a^*)$ and $c=\frac{1}{2i}(a-a^*))$, for more details see [11]. Let us that call $b=\Re(a)$ and $c=\Im(a)$ If no confusion can arise(note that $\Re(a)$ and $\Im(a)$ are belong to A).

We have also, properties (1)–(5) in A-2-inner product spaces and similarly these properties can be obtained as follow:

From (B) and (D), we have

$$\langle 0,y|z\rangle = \langle x,0|z\rangle = 0 \tag{10}$$

and also

$$\langle x,ya|z\rangle = \langle x,y|z\rangle a^*. \tag{11}$$

Using (B) and (E), we have

$$\langle z,z|x\mp y\rangle = \langle x\mp y,x\mp y|z\rangle = \langle x,x|z\rangle + \langle y,y|z\rangle \mp 2\Re\langle x,y|z\rangle \tag{12}$$

and

$$\Re\langle x,y|z\rangle = \frac{1}{4}[\langle z,z|x+y\rangle - \langle z,z|x-y\rangle], \tag{13}$$

$$\Im\langle x,y|z\rangle = \Re[-i\langle x,y|z\rangle] = \frac{1}{4}[\langle z,z|x+iy\rangle - \langle z,z|x-iy\rangle], \tag{14}$$

then

$$\langle x,y|z\rangle = \frac{1}{4}[\langle z,z|x+y\rangle - \langle z,z|x-y\rangle] + \frac{i}{4}[\langle z,z|x+iy\rangle - \langle z,z|x-iy\rangle]. \tag{15}$$

Using the above formula and (D), we have for any $a\in A$, that

$$\langle x,y|za\rangle = a\langle x,y|z\rangle a^*. \tag{16}$$

However, for $\alpha\in\mathbb{C}$, (16) reduces to

$$\langle x,y|\alpha z\rangle = |\alpha|^2\langle x,y|z\rangle. \tag{17}$$

Also, from (17) it follows that

$$\langle x,y|0\rangle = 0. \tag{18}$$

Now, we prove that there is an useful version of the Cauchy-Schwarz inequality:

PROPOSITION 1. *If E is a 2-semi inner product A-module and $x,y,z \in E$ then*

$$\langle y,x|z\rangle\langle x,y|z\rangle \leqslant \|\langle x,x|z\rangle\|\langle y,y|z\rangle. \tag{19}$$

Proof. Suppose, as we may without loss of generality, that $\|\langle x,x|z\rangle\| = 1$. For $a \in A$, we have

$$\begin{aligned}0 &\leqslant \langle xa-y, xa-y|z\rangle \\ &= a\langle x,x|z\rangle a^* - a\langle x,y|z\rangle - \langle y,x|z\rangle a^* + \langle y,y|z\rangle \\ &\leqslant aa^* - a\langle x,y|z\rangle - \langle y,x|z\rangle a^* + \langle y,y|z\rangle.\end{aligned}$$

(The last line comes from the fact (1.6.8 in [4]) that if c is a positive element of A then $aca^* \leqslant \|c\|aa^*$.) Now put $a = \langle y,x|z\rangle$ to get $aa^* \leqslant \langle y,y|z\rangle$, as required. □

COROLLARY 1. *let E be a 2-inner product A-module and $x,y \in E$ then*

$$\langle x,y|y\rangle = \langle y,x|y\rangle = 0.$$

DEFINITION 2. Let E be a linear space by A-rank greater than 1 which is a right A-module where A is a C^*-algebra and let $\|.|.\|$ be a real valued function in $E \times E$ satisfying the following conditions:

(I) $\|x|z\| \geqslant 0$ and $\|x|z\| = 0$ if and only if x,z are A-dependent,

(II) $\|z|x\| = \|x|z\|$,

(III) $\|\alpha x|z\| = |\alpha|\|x|z\|$ for any scalar $\alpha \in \mathbb{K}$,

(IV) $\|xa|z\| \leqslant \|a\|\|x|z\|$ for any $a \in A$,

(V) $\|x+x'|z\| \leqslant \|x|z\| + \|x'|z\|$.

$\|.|.\|$ is called an A-*2-norm* or C^*-*2-norm* on E and $(E, \|.|.\|)$ is called an A-*2-normed* or C^*- *2-normed space*. Note that an ordinary linear 2-normed space is a $\mathbb{C}$-2-normed space.

