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CENTERED CONVEX BODIES AND INEQUALITIES FOR CROSS–SECTION MEASURES

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Abstract. The purpose of this paper is to establish some new results on cross-section measures of centered convex bodies. More precisely, we show some connections between inequalities referring to cross-section measures and well-known affine isoperimetric inequalities. Based on this, we derive affine inequalities involving also new characterizations of ellipsoids. In addition, related results on three-dimensional zonoids are obtained. Some of our results are also interesting from the viewpoint of the geometry of finite dimensional real Banach spaces.

0. Introduction

For B a centered convex body in $\mathbb{R}^d$, $d \geqslant 2$, and u some direction vector from the unit sphere S^{d-1}, we denote by $\lambda_{d-1}(B|u^{\perp})$ and $\lambda_1(B|l_u)$ the $(d-1)$- and 1-dimensional Lebesgue measures of the orthogonal projections of B onto $(d-1)$-subspaces $u^{\perp}$ and 1-subspaces l_u, respectively. Then for any direction $u \in S^{d-1}$

$$1 \leqslant \frac{\lambda_{d-1}(B|u^{\perp})\lambda_1(B\cap l_u)}{\lambda(B)} \leqslant d$$

and

$$1 \leqslant \frac{\lambda_{d-1}(B\cap u^{\perp})\lambda_1(B|l_u)}{\lambda(B)} \leqslant d,$$

where $\lambda(B)$ is the volume of B (see [11], [7], and [14] for more general cases).

One could ask whether there is always a unit vector $u \in S^{d-1}$ such that

$$\frac{\lambda_{d-1}(B|u^{\perp})\lambda_1(B\cap l_u)}{\lambda(B)} \geqslant c$$

for some constant c, and then one should also find the maximum value of c.

Similarly, there is always a unit vector $u \in S^{d-1}$ such that

$$\frac{\lambda_{d-1}(B\cap u^{\perp})\lambda_1(B|l_u)}{\lambda(B)} \leqslant c$$

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for some constant c, and one should also find the minimum value of c.

Similar questions can be raised replacing projection by intersection and intersection by projection. Another interesting question is: when is the quantity

$$\frac{\lambda_{d-1}(B|u^{\perp})\lambda_1(B\cap l_u)}{\lambda(B)}$$

constant for all $u \in S^{d-1}$? Our goal is to discuss some of those (and related) problems for centered convex bodies in $\mathbb{R}^d$ and, thus, to derive some new affine inequalities involving characterizations of ellipsoids. In particular, some of our results refer to the class of zonoids, and some of them can also be interpreted in terms of the geometry of finite dimensional real Banach spaces (cf. [15]).

1. Definitions and preliminaries

Recall that a *convex body* K in $\mathbb{R}^d, d \geqslant 2$, is a compact, convex set with nonempty interior, and that K is said to be *centered* if it is symmetric with respect to the origin o of $\mathbb{R}^d$. As usual, S^{d-1} denotes the standard Euclidean unit sphere in $\mathbb{R}^d$. We write λ_i for the *i-dimensional Lebesgue measure* in $\mathbb{R}^d$, where $1 \leqslant i \leqslant d$, and instead of λ_d we simply write λ. We denote by $u^{\perp}$ the $(d-1)$-dimensional subspace orthogonal to $u \in S^{d-1}$, and by l_u the 1-subspace parallel to u. Some of the notions given in the following are usually defined and well-known for general convex bodies. Since, however, we need them only for centered convex bodies, we simplify things by introducing most of them only for the centered subcase. For a centered convex body K in $\mathbb{R}^d$, we define the *polar body* K° of K by $K^{\circ} = \{y \in \mathbb{R}^d : \langle x,y\rangle \leqslant 1, x \in K\}$ and identify $\mathbb{R}^d$ and its *dual space* $\mathbb{R}^{d*}$ by using the standard basis. In that case, λ_i and λ_i^* coincide in $\mathbb{R}^d$. The notation ε_i will stand for the volume of the standard unit Euclidean ball in $\mathbb{R}^i$. Thus, ε_d denotes the volume of the d-dimensional unit ball.

