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A MEAN VALUE THEOREM FOR THE CHEBYSHEV FUNCTIONAL

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Abstract. In the present paper we present a mean value theorem for the Chebyshev functional based on divided differences. This theorem is then used to obtain a new Chebyshev-Grüss type inequality.

1. Introduction

Let $f:[a,b]\to\mathbb{R}$ and $x_0,x_1,\ldots,x_n$ be $n+1$ distinct points in the interval $[a,b]$. The divided difference of the function f on the points $x_0,\ldots,x_n$ is defined by

$$[x_0,\ldots,x_n;f]=\sum_{i=0}^{n}\frac{f(x_i)}{l'(x_i)},$$

where $l(x)=\prod_{i=0}^{n}(x-x_i)$.

DEFINITION 1. ([3], pp. 15) Let $n\in\mathbb{N}$. We say that the function $f:[a,b]\to\mathbb{R}$ is n*-convex* if for any $n+1$ distinct points from $[a,b]$, its divided difference is non-negative, i.e.,

$$[x_0,\ldots,x_n;f]\geqslant 0,\forall\{x_0,\ldots,x_n\}\text{ distinct points from }[a,b].$$

We remark that the classical notion of convexity correponds to the case $n=2$ in the above definition. Let $f,g:[a,b]\to\mathbb{R}$ be two integrable functions. The Chebyshev functional $T(f,g)$ is defined by

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{(b-a)^2}\left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right). \tag{1}$$

The well-known Chebyshev inequality states that, [3],

$$T(f,g)\geqslant 0,$$

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whenever f and g are syncronous functions, i.e., $(f(x)-f(y))(g(x)-g(y)) \geqslant 0$, $\forall x,y \in [a,b]$. In [2], E.V. Atkinson showed that if f,g are convex functions which are twice differentiable on $[a,b]$ and

$$\int_a^b \left(x-\frac{a+b}{2}\right) g(x)dx = 0,$$

then

$$T(f,g) \geqslant 0.$$

A. Lupaş, [4], proved the following inequality

$$T(f,g) \geqslant \frac{12}{(b-a)^4}\left(\int_a^b \left(x-\frac{a+b}{2}\right) f(x)dx\right)\left(\int_a^b \left(x-\frac{a+b}{2}\right) g(x)dx\right) \quad (2)$$

for all $f,g:[a,b]\to\mathbb{R}$ convex functions, with equality when at least one of the functions is a polynomial of degree at most one on $[a,b]$. In [1], H. Belbachir and M. Rahmani, proved the following result

THEOREM 1. *If f,g are n-convex (n-concave) functions on $[a,b]$, then the following inequality holds*

$$T(f,g) \geqslant \sum_{k=1}^{n-1} \frac{2k+1}{(b-a)^2}\left(\int_a^b P_k\left(\frac{2x-a-b}{b-a}\right) f(x)dx\right)\left(\int_a^b P_k\left(\frac{2x-a-b}{b-a}\right) g(x)dx\right), \quad (3)$$

where P_n is the nth degree Legendre polynomial, satisfying $P_n(1)=1$,

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}\left(t^2-1\right)^n,\ n\geqslant 0.$$

Our goal is to generalize (3) for the weighted Chebyshev functional

$$T_w(f,g) = \int_a^b w(x)f(x)g(x)dx - \left(\int_a^b w(x)f(x)dx\right)\left(\int_a^b w(x)g(x)dx\right),$$

where $w:[a,b]\to\mathbb{R}_+$ is a weight function satisfying

$$\int_a^b w(x)dx = 1.$$

In obtaining a generalization of (3), we first establish a mean value theorem for the Chebyshev weighted functional.

2. Main results

In this section we plan to present a generalization of the inequality obtained by H. Belbachir and M. Rahmani. The new inequality uses general orthonormal polynomials and it is obtained by first giving a mean value theorem, which is usefull also as a standalone result. We start with the following lemma.

LEMMA 1. *Let* $w:(a,b)\to\mathbb{R}_+$ *be a weight function and* $g:[a,b]\to\mathbb{R}$ *be a continuous function satisfying the following properties:*

(a) $\int_a^b w(x)g(x)x^i dx=0,\ i=\overline{0,n-1},$

(b) g *changes sign exactly in* n *distinct points from* $[a,b]$.

Then for any continuous function $f:[a,b]\to\mathbb{R}$*, there exist* $n+1$ *distinct points* $\alpha_0,\dots,\alpha_n\in[a,b]$ *such that the identity*

$$\int_a^b w(x)f(x)g(x)dx=[\alpha_0,\dots,\alpha_n;f]\int_a^b w(x)x^n g(x)dx \tag{4}$$

holds.

