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VECTOR MAJORIZATION AND SCHUR–CONCAVITY OF SOME SUMS GENERATED BY THE JENSEN AND JENSEN–MERCER FUNCTIONALS

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Abstract. In this paper we study a vector majorization ordering for comparing two m-tuples of vectors of a real linear space. This extends the classical approach of (scalar) majorization theory for comparing m-tuples of scalars in $\mathbf{R}$. We prove a Sherman type inequality for a vector-valued $\leqslant_C$-convex function f, where $\leqslant_C$ is a cone ordering. In consequence, we obtain a Hardy-Littlewood-Pólya-Karamata type inequality generated by m-tuples of vectors in a vector space. As applications, we present majorization generalizations of the superadditivity properties of the Jensen and Jensen-Mercer functionals generated by a convex function f. In addition, we show that some sums generated by the Jensen and Jensen-Mercer functionals are Schur-concave with respect to their weight vectors. We also give interpretations of the obtained results for tridiagonal doubly stochastic matrices and doubly stochastic circular matrices.

1. Introduction

Let $f : I \to \mathbf{R}$ be a function on an interval $I \subset \mathbf{R}$, $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in I^k$ and $\mathbf{p} \in \mathscr{P}_k^0$, where

$$\mathscr{P}_k^0 = \{\mathbf{p} = (p_1, p_2, \ldots, p_k) \in \mathbf{R}^k : p_i \geqslant 0,\ P_k > 0\} \quad \text{with} \quad P_k = \sum_{i=1}^k p_i.$$

The *Jensen functional* is defined by

$$J(f,\mathbf{x},\mathbf{p}) = \sum_{i=1}^k p_i f(x_i) - P_k f\left(\frac{1}{P_k}\sum_{i=1}^k p_i x_i\right) \tag{1}$$

(see [8]). We adopt the convention that $0 f(\frac{0}{0}) = 0$.

Equation (1) can be rewritten in the form

$$J(f,\mathbf{x},\mathbf{p}) = \langle \mathbf{p}, f(\mathbf{x})\rangle - \langle \mathbf{p}, \mathbf{e}\rangle f\left(\frac{\langle \mathbf{p}, \mathbf{x}\rangle}{\langle \mathbf{p}, \mathbf{e}\rangle}\right), \tag{2}$$

where $f(\mathbf{x}) = (f(x_1), f(x_2), \ldots, f(x_k))$, $\mathbf{e} = (1,1,\ldots,1) \in \mathbf{R}^k$, and $\langle \cdot,\cdot \rangle$ is the standard inner product on $\mathbf{R}^k$.

By Jensen's inequality,

$$J(f,\mathbf{x},\mathbf{p}) \geqslant 0 \quad \text{for a convex function } f.$$

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THEOREM A. [8] *If $f : I \to \mathbf{R}$ is a convex function then the function $\mathbf{p} \to J(f,\mathbf{x},\mathbf{p})$ is superadditive for any $\mathbf{x} \in I^k$, i.e.,*

$$J(f,\mathbf{x},\mathbf{p}+\mathbf{q}) \geqslant J(f,\mathbf{x},\mathbf{p}) + J(f,\mathbf{x},\mathbf{q}) \quad \textit{for } \mathbf{p},\mathbf{q} \in \mathscr{P}_k^0. \tag{3}$$

In consequence, the function $\mathbf{p} \to J(f,\mathbf{x},\mathbf{p})$ is monotone for any $\mathbf{x} \in I^k$, i.e.,

$$J(f,\mathbf{x},\mathbf{p}+\mathbf{q}) \geqslant J(f,\mathbf{x},\mathbf{p}) \quad \textit{for } \mathbf{p},\mathbf{q} \in \mathscr{P}_k^0.$$

It is not hard to verify that property (3) is equivalent to the subadditivity of the function

$$\mathbf{p} \to \langle \mathbf{p},\mathbf{e}\rangle f\left(\frac{\langle \mathbf{p},\mathbf{x}\rangle}{\langle \mathbf{p},\mathbf{e}\rangle}\right), \quad \mathbf{p} \in \mathscr{P}_k^0.$$