For K a centered convex body in $\mathbb{R}^d$ and $u \in S^{d-1}$, the *support function of* K is defined by $h_K(u) = \sup\{\langle u,y\rangle : y \in K\}$, and its *radial function* $\rho_K(u)$ by $\rho_K(u) = \max\{\alpha \geqslant 0 : \alpha u \in K\}$. We always have $h_{\alpha K} = \alpha h_K$ and $\rho_{\alpha K} = \alpha\rho_K$; only these are needed here. In addition, it is well known that $h_K(u) \leqslant h_L(u)$ for any direction $u \in S^{d-1}$ if and only if $K \subseteq L$. We also mention that for centered K the relations $K^{\circ\circ} = K$,

$$\rho_{K^{\circ}}(u) = \frac{1}{h_K(u)}, \ u \in S^{d-1}, \tag{1}$$

and

$$\lambda(K) = \frac{1}{d}\int_{S^{d-1}} \rho_K(u)^d du \tag{2}$$

hold.

The *projection body* ΠK of a centered convex body K in $\mathbb{R}^d$ is defined by $h_{\Pi K}(u) = \lambda_{d-1}(K|u^{\perp})$ for each $u \in S^{d-1}$, where $K|u^{\perp}$ is the orthogonal projection of K onto $u^{\perp}$ and $\lambda_{d-1}(K|u^{\perp})$ is called the *$(d-1)$-dimensional outer cross-section measure* of K at u. The *intersection body* IK of $K \subset \mathbb{R}^d$ is defined by $\rho_{IK}(u) = \lambda_{d-1}(K \cap u^{\perp})$

for each $u \in S^{d-1}$. By notation it is clear that $\omega(K,u) = \lambda_1(K|l_u)$ denotes the 1-*dimensional outer cross-section measure* or *width* of K at u. If K is a centered convex body in $\mathbb{R}^d$ and S is a subspace, then we also have

$$K^\circ \cap S = (K|S)^\circ. \tag{3}$$

For centered convex bodies K and L, the *dual mixed volume* for all i is defined as

$$\tilde{V}_i(K[d-i],L[i]) = \frac{1}{d}\int_{S^{d-1}} \rho_K(u)^{d-i}\rho_L(u)^i du,$$

where the symbols $K[\cdot]$, $L[\cdot]$ used for describing *mixed volumes* are taken from [13], p. 279.

A convex body K in $\mathbb{R}^d$ is called a *zonoid* if it can be approximated in the Hausdorff metric by finite vector sums of line segments (i.e., by zonotopes). For this very interesting class of convex bodies we refer to the surveys [16] and [3].

If we choose a centered convex body B in $\mathbb{R}^d$ as the unit ball of a finite dimensional real Banach space $(\mathbb{R}^d, ||\cdot||)$, then some of our results can be fruitfully interpreted also in the setting of Minkowski geometry (see the monograph [15]). E.g., note that $\hat{I}_B = \dfrac{\varepsilon_d}{\lambda(B^\circ)}\dfrac{\Pi B^\circ}{\varepsilon_{d-1}}$ is the *normalized solution of the isoperimetric problem* for the Holmes-Thompson measure (see [15], sec. 5.4).

2. Some affine inequalities

The *Petty projection inequality* states that if K is a convex body in $\mathbb{R}^d$, then

$$\lambda(K)^{d-1}\lambda((\Pi K)^\circ) \leqslant \left(\frac{\varepsilon_d}{\varepsilon_{d-1}}\right)^d,$$

with equality if and only if K is an ellipsoid.

The *Busemann intersection inequality* says that for K a star body (i.e., K is star-shaped with respect to the origin) in $\mathbb{R}^d$

$$\lambda(IK) \leqslant \frac{\varepsilon_{d-1}^d}{\varepsilon_d^{d-2}}\lambda(K)^{d-1}$$

holds, with equality if and only if K is an ellipsoid centered at the origin.