Using(V), we have for any points $x,y \in E$ and any $a \in A$,

$$\|x|y\| = \|x|y+xa\|.$$

Let E be a 2-inner product A-module. For x,z in E we define $\|x|z\| = \|\langle x,x|z\rangle\|^{\frac{1}{2}}$. It follows from proposition 1 that

$$\|\langle x,y|z\rangle\| \leqslant \|x|z\|\|y|z\| \tag{20}$$

and it is easy to deduce that $\|.|.\|$ is an A-2-norm on E (using the formulas $\|ab\| \leqslant \|a\|\|b\|$ and $\|a\| = \|a^*\|$ where $a,b \in A$, we have (IV) and by (20) we have the condition (V)).

A 2-inner product A-module which is complete with respect to its A-2-norm is called a *2-Hilbert A-module*, or a *2-Hilbert C^*-module* over the C^*-algebra A. (Note that, convergence in A-2-normed space is the same thing as convergence in linear 2-normed space.)

EXAMPLE 1. Let A be a commutative unital C^*-algebra. By the commutative Gelfand-Naimark theorem we can identify A with $C(X)$, the algebra of continuous complex-valued functions on a compact Hausdorff space X. If X were an euclidean manifold then one would analyse it by geometric techniques, among the most important of which is the study of vector bundles over X. A vector bundle E can be described as follows. Take a fixed euclidean space H (note that, for example, if $(H,\langle .,.\rangle)$ is a Hilbert space a continuous 2-inner product is defined on H by $\langle a,b|c\rangle = \langle a,b\rangle \|c\|^2 - \langle a,c\rangle\langle b,c\rangle$, $a,b,c \in H$) and for each t in X let H_t be a subspace of H. Let E be the space of all continuous functions ξ from X to H such that, for all t in X, $\xi(t) \in H_t$. Then E is naturally endowed with a $C(X)$-valued 2-semi inner product. Namely, if $\xi,\eta \in E$, then we define $\langle \xi,\eta|\gamma\rangle$ to be the function $t \mapsto \langle \xi(t),\eta(t)|\gamma(t)\rangle_H$ Also, E has the structure of a $C(X)$-module: given ξ in E and f in $C(X)$ we define ξf to be the pointwise product $t \mapsto \xi(t)f(t)$, which is an element of E.

EXAMPLE 2. Recall that every C^*-algebra A itself is a Hilbert A-module, but note that A is not a nontrivial 2-pre-Hilbert A-module. Indeed, if A is a 2-pre-Hilbert A-module then by (C), (11), (16) and (A),

$$\langle a,b|c\rangle = ac\langle 1,1|1\rangle c^* b^* = 0 \quad (a,b,c \in A).$$

In the following example we show that, every 2-semi-inner product A-module is an semi-inner product A-module.

EXAMPLE 3. Let E be a 2-semi inner product A-module, then E is a semi-inner product A-module if we define

$$\langle x,y\rangle := \langle x,y|z\rangle + \langle x,y|z'\rangle,$$

where $x,y,z \in E$ and z,z' are A-module independent.

EXAMPLE 4. Let E be a 2-inner product A-module, then we can construct a quotient of E that is a 2-inner product A-module, using proposition 1. In fact Let A be a unital C^*-algebra, B a C^*-subalgebra of A containing the identity, and $\psi : A \to B$ a linear norm reducing idempotent. Such a map is called a conditional expectation from A to B. A conditional expectation is always a positive map and satisfies

$$\psi(bac) = b\psi(a)c \qquad (a \in A, b,c \in B)$$

(see [12]). Suppose that E is a 2-semi-inner product A-module with A-valued 2-semi-inner product $\langle .,.|.\rangle_A$. Then E is a 2-semi-inner product B-module under B-valued 2-semi-inner product given by

$$\langle x,y|z\rangle_B = \psi(\langle x,y|z\rangle_A).$$

3. A Grüss' type inequality in A-2-inner product spaces

The Grüss inequality have been studied by many authors in different spaces, see [5], [7], [6], [1] and [9]. This section is a version of section 3 in [6] for A-2-inner product spaces.