That is,

$$\langle \mathbf{p}+\mathbf{q},\mathbf{e}\rangle f\left(\frac{\langle \mathbf{p}+\mathbf{q},\mathbf{x}\rangle}{\langle \mathbf{p}+\mathbf{q},\mathbf{e}\rangle}\right) \leqslant \langle \mathbf{p},\mathbf{e}\rangle f\left(\frac{\langle \mathbf{p},\mathbf{x}\rangle}{\langle \mathbf{p},\mathbf{e}\rangle}\right) + \langle \mathbf{q},\mathbf{e}\rangle f\left(\frac{\langle \mathbf{q},\mathbf{x}\rangle}{\langle \mathbf{q},\mathbf{e}\rangle}\right) \quad \text{for } \mathbf{p},\mathbf{q} \in \mathscr{P}_k^0.$$

Consider a function $f : [a,b] \to \mathbf{R}$ and a k-tuple $\mathbf{x} = (x_1,x_2,\ldots,x_k) \in [a,b]^k$, where $[a,b] \subset \mathbf{R}$ is an interval. Let $\mathbf{p} = (p_1,p_2,\ldots,p_k) \in \mathscr{P}_k^0$. The *Jensen-Mercer functional* is defined by

$$M(f,\mathbf{x},\mathbf{p}) = P_k(f(a)+f(b)) - \sum_{i=1}^k p_i f(x_i) - P_k f\left(a+b-\frac{1}{P_k}\sum_{i=1}^k p_i x_i\right)$$

(see [9]). Clearly,

$$M(f,\mathbf{x},\mathbf{p}) = (f(a)+f(b))\langle \mathbf{p},\mathbf{e}\rangle - \langle \mathbf{p}, f(\mathbf{x})\rangle - \langle \mathbf{p},\mathbf{e}\rangle f\left(\frac{(a+b)\langle \mathbf{p},\mathbf{e}\rangle - \langle \mathbf{p},\mathbf{x}\rangle}{\langle \mathbf{p},\mathbf{e}\rangle}\right),$$

where $f(\mathbf{x}) = (f(x_1), f(x_2),\ldots,f(x_k))$ and $\mathbf{e} = (1,1,\ldots,1) \in \mathbf{R}^k$.

Mercer's inequality [12, 13] says that

$$M(f,\mathbf{x},\mathbf{p}) \geqslant 0 \quad \text{for a convex function } f.$$

THEOREM B. [9] *If $f : I = [a,b] \to \mathbf{R}$ is a convex function then the function $\mathbf{p} \to M(f,\mathbf{x},\mathbf{p})$ is superadditive for any $\mathbf{x} \in I^k$, i.e.,*

$$M(f,\mathbf{x},\mathbf{p}+\mathbf{q}) \geqslant M(f,\mathbf{x},\mathbf{p}) + M(f,\mathbf{x},\mathbf{q}) \quad \textit{for } \mathbf{p},\mathbf{q} \in \mathscr{P}_k^0. \tag{4}$$

In consequence, the function $\mathbf{p} \to M(f,\mathbf{x},\mathbf{p})$ is monotone for any $\mathbf{x} \in I^k$, i.e.,

$$M(f,\mathbf{x},\mathbf{p}+\mathbf{q}) \geqslant M(f,\mathbf{x},\mathbf{p}) \quad \textit{for } \mathbf{p},\mathbf{q} \in \mathscr{P}_k^0.$$