Petty's conjectured projection inequality (see [10]) states that if K is a convex body in $\mathbb{R}^d$ with $d \geqslant 3$, then

$$\lambda(\Pi K) \geqslant \frac{\varepsilon_{d-1}^d}{\varepsilon_d^{d-2}}\lambda(K)^{d-1},$$

with equality if and only if K is an ellipsoid. It can be proven that this inequality is stronger than Petty's projection inequality. In [1] it was shown that Petty's conjectured inequality is true for cylindrical convex bodies in $\mathbb{R}^3$.

We also mention that if K is a convex body and L is a star body in $\mathbb{R}^d$ with $o \in L$, then

$$V(K[d-1], \Gamma L) = \frac{2}{(d+1)\lambda(L)} \tilde{V}_{-1}(L[d+1], (\Pi B)^\circ[-1]), \tag{4}$$

where ΓL is the *centroid body* of L (see [2] and, again for notation, [13], p. 279).

3. Cross-section measures

The following theorem was proved already in [6]. However, we present it with a shorter proof based on Petty's projection inequality.

THEOREM 1. *Let B be a centered convex body in $\mathbb{R}^d$. Then there exists a unit vector $u \in S^{d-1}$ such that*

$$\frac{\lambda_{d-1}(B|u^\perp)\lambda_1(B \cap l_u)}{\lambda(B)} \geqslant \frac{2\varepsilon_{d-1}}{\varepsilon_d}.$$

Furthermore, equality holds for every $u \in S^{d-1}$ if and only if B is an ellipsoid.

Proof. Assume that for every $u \in S^{d-1}$

$$\frac{\lambda_{d-1}(B|u^\perp)\lambda_1(B \cap l_u)}{\lambda(B)} < \frac{2\varepsilon_{d-1}}{\varepsilon_d}.$$

Then

$$2h_{\Pi B}(u)\rho_B(u) < \lambda(B)\frac{2\varepsilon_{d-1}}{\varepsilon_d},$$

and thus

$$\rho_B(u) < \rho_{(\Pi B)^\circ}(u)\lambda(B)\frac{\varepsilon_{d-1}}{\varepsilon_d}.$$

Integrating the last inequality, we obtain

$$\int_{S^{d-1}} \rho_B(u)^d du < \lambda(B)^d \left(\frac{\varepsilon_{d-1}}{\varepsilon_d}\right)^d \int_{S^{d-1}} \rho_{(\Pi B)^\circ}(u)^d du.$$

Hence

$$\lambda(B) < \lambda((\Pi B)^\circ)\lambda(B)^d \left(\frac{\varepsilon_{d-1}}{\varepsilon_d}\right)^d,$$

which is in contradiction to Petty's projection inequality. One can also see that equality for all $u \in S^{d-1}$ holds if and only if B is an ellipsoid (see also [8]). □

If we set K to be a centered convex body B with $\lambda(B) = \varepsilon_d$ in the Busemann intersection inequality, we obtain

$$\lambda(IB) \leqslant \varepsilon_d \varepsilon_{d-1}^d$$

with equality if and only if B is an ellipsoid.

Our next theorem is a new result.

THEOREM 2. *Let B be a centered convex body in $\mathbb{R}^d$ with $\lambda(B)=\varepsilon_d$. Then there exists a unit vector $u\in S^{d-1}$ such that*

$$\lambda_{d-1}(B\cap u^{\perp})\lambda_1(B^{\circ}|l_u)\leqslant 2\varepsilon_{d-1}.$$

Furthermore, equality for all $u\in S^{d-1}$ holds if and only if B is an ellipsoid.