Throughout this section, let E be a 2-inner product A-module over the C^*-algebra A (with unit 1_A) equipped to A-2-norm $\|x|z\| = \|\langle x,x|z\rangle\|^{\frac{1}{2}}$ $(x,z \in E)$. We assume that $x1_A = x$ for every $x \in E$.

The following lemma holds.

LEMMA 1. *Let E be a 2-inner product A-module over C^* algebra A, $y,x,z,Y \in E$ and $a \neq d$. Then*

$$\Re\langle Y-x, x-y|z\rangle \geqslant 0$$

if and only if

$$\left\langle x-\frac{y+Y}{2}, x-\frac{y+Y}{2}\Big|z\right\rangle \leqslant \frac{1}{4}\langle Y-y, Y-y|z\rangle.$$

Proof. Define

$$I_1 := \Re\langle Y-x, x-y|z\rangle, \qquad I_2 := \frac{1}{4}\langle Y-y, Y-y|z\rangle - \left\langle x-\frac{y+Y}{2}, x-\frac{y+Y}{2}\Big|z\right\rangle.$$

A simple calculation shows that

$$I_1 = I_2 = \Re[\langle x,y|z\rangle + \langle Y,x|z\rangle] - \Re\langle Y,y|z\rangle - \langle x,x|z\rangle$$

and thus, obviously, $I_1 \geqslant 0$ iff $I_2 \geqslant 0$ showing the required equivalence. □

The following corollary is obvious.

COROLLARY 2. *Let $x,z,e \in E$ and $a,b \in A$ with $a \neq b$. Then*

$$\Re\langle eb-x, x-ea|z\rangle \geqslant 0$$

if and only if

$$\left\langle x-e\frac{a+b}{2}, x-e\frac{a+b}{2}\Big|z\right\rangle \leqslant \frac{1}{4}(b-a)\langle e,e|z\rangle(b-a)^*.$$

COROLLARY 3. *Let $x,z,e \in E$, $\langle e,e|z\rangle \neq 0$ be an idempotent and $a,b \in A$ with $a \neq b$. If*

$$\Re\langle eb-x, x-ea|z\rangle \geqslant 0$$

then

$$\left\|x-e\frac{a+b}{2}\Big|z\right\| \leqslant \frac{1}{2}\|b-a\|.$$

Proof. Since $\langle e,e|z\rangle \neq 0$ is idempotent, we have, $\|e,z\| = \|\langle e,e|z\rangle\|^{\frac{1}{2}} = 1$. Therefore the desired inequality is obtained. □

LEMMA 2. *Let E be a 2-inner product A-module and $x,z,e \in E$. If $\langle e,e|z\rangle \neq 0$ is idempotent, then $e\langle e,e|z\rangle = za+e$ for some $a \in A$, and therefore*

$$\langle e,e|z\rangle\langle e,x|z\rangle = \langle e,x|z\rangle, \quad \langle x,e|z\rangle = \langle x,e|z\rangle\langle e,e|z\rangle.$$

Proof. Observe that the equality

$$\begin{aligned}\langle e\langle e,e|z\rangle - e,e\langle e,e|z\rangle - e|z\rangle \\ &= \langle e,e|z\rangle\langle e,e|z\rangle\langle e,e|z\rangle - \langle e,e|z\rangle\langle e,e|z\rangle - \langle e,e|z\rangle\langle e,e|z\rangle + \langle e,e|z\rangle \\ &= 0,\end{aligned}$$

implies that $e\langle e,e|z\rangle - e = za$ or $z = (e\langle e,e|z\rangle - e)b$, for some $a,b \in A$.

If $e\langle e,e|z\rangle - e = za$, then

$$\begin{aligned}\langle e,e|z\rangle\langle e,x|z\rangle &= \langle e\langle e,e|z\rangle,x|z\rangle \\ &= \langle za+e,x|z\rangle \\ &= \langle za,x|z\rangle + \langle e,x|z\rangle \\ &= \langle e,x|z\rangle.\end{aligned}$$

Similarly, $\langle x,e|z\rangle = \langle x,e|z\rangle\langle e,e|z\rangle$.

But, if $z = (e\langle e,e|z\rangle - e)b$, then $z = e(\langle e,e|z\rangle b - b)$, that means e,z are A-dependent or $\langle e,e|z\rangle = 0$. Therefore, this case does not occur. $\square$

The following lemma also holds.