It is easy to check that inequality (4) is equivalent to the subadditivity of the function

$$\mathbf{p} \to \langle \mathbf{p},\mathbf{e}\rangle f\left(\frac{\langle \mathbf{p},\mathbf{y}\rangle}{\langle \mathbf{p},\mathbf{e}\rangle}\right), \quad \mathbf{p} \in \mathscr{P}_k^0,$$

where $\mathbf{y} = (a+b)\mathbf{e} - \mathbf{x}$, i.e.,

$$\langle \mathbf{p}+\mathbf{q}, \mathbf{e}\rangle f\left(\frac{\langle \mathbf{p}+\mathbf{q}, \mathbf{y}\rangle}{\langle \mathbf{p}+\mathbf{q}, \mathbf{e}\rangle}\right) \leqslant \langle \mathbf{p}, \mathbf{e}\rangle f\left(\frac{\langle \mathbf{p}, \mathbf{y}\rangle}{\langle \mathbf{p}, \mathbf{e}\rangle}\right) + \langle \mathbf{q}, \mathbf{e}\rangle f\left(\frac{\langle \mathbf{q}, \mathbf{y}\rangle}{\langle \mathbf{q}, \mathbf{e}\rangle}\right) \quad \text{for } \mathbf{p}, \mathbf{q} \in \mathscr{P}_k^0.$$

To motivate our studies, for a given vector $\mathbf{c} \in \mathscr{P}_k^0$, consider two sum decompositions

$$\mathbf{c} = \mathbf{p}_1 + \mathbf{p}_2 + \ldots + \mathbf{p}_m \quad \text{and} \quad \mathbf{c} = \mathbf{q}_1 + \mathbf{q}_2 + \ldots + \mathbf{q}_n,$$

where $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m \in \mathscr{P}_k^0$ and $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n \in \mathscr{P}_k^0$. It follows from Theorem **A** that

$$J(f, \mathbf{x}, \mathbf{c}) \geqslant J(f, \mathbf{x}, \mathbf{p}_1) + J(f, \mathbf{x}, \mathbf{p}_2) + \ldots + J(f, \mathbf{x}, \mathbf{p}_m)$$

and

$$J(f, \mathbf{x}, \mathbf{c}) \geqslant J(f, \mathbf{x}, \mathbf{q}_1) + J(f, \mathbf{x}, \mathbf{q}_2) + \ldots + J(f, \mathbf{x}, \mathbf{q}_n).$$

It is of interest to know when the following inequality (5) holds:

$$\begin{aligned} &J(f, \mathbf{x}, \mathbf{p}_1) + J(f, \mathbf{x}, \mathbf{p}_2) + \ldots + J(f, \mathbf{x}, \mathbf{p}_m) \\ &\geqslant J(f, \mathbf{x}, \mathbf{q}_1) + J(f, \mathbf{x}, \mathbf{q}_2) + \ldots + J(f, \mathbf{x}, \mathbf{q}_n). \end{aligned} \tag{5}$$

This can be helpful for constructing refined Jensen type inequalities.

Likewise, in accordance to Theorem **B**, one has

$$M(f, \mathbf{x}, \mathbf{c}) \geqslant M(f, \mathbf{x}, \mathbf{p}_1) + M(f, \mathbf{x}, \mathbf{p}_2) + \ldots + M(f, \mathbf{x}, \mathbf{p}_m)$$

and

$$M(f, \mathbf{x}, \mathbf{c}) \geqslant M(f, \mathbf{x}, \mathbf{q}_1) + M(f, \mathbf{x}, \mathbf{q}_2) + \ldots + M(f, \mathbf{x}, \mathbf{q}_n).$$

In this context it would be nice to know when the inequality (6) is met:

$$\begin{aligned} &M(f, \mathbf{x}, \mathbf{p}_1) + M(f, \mathbf{x}, \mathbf{p}_2) + \ldots + M(f, \mathbf{x}, \mathbf{p}_m) \\ &\geqslant M(f, \mathbf{x}, \mathbf{q}_1) + M(f, \mathbf{x}, \mathbf{q}_2) + \ldots + M(f, \mathbf{x}, \mathbf{q}_n). \end{aligned} \tag{6}$$

And this can be utilized in deriving refined Jensen-Mercer type inequalities.