Proof. Assume that for every $u\in S^{d-1}$

$$\lambda_{d-1}(B\cap u^{\perp})\lambda_1(B^{\circ}|l_u)>2\varepsilon_{d-1}$$

holds. This is equivalent to

$$2\rho_{IB}(u)h_{B^{\circ}}(u)>2\varepsilon_{d-1}.$$

Again using (1) and integrating both sides, we obtain

$$\int_{S^{d-1}}\rho_{IB}(u)^d du>\varepsilon_{d-1}^d\int_{S^{d-1}}\rho_B(u)^d du.$$

Therefore

$$\lambda(IB)>\varepsilon_{d-1}^d\lambda(B)=\varepsilon_{d-1}^d\varepsilon_d,$$

which is in contradiction to the Busemann intersection inequality. It is also easy to see that equality for all $u\in S^{d-1}$ holds if and only if B is a centered ellipsoid. □

COROLLARY 3. *If B is a centered convex body in $\mathbb{R}^d$ with $\lambda(B)=\varepsilon_d$, then there is a unit vector $u\in S^{d-1}$ such that*

$$\lambda_{d-1}(B\cap u^{\perp})\leqslant\frac{\varepsilon_{d-1}}{2}\lambda_1(B\cap l_u).$$

Furthermore, equality for all $u\in S^{d-1}$ holds if and only if B is an ellipsoid.

Proof. We can rewrite this inequality as $\lambda_{d-1}(B\cap u^{\perp})\leqslant\varepsilon_{d-1}\rho_B(u)$ which is also equivalent to $\lambda_{d-1}(B\cap u^{\perp})h_{B^{\circ}}(u)\leqslant\varepsilon_{d-1}$. The result follows from the above theorem. □

COROLLARY 4. *Let B be a centered convex body in $\mathbb{R}^d$. Then there exists a unit vector $u\in S^{d-1}$ such that*

$$\lambda_{d-1}(B\cap u^{\perp})\leqslant\frac{\varepsilon_{d-1}}{2\varepsilon_d}\varepsilon_d^{2/d}\lambda_1(B\cap l_u)\lambda(B)^{1-2/d}.$$

Furthermore, equality for all $u\in S^{d-1}$ holds if and only if B is an ellipsoid.

Petty's conjectured inequality has been described as one of the major open problems in the area of affine isoperimetric inequalities (see [5] and the survey [4]). Setting K to be a centered convex body B with $\lambda(B)=\varepsilon_d$, the conjectured inequality becomes

$$\lambda(\Pi B)\geqslant\varepsilon_d\varepsilon_{d-1}^d$$

with equality if and only if B is an ellipsoid.

OBSERVATION 5. Let B be a centered convex body in $\mathbb{R}^d$ with $\lambda(B) = \varepsilon_d$. If Petty's conjectured inequality is true, then there exists $u \in S^{d-1}$ such that

$$\lambda_{d-1}(B|u^{\perp})\lambda_1(B^{\circ} \cap l_u) \geqslant 2\varepsilon_{d-1}.$$

Furthermore, equality for all u would hold if and only B is an ellipsoid.

Proof. Assume that for all $u \in S^{d-1}$

$$\lambda_{d-1}(B|u^{\perp})\lambda_1(B^{\circ} \cap l_u) < 2\varepsilon_{d-1}$$

holds, that is,

$$h_{\Pi B}(u)\rho_{B^{\circ}}(u) < \varepsilon_d.$$

This inequality is equivalent to

$$h_{\Pi B}(u) \leqslant \varepsilon_{d-1} h_B(u)$$

for all $u \in S^{d-1}$. Using known properties of support functions, we obtain $\Pi B \subset \varepsilon_{d-1} B$. Therefore, $\lambda(\Pi B) < \varepsilon_{d-1}^d \lambda(B)$, which is a contradiction to Petty's conjectured inequality. $\square$

By [1], Observation 5 is an established fact for cylindrical bodies in $\mathbb{R}^3$.