LEMMA 3. *Let E be a 2-inner product A-module, $x,z,e \in E$ and $\langle e,e|z\rangle \neq 0$ be an idempotent. Then*

$$0 \leqslant \langle x,x|z\rangle - \langle x,e|z\rangle\langle e,x|z\rangle$$

and

$$\|\langle x,x|z\rangle - \langle x,e|z\rangle\langle e,x|z\rangle\| = \inf_{a\in A} \|x-ea|z\|^2. \tag{21}$$

Proof. Observe, for any $a \in A$, that

$$\begin{aligned}\langle x-ea,x-e\langle x,e|z\rangle|z\rangle &= \langle x,x|z\rangle - \langle x,e|z\rangle\langle e,x|z\rangle - a\langle e,x|z\rangle + a\langle e,e|z\rangle\langle e,x|z\rangle \\ &= \langle x,x|z\rangle - \langle x,e|z\rangle\langle e,x|z\rangle.\end{aligned}$$

This implies that

$$\langle x,x|z\rangle - \langle x,e|z\rangle\langle e,x|z\rangle = \langle x-e\langle x,e|z\rangle,x-e\langle x,e|z\rangle|z\rangle \geqslant 0.$$

Also,we have

$$\|\langle x,x|z\rangle - \langle x,e|z\rangle\langle e,x|z\rangle\| = \|\langle x-ea,x-e\langle x,e|z\rangle|z\rangle\|.$$

Using (20), we have

$$\begin{aligned}\|\langle x,x|z\rangle-\langle x,e|z\rangle\langle e,x|z\rangle\|^2 &= \|\langle x-ea,x-e\langle x,e|z\rangle|z\rangle\|^2\\ &\leqslant \|x-ea|z\|^2\|x-e\langle x,e|z\rangle|z\|^2\\ &= \|x-ea|z\|^2\|\langle x,x|z\rangle-\langle x,e|z\rangle\langle e,x|z\rangle\|\end{aligned}$$

which gives the bound

$$\|\langle x,x|z\rangle-\langle x,e|z\rangle\langle e,x|z\rangle\| \leqslant \|x-ea|z\|^2, \quad a\in A. \tag{22}$$

Taking the infimum in (22) over $a\in A$, we deduce

$$\|\langle x,x|z\rangle-\langle x,e|z\rangle\langle e,x|z\rangle\| \leqslant \inf_{a\in A}\|x-ea|z\|^2.$$

Since, for $a_0=\langle x,e|z\rangle$, we get $\|x-ea_0|z\|^2=\|\langle x,x|z\rangle-\langle x,e|z\rangle\langle e,x|z\rangle\|$, then the representation (21) is proved. □

Now we can prove the Grüss inequality in A-2-inner product spaces.

THEOREM 2. *Let E be a 2-inner product A-module, $e,z\in E$ and $\langle e,e|z\rangle\neq 0$ be an idempotent. If $a,b,c,d\in A$ and x,y are vectors in E such that the conditions*

$$\Re\langle ec-x,x-ea|z\rangle\geqslant 0, \quad \Re\langle ed-y,y-eb|z\rangle\geqslant 0 \tag{23}$$

hold or, if the following assumptions

$$\left\|x-e\frac{a+c}{2}\middle|z\right\|\leqslant\frac{1}{2}\|c-a\|, \quad \left\|y-e\frac{b+d}{2}\middle|z\right\|\leqslant\frac{1}{2}\|d-b\| \tag{24}$$

are valid, then one has the inequality

$$\|\langle x,y|z\rangle-\langle x,e|z\rangle\langle e,y|z\rangle\|\leqslant\frac{1}{4}\|c-a\|.\|d-b\|. \tag{25}$$

Furthermore, if there is a non zero element m in E such that $\langle m,e|z\rangle=0$ and $\langle m,m|z\rangle=1$, then the constant $\frac{1}{4}$ is best possible.