One of our aims is to provide a condition on the vector tuples $(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m)$ and $(\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n)$ for the inequalities (5)–(6) to hold. So, for the Jensen and Jensen-Mercer functionals, we are interested in inequalities of the Hardy-Littlewood-Pólya-Karamata type generated by tuples of vectors. A wider class of Sherman type inequalities will be also considered.

To this end, in Section 2 we investigate a *vector majorization* relation for comparing tuples of vectors $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)$ and $(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n)$. This extends the classical approach of (scalar) majorization theory for comparing n-tuples of scalars in $\mathbf{R}$ [10].

In Sections 3 and 4 we demonstrate majorization generalizations of Theorems **A** and **B** for the Jensen and Jensen-Mercer functionals, respectively. As applications, we interpret the obtained results for tridiagonal doubly stochastic matrices and doubly stochastic circular matrices.

2. Vector majorization

Unless stated otherwise, throughout this section $\mathscr{V}$ and $\mathscr{W}$ are real linear spaces and $\mathscr{C} \subset \mathscr{W}$ is a convex cone.

Recall that a nonempty subset $\mathscr{C}$ of $\mathscr{W}$ is said to be a *convex cone* if (i) $\mathbf{a},\mathbf{b} \in \mathscr{C}$ implies $\mathbf{a}+\mathbf{b} \in \mathscr{C}$, and (ii) $\mathbf{a} \in \mathscr{C}$ and $0 \leqslant t \in \mathbf{R}$ imply $t\,\mathbf{a} \in \mathscr{C}$.

If $\mathscr{C} \subset \mathscr{W}$ is a convex cone then the relation $\leqslant_{\mathscr{C}}$ on $\mathscr{W}$ defined by: for $\mathbf{a},\mathbf{b} \in \mathscr{W}$,

$$\mathbf{b} \leqslant_{\mathscr{C}} \mathbf{a} \quad \text{iff} \quad \mathbf{a}-\mathbf{b} \in \mathscr{C},$$

is called the *cone ordering* (induced by $\mathscr{C}$).

An $m \times n$ real matrix $\mathbf{A} = (a_{ij})$ is said to be *column stochastic* if $a_{ij} \geqslant 0$ for $i = 1,2,\ldots,m,\ j = 1,2,\ldots,n$, and all column sums of $\mathbf{A}$ are equal to 1, i.e., $\sum\limits_{i=1}^{m} a_{ij} = 1$ for $j = 1,2,\ldots,n$.

An $n \times n$ real matrix $\mathbf{A} = (a_{ij})$ is said to be *doubly stochastic* if $a_{ij} \geqslant 0$ for $i = 1,2,\ldots,n,\ j = 1,2,\ldots,n$, and all row and column sums of $\mathbf{A}$ are equal to 1, i.e., $\sum\limits_{j=1}^{n} a_{ij} = 1$ for $i = 1,2,\ldots,n$, and $\sum\limits_{i=1}^{n} a_{ij} = 1$ for $j = 1,2,\ldots,n$.

In other words, an $m \times n$ matrix $\mathbf{A}$ is column stochastic iff $\mathbf{A} \geqslant 0$ (entrywise) and $\mathbf{eA} = \mathbf{e}$, where $\mathbf{e} = (1,\ldots,1)$ is the vector of ones of an appropriate dimension. Likewise, an $n \times n$ matrix $\mathbf{A}$ is doubly stochastic iff $\mathbf{A} \geqslant 0$ (entrywise) and $\mathbf{eA} = \mathbf{e} = \mathbf{eA}^T$.