Proof. Let $x_0,\dots,x_{n-1}$ be the points where $g(x)$ changes sign. Let $L_{n-1}(f;x_0,\dots,x_{n-1})(x)$ be the Lagrange polynomial which interpolates the function $f(x)$ in the points $x_0,\dots,x_{n-1}$. Using the remainder formula from Lagrange polynomial interpolation, we have

$$f(x)=L_{n-1}(f;x_0,\dots,x_{n-1})(x)+l(x)\,[x,x_0,\dots,x_{n-1};f], \tag{5}$$

where $l(x)=(x-x_0)\dots(x-x_{n-1})$. It follows from (5) and assumption *(a)* above, that

$$\int_a^b w(x)f(x)g(x)dx=\int_a^b [x,x_0,\dots,x_{n-1};f]\,w(x)l(x)g(x)dx. \tag{6}$$

Since the function $w(x)l(x)g(x)$ doesn't change sign in $[a,b]$ we can apply the integral mean value theorem in (6) to obtain

$$\int_a^b w(x)f(x)g(x)dx=[\alpha_0,x_0,\dots,x_{n-1};f]\int_a^b w(x)l(x)g(x)dx, \tag{7}$$

for some $\alpha_0:=\alpha_0(f,g)\in[a,b]$. By using the fact that

$$\int_a^b w(x)l(x)g(x)dx=\int_a^b w(x)x^n g(x)dx,$$

(which is true due to assumption *(a)*) in (7), we obtain the desired result. □

The following definition was given by T. Popoviciu in [8].

DEFINITION 2. Let S be a linear subspace of $C[a,b]$ and $A:S\to\mathbb{R}$ be a linear functional. The functional A is called P_n-simple ($n\in\mathbb{Z}$, $n\geqslant -1$), if the following requirments hold:

(i) $A(e_{n+1})\neq 0$, where $e_i:[a,b]\to\mathbb{R}$, $e_i(x)=x^i$,

(ii) for any function $f\in S$, there exist distinct points $t_i:=t_i(f)\in[a,b]$, $i=1,\dots,n+2$ such that

$$A(f)=A(e_{n+1})\,[t_1,\dots,t_{n+2};f]. \tag{8}$$

Before giving the main theorem we list two results that will be used in the proof. The following theorem was proved in [6].

THEOREM 2. *Let $A : C[a,b] \to \mathbb{R}$ be a linear bounded functional, different from the null functional. If $A(f) \geqslant 0$ for any convex function $f \in C[a,b]$ of order $n+1$, then A is a P_n-simple functional.*

The next result was proved in A. Lupaş's PhD thesis, [5] and shows that it is enough to prove the P_n-simple property on $C^{(n+1)}[a,b]$ in order to get the same property on the larger space $C[a,b]$. For a proof of this result one may also look at [7].

THEOREM 3. (A. Lupaş, [5]) *Let $A : C[a,b] \to \mathbb{R}$ be a linear bounded functional. If A is P_n-simple on $C^{(n+1)}[a,b]$, then A is P_n-simple on $C[a,b]$.*

The main result, which establishes a mean value theorem for the weighted Chebyshev functional is given in the following theorem.

THEOREM 4. *Let $f,g \in C[a,b]$ and P_k be the orthonormal polynomial of degree k, $k = 0,1,\ldots,n$ ($n \geqslant 2$) with respect to the weight function $w(x)$. Then there exists a set of $n+1$ distinct points $a_i := a_i(f,g)$, $i = \overline{0,n}$ and another one of $n+1$ distinct points $b_i := b_i(f,g)$, $i = \overline{0,n}$ such that*

$$T_w(f,g) = \sum_{k=1}^{n-1} \langle P_k, f\rangle_w \langle P_k, g\rangle_w + \frac{1}{A_n^2}[a_0,\ldots,a_n;f]\,[b_0,\ldots,b_n;g], \tag{9}$$

where A_n is the coefficient of x^n in $P_n(x)$ and $\langle u,v\rangle_w = \int_a^b w(x)u(x)v(x)dx$ for $u,v \in C[a,b]$.

Proof. For $g \in C[a,b]$ fixed, we consider the functional $A_g : C[a,b] \to \mathbb{R}$, given by

$$A_g(f) = \int_a^b w(x)g(x)\left[f(x) - \sum_{k=0}^{n-1} \langle P_k, f\rangle_w P_k(x)\right] dx.$$

We have

$$A_g(e_i) = 0,\ i = \overline{0,n-1}.$$

In order to get the desired result it is enough to show (as it can be seen below) that A_g is a P_{n-1}–simple functional. To prove this claim we take the following steps.

(i) We show that $A_g(f)$ is a P_{n-1}–simple functional, by using Theorem 2, for the case when g is a fixed n-convex function.

(ii) We extend the result (i) for arbitrary $g \in C^{(n)}[a,b]$.

(iii) We use (ii) in accordance with Theorem 3, to get the claim for all $g \in C[a,b]$.

We will now show that $A_g(\cdot)$ is a P_n-simple functionals. For this we will use the characterization given by Theorem 2. Let $f \in C[a,b]$ be an n-convex function.