Proof. We have

$$\|\langle x,y|z\rangle-\langle x,e|z\rangle\langle e,y|z\rangle\|=\|\langle x-e\langle x,e|z\rangle,y-e\langle y,e|z\rangle|z\rangle\|,$$

now, use inequality (20) for the vectors $x-e\langle x,e|z\rangle$, $y-e\langle y,e|z\rangle$, then we have the following inequality

$$\|\langle x,y|z\rangle-\langle x,e|z\rangle\langle e,y|z\rangle\|\leqslant\|x-e\langle x,e|z\rangle|z\|.\|y-e\langle y,e|z\rangle|z\|. \tag{26}$$

Using Lemma 3 and the conditions (24) we obviously have that

$$\|x-e\langle x,e|z\rangle|z\|=\inf_{a\in A}\|x-ea|z\|\leqslant\left\|x-e\frac{a+c}{2}\middle|z\right\|\leqslant\frac{1}{2}\|c-a\|$$

and

$$\|y - e\langle y, e|z\rangle|z\| = \inf_{a\in A}\|y - ea|z\| \leqslant \left\|y - e\frac{b+d}{2}\Big|z\right\| \leqslant \frac{1}{2}\|d-b\|$$

and so, by (26) the desired inequality (25) is obtained.

To prove the sharpness of the constant $\frac{1}{4}$, assume that (25) holds with $x = y$ and a constant $C > 0$, i.e.,

$$\|\langle x,x|z\rangle - \langle x,e|z\rangle\langle e,x|z\rangle\| \leqslant C\|c-a\|^2, \tag{27}$$

provided x,e,z,a and c satisfy the hypothesis of the theorem.

If we define

$$x = e\frac{a+c}{2} + m\frac{c-a}{2},$$

then we have

$$\left\|\left\langle x - e\frac{a+c}{2}\Big|z\right\rangle\right\| = \frac{1}{2}\|c-a\|$$

and thus the condition (24) is fulfilled. From (27) we have

$$\|\langle x,x|z\rangle - \langle x,e|z\rangle\langle e,x|z\rangle\| \leqslant C\|c-a\|^2$$

and, since

$$\|\langle x,x|z\rangle - \langle x,e|z\rangle\langle e,x|z\rangle\| = \frac{1}{4}\|c-a\|^2,$$

then, we get

$$\frac{1}{4}\|c-a\|^2 \leqslant C\|c-a\|^2,$$

for $a \neq c$, which implies that $C \geqslant \frac{1}{4}$, and the proof is completed. □

4. A refinement of Grüss inequality in A-2-inner product spaces

This section is a new version of Theorem 2 in [9] for A-2-inner product spaces. The following result improving (25) holds.

THEOREM 3. *Let E be a 2-inner product A-module. If $x,y,e,z \in E$, $\langle e,e|z\rangle \neq 0$ is an idempotent and $a,b,c,d \in A$ such that*

$$\left\|x - e\frac{a+c}{2}\Big|z\right\| \leqslant \frac{1}{2}\|c-a\|, \quad \left\|y - e\frac{b+d}{2}\Big|z\right\| \leqslant \frac{1}{2}\|d-b\|$$

hold, then one has the inequality

$$\begin{aligned}
&\|\langle x,y|z\rangle - \langle x,e|z\rangle\langle e,y|z\rangle\| \\
&\leqslant \frac{1}{4}\|c-a\|\|d-b\| - \left(\frac{1}{4}\|c-a\|^2 - \left\|x - e\frac{a+c}{2}\Big|z\right\|^2\right)^{\frac{1}{2}} \\
&\quad \times \left(\frac{1}{4}\|d-b\|^2 - \left\|y - e\frac{b+d}{2}\Big|z\right\|^2\right)^{\frac{1}{2}} \\
&\leqslant \frac{1}{4}\|c-a\|\|d-b\|.
\end{aligned}$$

Furthermore, if there is a non zero element m in E such that $\langle m,e|z\rangle = 0$ and $\langle m,m|z\rangle = 1$, then the constant $\frac{1}{4}$ is best possible.