Given an $m \times n$ real matrix $\mathbf{A} = (a_{ij})$ and $(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_m) \in \mathscr{V}^m$ and $(\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_n) \in \mathscr{V}^n$, we write

$$(\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_n) = (\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_m)\mathbf{A} \tag{7}$$

in the usual sense

$$\mathbf{y}_j = a_{1j}\mathbf{x}_1 + a_{2j}\mathbf{x}_2 + \ldots + a_{mj}\mathbf{x}_m \quad \text{for } j = 1,2,\ldots,n. \tag{8}$$

We say that n-tuple $\mathbf{Y} = (\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_n) \in \mathscr{V}^n$ is *pre-majorized* by m-tuple $\mathbf{X} = (\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_m) \in \mathscr{V}^m$, written as $\mathbf{Y} \prec_p \mathbf{X}$, if there exists an $m \times n$ column stochastic matrix $\mathbf{A} = (a_{ij})$ such that

$$(\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_n) = (\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_m)\mathbf{A}. \tag{9}$$

(See [5] for a similar concept of *matrix majorization* induced by row stochastic matrix $\mathbf{A}$.)

We say that n-tuple $(\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_n) \in \mathscr{V}^n$ is *majorized* by n-tuple $(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n) \in \mathscr{V}^n$, written as $\mathbf{Y} \prec \mathbf{X}$, if there exists an $n \times n$ doubly stochastic matrix $\mathbf{A} = (a_{ij})$ such that

$$(\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_n) = (\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)\mathbf{A}. \tag{10}$$

It is known [10, p. 33] that if $\mathscr{V} = \mathbf{R}$ then $\prec$ reduces to the usual (scalar) majorization ordering for n-tuples in $\mathbf{R}^n$ defined equivalently as follows (see [10, p. 8]).

A vector $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbf{R}^n$ is said to be *majorized* by a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$, written as $\mathbf{y} \prec \mathbf{x}$, if

$$\sum_{i=1}^{l} y_{[i]} \leqslant \sum_{i=1}^{l} x_{[i]} \quad \text{for } l = 1, 2, \ldots, n$$

with equality for $l = n$. Here $x_{[i]}$ and $y_{[i]}$ are the ith largest entry of $\mathbf{x}$ and $\mathbf{y}$, respectively.

A function $f : \mathscr{I} \to \mathscr{W}$ is said to be $\leqslant_{\mathscr{C}}$*-convex* on a convex set $\mathscr{I} \subset \mathscr{V}$, if

$$f(\alpha \mathbf{x} + (1-\alpha)\mathbf{y}) \leqslant_{\mathscr{C}} \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) \quad \text{for } \mathbf{x}, \mathbf{y} \in \mathscr{I}, \ \alpha \in [0,1].$$

A function $f : \mathscr{I} \to \mathscr{W}$ is said to be $\leqslant_{\mathscr{C}}$-concave if $-f$ is $\leqslant_{\mathscr{C}}$-convex (cf. [3, pp. 72–73]).

In the forthcoming theorem we present a Sherman type inequality (12) for tuples of vectors (cf. [17], see also [1, 4, 16]). Here we extend the classical (scalar) result to the vector case. However, the idea of the proof remains the same [16].

THEOREM 2.1. *Let* $f : \mathscr{I} \to \mathscr{W}$ *be a* $\leqslant_{\mathscr{C}}$*-convex function defined on a convex set* $\mathscr{I} \subset \mathscr{V}$*. Let* $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m) \in \mathscr{I}^m$, $\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n) \in \mathscr{I}^n$, $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \mathbf{R}_+^m$ *and* $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbf{R}_+^n$.

If

$$\mathbf{Y} = \mathbf{X}\mathbf{A} \quad \text{and} \quad \mathbf{a} = \mathbf{b}\mathbf{A}^T \tag{11}$$

for some $m \times n$ *column stochastic matrix* $\mathbf{A} = (a_{ij})$*, then*

$$\sum_{j=1}^{n} b_j f(\mathbf{y}_j) \leqslant_{\mathscr{C}} \sum_{i=1}^{m} a_i f(\mathbf{x}_i). \tag{12}$$

If f *is* $\leqslant_{\mathscr{C}}$*-concave, then the inequality* (12) *is reversed.*

Proof. In light of (11) we have

$$(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n) = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)\mathbf{A}.$$