Let Π_m denote the set of all polynomials of degree at most m. We can consider without loss of generalization that $f \notin \Pi_{n-1}$, since it is easy to see that for $f \in \Pi_{n-1}$, $A_g(f) = 0$. For $f \notin \Pi_{n-1}$, the function $h(x) = f(x) - \sum_{k=0}^{n-1} \langle P_k, f \rangle_w P_k(x)$ has the following properties

(a) $\int_a^b w(x)x^k h(x)dx = 0,\ k = 0,1,\ldots,n-1$,

(b) the function $h(x)$ changes sign in exactly n distinct points from (a,b).

From Lemma 1, there exist the points $\alpha_0,\ldots,\alpha_n \in [a,b]$ such that

$$\begin{aligned} A_g(f) &= [\alpha_0,\ldots,\alpha_n;g] \int_a^b w(x)x^n \left[f(x) - \sum_{k=0}^{n-1} \langle P_k, f \rangle_w P_k(x) \right] dx \\ &= [\alpha_0,\ldots,\alpha_n;g] \frac{1}{A_n} \int_a^b w(x)P_n(x)f(x)dx. \end{aligned}$$

If g is an n-convex function, using again Lemma 1, we get that

$$A_g(f) \geqslant 0.$$

From Theorem 2, it follows that A_g is a P_{n-1}-simple functional and so there exist $n+1$ distinct points such that

$$A_g(f) = [\alpha_0,\ldots,\alpha_n;g]\,[\beta_0,\ldots,\beta_n;f]\,\frac{1}{A_n^2}.$$

Since, on the other hand

$$A_g(f) = T_w(f,g) - \sum_{k=1}^{n-1} \langle f, P_k \rangle_w \langle f, P_k \rangle_w$$

and therefore the theorem is proved when g is an n-convex function. To get the result all over $C[a,b]$, we prove that this holds for arbitrary $g \in C^{(n)}[a,b]$ and use Theorem 3 to extend the result to any function $g \in C[a,b]$.

Therefore let $g \in C^{(n)}[a,b]$. We note that the function

$$u = g + \frac{\left\| g^{(n)} \right\|_\infty P_n}{n!A_n}$$

is an n-convex function. We have

$$A_u(f) = T_w(f,g) - \sum_{k=1}^{n-1} \langle f, P_k \rangle_w \langle g, P_k \rangle_w + \frac{\left\| g^{(n)} \right\|_\infty P_n}{n!A_n} \int_a^b w(x)f(x)P_n(x)dx. \qquad (10)$$

On the other hand, since A_u is P_{n-1}–simple, there exist points $c_0,\dots,c_n$ such that

$$A_u(f)=\left([c_0,\dots,c_n;g]+\frac{\left\|g^{(n)}\right\|_\infty}{n!}\right)\frac{1}{A_n}\int_a^b w(x)f(x)P_n(x)dx. \tag{11}$$

From (10) and (11), we get

$$T_w(f,g)=\frac{1}{A_n}[c_0,\dots,c_n;g]\int_a^b w(x)f(x)P_n(x)dx$$

and using now Lemma 1, the proof is finished. □

REMARKS. In what follows, we briefly discuss some of the conclusions derived from the above result.

a) It follows from (9), that when f and g are n-convex functions we obtain

$$T_w(f,g)\geqslant\sum_{k=1}^{n-1}\langle P_k,f\rangle_w\langle P_k,g\rangle_w$$

which is a generalization of the result obtained by H. Belbachir and M. Rahmani. When the weight function $w=1$, the inequality (3) is recovered.

b) If $f,g\in C^{(n)}[a,b]$, it follows that there exist the points θ and η in $[a,b]$ such that

$$T_w(f,g)=\sum_{k=1}^{n-1}\langle P_k,f\rangle_w\langle P_k,g\rangle_w+\frac{f^{(n)}(\theta)g^{(n)}(\theta)}{(n!A_n)^2},$$

where to get the above identity, we have used (9) and the mean value theorem for divided differences of order n. It follows that for $f,g\in C^{(n)}[a,b]$, we can write the following inequality.

$$\left|T_w(f,g)-\sum_{k=1}^{n-1}\langle P_k,f\rangle_w\langle P_k,g\rangle_w\right|\leqslant\frac{\left\|f^{(n)}\right\|_\infty\left\|g^{(n)}\right\|}{(n!A_n)^2}$$

c) Let $g:=P_n$ for some $n\in\mathbb{N}^*$. Then, $[b_0,\dots,b_n;g]=A_n$ and the sum in the right hand side of (9) is 0 due to orthogonality. Therefore, we recover the identity

$$\langle f,P_n\rangle_w=\frac{[a_0,\dots,a_n;f]}{A_n}.$$

Conclusions

In this paper we have presented a mean value theorem for weighted Chebyshev functionals based on divided differences. This result can then be used to obtain generalizations of an inequality obtained by H. Belbachir and M. Rahmani. As pointed out in [9], these Chebyshev-Grüss type inequalities as well as the mean value theorem obtained here may be successfully used in statistics if the weight function is associated with a probability density function. In this case the inner products $\langle f,g\rangle_w$ can be seen as the expectation of the transformed random variable $f(X)g(X)$ obtained from an initial random variable X.

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