In [12], it was proved that if B is a three-dimensional zonoid, then

$$\lambda(\Pi B) \leqslant 2^3 \lambda(B)^2. \tag{5}$$

Equality holds if and only if B can be written as the Minkowski sum of five line segments or as the sum of a cylinder and a line segment. By this result we obtain the following

THEOREM 6. *Let B be a centered zonoid in $\mathbb{R}^3$. Then there exists a unit vector $u \in S^2$ such that*

$$\lambda_2(B|u^{\perp}) \leqslant \lambda_1(B|l_u)\lambda(B)^{1/3}.$$

Furthermore, if equality holds for all $u \in S^2$, then either B can be written as the Minkowski sum of five line segments or as the sum of a cylinder and a line segment.

Proof. Assume that for all $u \in S^2$ the relation

$$\lambda_2(B|u^{\perp}) > \lambda_1(B|l_u)\lambda(B)^{1/3}$$

holds; that is,

$$h_{\Pi B}(u) > 2h_B(u)\lambda(B)^{1/3}.$$

Thus

$$2\lambda(B)^{1/3} B \subset \Pi B,$$

and therefore

$$2^3 \lambda(B)^2 < \lambda(\Pi B),$$

which contradicts (5). The equality case follows from (5) as well. $\square$

COROLLARY 7. *Let B be a centered zonoid in $\mathbb{R}^3$. Then there exists a unit vector $u \in S^2$ such that*

$$\lambda_2(B|u^\perp)\lambda_1(B^\circ \cap l_u) \leqslant 2^2 \lambda(B)^{1/3}.$$

Furthermore, if equality holds for all $u \in S^2$, then either B can be written as the Minkowski sum of five line segments or as the sum of a cylinder and a line segment.

Also in [12] it was shown that for all three-dimensional zonoids B

$$\lambda((\Pi B)^\circ)\lambda(B)^2 \geqslant \frac{4}{3} \tag{6}$$

holds, with equality if and only B is a parallelepiped (i.e., an affine cube). In general, the problem of finding the sharp lower bound for $\lambda((\Pi B)^\circ)\lambda(B)^{d-1}$ among all centrally symmetric convex bodies is still an open question. In [6], Makai and Martini conjectured that this sharp bound is attained if and only if B is a parallelepiped.

THEOREM 8. *Let B be a centered zonoid in $\mathbb{R}^3$. Then there exists a unit vector $u \in S^2$ such that*

$$\frac{\lambda_2(B|u^\perp)\lambda_1(B \cap l_u)}{\lambda(B)} \leqslant \sqrt[3]{6}.$$

Furthermore, if equality holds for all $u \in S^2$, then B is a parallelepiped.

Proof. Assume that for all $u \in S^2$

$$\frac{\lambda_2(B|u^\perp)\lambda_1(B \cap l_u)}{\lambda(B)} > \sqrt[3]{6}$$

holds, that is,

$$2h_{\Pi B}(u)\rho_B(u) > \sqrt[3]{6}\lambda(B).$$

Hence,

$$2\rho_B(u) > \sqrt[3]{6}\rho_{(\Pi B)^\circ}(u)\lambda(B).$$

Integrating both sides of the last inequality, we obtain

$$2^3\lambda(B) > 6\lambda((\Pi B)^\circ)\lambda(B)^3,$$

which is in contradiction to (6), and the equality case follows easily from (6), too. □

The d-dimensional analogue of the upper bound in Theorem 8 above is not confirmed until now. So we can only formulate the following open problem which is also related to the Makai-Martini conjecture mentioned above.

PROBLEM 9. Let B a centered convex body in $\mathbb{R}^d$ with $d \geqslant 3$. Is there a unit vector u such that

$$\frac{\lambda_{d-1}(B|u^\perp)\lambda_1(B \cap l_u)}{\lambda(B)} \leqslant \sqrt[d]{d!}?$$

PROPOSITION 10. *If B is a centered zonoid in $\mathbb{R}^d$ with $d \geqslant 3$, then*

$$\lambda(\Pi B)\lambda(B)^{1-d} \geqslant \frac{2^d}{d!}.$$

Proof. In [2] (Theorem 9.3.3) it has been shown that if K_1 is a convex body and K_2 is a zonoid in $\mathbb{R}^d$, then

$$\left(\frac{\lambda(K_2)}{\lambda(K_1)}\right)^{1/d} \geqslant \min_{u \in S^{d-1}} \frac{\lambda_1(K_2|l_u)}{\lambda_1(K_1|l_u)},$$

with equality if and only if K_1 and K_2 are homothetic.