Proof. A simple calculation shows that

$$\langle ea - e\langle x,e|z\rangle, e\langle x,e|z\rangle - ec|z\rangle - \langle ea - x, x - ec|z\rangle = \langle x,x|z\rangle - \langle x,e|z\rangle\langle e,x|z\rangle,$$

Therefore

$$\Re\langle ea - e\langle x,e|z\rangle, e\langle x,e|z\rangle - ec|z\rangle - \Re\langle ea - x, x - ec|z\rangle = \langle x,x|z\rangle - \langle x,e|z\rangle\langle e,x|z\rangle.$$

Since for any $a,b \in A, \Re(a) = \frac{1}{2}(a+a^*)$ and $\langle a,b|z\rangle + \langle b,a|z\rangle \leqslant \frac{1}{2}\langle a+b, a+b|z\rangle$, so

$$\Re\langle ea - e\langle x,e|z\rangle, e\langle x,e|z\rangle - ec|z\rangle \leqslant \frac{1}{4}(a-c)\langle e,e|z\rangle(a-c)^*.$$

As in the proof of Corollary 2

$$\Re\langle ea - x, x - ec|z\rangle = \frac{1}{4}(a-c)\langle e,e|z\rangle(a-c)^* - \left\langle x - e\frac{a+c}{2}, x - e\frac{a+c}{2}\middle| z\right\rangle,$$

Therefore

$$\langle x,x|z\rangle - \langle x,e|z\rangle\langle e,x|z\rangle \leqslant \left\langle x - e\frac{a+c}{2}, x - e\frac{a+c}{2}\middle| z\right\rangle.$$

Similarly

$$\langle y,y|z\rangle - \langle y,e|z\rangle\langle e,y|z\rangle \leqslant \left\langle y - e\frac{b+d}{2}, y - e\frac{b+d}{2}\middle| z\right\rangle.$$

We obtain

$$\|\langle x,x|z\rangle - \langle x,e|z\rangle\langle e,x|z\rangle\|\,\|\langle y,y|z\rangle - \langle y,e|z\rangle\langle e,y|z\rangle\| \leqslant \left\|\left\langle x - e\frac{a+c}{2}\middle| z\right\rangle\right\|^2 \left\|\left\langle y - e\frac{b+d}{2}\middle| z\right\rangle\right\|^2.$$

Finally, using the elementary inequality for real numbers

$$(m^2 - n^2)(p^2 - q^2) \leqslant (mp - nq)^2$$

on

$$m = \frac{1}{2}\|c-a\|, \qquad n = \left[\frac{1}{4}\|c-a\|^2 - \left\|x - e\frac{a+c}{2}\middle| z\right\|^2\right]^{\frac{1}{2}},$$

$$p = \frac{1}{2}\|d-b\|, \qquad q = \left[\frac{1}{4}\|d-b\|^2 - \left\|y - e\frac{b+d}{2}\middle| z\right\|^2\right]^{\frac{1}{2}},$$

we get

$$\begin{aligned}
&\left\|x - e\frac{a+c}{2}\middle| z\right\| \left\|y - e\frac{b+d}{2}\middle| z\right\| \\
&\leqslant \frac{1}{4}\|c-a\|\|d-b\| - \left(\frac{1}{4}\|c-a\|^2 - \left\|x - e\frac{a+c}{2}\middle| z\right\|^2\right)^{\frac{1}{2}} \\
&\quad \times \left(\frac{1}{4}\|d-b\|^2 - \left\|y - e\frac{b+d}{2}|z\right\|^2\right)^{\frac{1}{2}} \\
&\leqslant \frac{1}{4}\|c-a\|\|d-b\|.
\end{aligned}$$

The fact that $\frac{1}{4}$ is the best constant can be proven in a similar manner to the one in the previous theorem. The details are omitted. □

5. Some companion inequality

The following companion of Grüss inequality in A-2-inner product spaces holds.

THEOREM 4. *Let $(E,\langle .,.|.\rangle)$ be a A-2-inner product space and $e,z \in E$ such that $\langle e,e|z\rangle \neq 0$ is an idempotent. If $a,b \in A$ and $x,y \in E$ are so that*

$$\Re\left\langle eb - \frac{x+y}{2}, \frac{x+y}{2} - ea \middle| z\right\rangle \geqslant 0 \tag{28}$$

or,

$$\left\|\frac{x+y}{2} - e\frac{a+b}{2}\middle| z\right\| \leqslant \frac{1}{2}\|b-a\|, \tag{29}$$

then we have the inequality

$$\|\Re[\langle x,y|z\rangle - \langle x,e|z\rangle\langle e,y|z\rangle]\| \leqslant \frac{1}{4}\|b-a\|^2. \tag{30}$$

Furthermore, if there is a non zero element m in E such that $\langle m,e|z\rangle = 0$ and $\langle m,m|z\rangle = 1$, then the constant $\frac{1}{4}$ is best possible.

Proof. Using (13), we have

$$\Re\langle w,u|z\rangle \leqslant \frac{1}{4}\langle w+u, w+u|z\rangle. \tag{31}$$

Since

$$\langle x,y|z\rangle - \langle x,e|z\rangle\langle e,y|z\rangle = \langle x - e\langle x,e,|z\rangle, y - e\langle y,e|z\rangle|z\rangle,$$

then, using(31), we may write

$$\begin{aligned}
&\Re[\langle x,y|z\rangle - \langle x,e|z\rangle\langle e,y|z\rangle] \\
&\quad = \Re[\langle x - e\langle x,e|z\rangle, y - e\langle y,e|z\rangle|z\rangle \\
&\quad \leqslant \frac{1}{4}\langle x - e\langle x,e|z\rangle + y - e\langle y,e|z\rangle, x - e\langle x,e|z\rangle + y - e\langle y,e|z\rangle|z\rangle \\
&\quad = \left\langle \frac{x+y}{2} - e\left\langle \frac{x+y}{2}, e\middle| z\right\rangle, \frac{x+y}{2} - e\left\langle \frac{x+y}{2}, e\middle| z\right\rangle\middle| z\right\rangle \\
&\quad = \left\langle \frac{x+y}{2}, \frac{x+y}{2}\middle| z\right\rangle - \left\langle \frac{x+y}{2}, e\middle| z\right\rangle\left\langle e, \frac{x+y}{2}\middle| z\right\rangle.
\end{aligned}$$

Therefore,

$$\|\Re[\langle x,y|z\rangle - \langle x,e|z\rangle\langle e,y|z\rangle]\| \leqslant \left\|\left\langle \frac{x+y}{2}, \frac{x+y}{2}\middle| z\right\rangle - \left\langle \frac{x+y}{2}, e\middle| z\right\rangle\left\langle e, \frac{x+y}{2}\middle| z\right\rangle\right\|. \tag{32}$$

If we apply the Grüss inequality (25) for $\frac{x+y}{2}$, then we get

$$\left\|\left\langle \frac{x+y}{2}, \frac{x+y}{2} \middle| z\right\rangle - \left\langle \frac{x+y}{2}, e \middle| z\right\rangle \left\langle e, \frac{x+y}{2} \middle| z\right\rangle\right\| \leqslant \frac{1}{4}\|b-a\|^2. \tag{33}$$

Making use of (32) and(33) we deduce (30).

The fact that $\frac{1}{4}$ is best possible constant in (30) follows by the fact that if in (28) we choose $x=y$, then it because $\Re\langle eb-x, x-ea|z\rangle \geqslant 0$, implying $\|\langle x,x|z\rangle - \langle x,e|z\rangle\langle e,x|z\rangle\|^2 \leqslant \frac{1}{4}\|b-a\|^2$, for which, by Grüss inequality in A-2-inner product space, we know that the constant $\frac{1}{2}$ is best possible. □

The following corollary might be of interest if one wanted to evaluate the norm of

$$\|\Re[\langle x,y|z\rangle - \langle x,e|z\rangle\langle e,y|z\rangle]\|.$$

COROLLARY 4. *Let $(E, \langle .,.|.\rangle)$ be a A-2-inner product space and $e,z \in E$, such that $\langle e,e|z\rangle \neq 0$ is an idempotent. If $a,b \in A$ and $x,y \in E$ are so that*

$$\Re\left\langle eb - \frac{x\pm y}{2}, \frac{x\pm y}{2} - ea \middle| z\right\rangle \geqslant 0 \tag{34}$$

or,

$$\left\|\frac{x\pm y}{2} - e\frac{a+b}{2}\middle| z\right\| \leqslant \frac{1}{2}\|b-a\|, \tag{35}$$

then we have the inequality

$$\|\Re[\langle x,y|z\rangle - \langle x,e|z\rangle\langle e,y|z\rangle]\| \leqslant \frac{1}{4}\|b-a\|^2. \tag{36}$$

Furthermore, if there is a non zero element m in E such that $\langle m,e|z\rangle = 0$ and $\langle m,m|z\rangle = 1$, then the constant $\frac{1}{4}$ is best possible.

Proof. Apply Theorem 4 for $-y$ instead of y. □

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