We set K_1 to be a centered convex body B and K_2 to be $\hat{I}_B$, which is always a zonoid. Using the above result, we obtain

$$\left(\frac{\lambda(\hat{I}_B)}{\lambda(B)}\right)^{1/d} \geqslant \min_{u \in S^{d-1}} \frac{\lambda_1(\hat{I}_B|l_u)}{\lambda_1(B|l_u)} = \frac{1}{2} \min_{u \in S^{d-1}} \frac{2\omega(\hat{I}_B, u)}{\omega(B,u)} = \frac{1}{2}\omega_B(\hat{I}_B).$$

In [9] it was established that $\omega_B(\hat{I}_B) \geqslant \dfrac{\varepsilon_d}{\varepsilon_{d-1}}$. Therefore

$$\lambda(\hat{I}_B) \geqslant \left(\frac{\varepsilon_d}{2\varepsilon_{d-1}}\right)^d \lambda(B).$$

If we set $\hat{I}_B = \dfrac{\varepsilon_d \Pi B^\circ}{\lambda(B^\circ)\varepsilon_{d-1}}$ (recall that $\hat{I}_B = \dfrac{\varepsilon_d}{\lambda(B^\circ)} \dfrac{\Pi B^\circ}{\varepsilon_{d-1}}$ is the normalized solution of the isoperimetric problem for the Holmes-Thompson measure), then we obtain

$$2^d \lambda(\Pi B^\circ) \geqslant \lambda(B)\lambda(B^\circ)^d.$$

Using the fact that for zonoids $\lambda(B)\lambda(B^\circ) \geqslant \dfrac{4^d}{d!}$ (the Mahler-Reisner inequality) holds, we establish the result of the proposition. □

One of the challenging open problems referring to affine isoperimetric inequalities is the question whether a centered convex body B in $\mathbb{R}^d$, $d \geqslant 3$, must be an ellipsoid if B and ΠB° are homothetic (see [2], [13], and [15]). Equivalently, if B and $\hat{I}_B$ are homothetic, then B must be an ellipsoid. If, apart from an ellipsoid, such a centered convex body B in $\mathbb{R}^d$, $d \geqslant 3$, exists, then for all $u \in S^{d-1}$

$$\frac{\lambda_{d-1}(B|u^{d-1})\lambda_1(B \cap l_u)}{\lambda(B)} = c$$

holds, where c is a fixed constant with $c \geqslant \dfrac{2\varepsilon_{d-1}}{\varepsilon_d}$ (see [8]).

Since dilation does not change the left side of the above equality, we could set $\lambda(B) = 1$. Then for all $u \in S^{d-1}$ we obtain

$$\lambda_{d-1}(B|u^\perp)\lambda_1(B \cap l_u) = 2h_{\Pi B}(u)\rho_B(u) = 2\rho_B(u)\rho^{-1}_{(\Pi B)^\circ}(u) = c.$$

Therefore,

$$c\rho_B(u)^d = 2\rho_B(u)^{d+1}\rho_{(\Pi B)^\circ)}(u)^{-1}.$$

Integrating both sides of the last equality, we establish

$$c\lambda(B) = 2\tilde{V}_{-1}(B[d+1], (\Pi B)^\circ[-1]).$$

Thus, we have

$$c = 2\tilde{V}_{-1}(B[d+1], (\Pi B)^\circ[-1]) = (d+1)V(B[d-1], \Gamma B[1]).$$

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