Thanks to (8) we can see that $\mathbf{y}_j = \sum_{i=1}^{m} a_{ij} \mathbf{x}_i$ with $\sum_{i=1}^{m} a_{ij} = 1$, $j = 1, 2, \ldots, n$, and $a_{ij} \geqslant 0$. Hence, by the $\leqslant_{\mathscr{C}}$-convexity of f,

$$f(\mathbf{y}_j) = f\left(\sum_{i=1}^{m} a_{ij} \mathbf{x}_i\right) \leqslant_{\mathscr{C}} \sum_{i=1}^{m} a_{ij} f(\mathbf{x}_i) \quad \text{for } j = 1, 2, \ldots, n.$$

Therefore

$$\sum_{j=1}^{n} b_j f(\mathbf{y}_j) \leqslant_{\mathscr{C}} \sum_{j=1}^{n} b_j \sum_{i=1}^{m} a_{ij} f(\mathbf{x}_i),$$

which means

$$\sum_{j=1}^{n} b_j f(\mathbf{y}_j) \leqslant_{\mathscr{C}} \sum_{i=1}^{m} \sum_{j=1}^{n} b_j a_{ij} f(\mathbf{x}_i).$$

However, $\mathbf{a} = \mathbf{b}\mathbf{A}^T$ by (11). So, we get $a_i = \sum_{j=1}^{n} b_j a_{ij}$, $i = 1, 2, \ldots, m$.

In consequence, we conclude that

$$\sum_{j=1}^{n} b_j f(\mathbf{y}_j) \leqslant_{\mathscr{C}} \sum_{i=1}^{m} \left(\sum_{j=1}^{n} b_j a_{ij} \right) f(\mathbf{x}_i) = \sum_{i=1}^{m} a_i f(\mathbf{x}_i).$$

This completes the proof of inequality (12). □

It is worth emphasing that the above requirement (11) leads to the notion of *weighted majorization* for pairs $(\mathbf{X}, \mathbf{a})$ and $(\mathbf{Y}, \mathbf{b})$ (see [4]).

We now illustrate Sherman type inequality (12) by providing an example.

EXAMPLE 2.2. Here we extend a result in [2].

Let $\mathscr{V} = \mathscr{W} = \mathbf{H}_k$ be the real linear space of $k \times k$ Hermitan matrices, with the Loewner cone $\mathscr{C} = \mathbf{L}^k$ of all positive semidefinite matrices in $\mathbf{H}_k$, and with the Loewner ordering $\leqslant$ induced by $\mathbf{L}^k$.

Let f be a real operator convex function defined on an interval $[0, \infty) \subset \mathbf{R}$. As usual, we extend the action of f on $\mathbf{L}^k$ via the standard action of f on the (nonnegative) eigenvalues of the matrices in $\mathbf{L}^k$. Thus $f : \mathscr{I} \to \mathbf{H}_k$ with $\mathscr{I} = \mathbf{L}^k$.

We consider the matrices $\mathbf{A}$ and $\mathbf{A}^T$ of sizes $m \times (m+1)$ and $(m+1) \times m$, respectively, where

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2} & 0 & \ldots & 0 & \frac{1}{2} & \frac{1}{m} \\ \frac{1}{2} & \frac{1}{2} & \ldots & 0 & 0 & \frac{1}{m} \\ 0 & \frac{1}{2} & \ldots & 0 & 0 & \frac{1}{m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \frac{1}{2} & 0 & \frac{1}{m} \\ 0 & 0 & \ldots & \frac{1}{2} & \frac{1}{2} & \frac{1}{m} \end{pmatrix} \quad \text{and} \quad \mathbf{A}^T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \ldots & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \ldots & 0 & \frac{1}{2} \\ \frac{1}{m} & \frac{1}{m} & \frac{1}{m} & \ldots & \frac{1}{m} & \frac{1}{m} \end{pmatrix}.$$

For any positive vector $\mathbf{b} = (b_1, b_2, \ldots, b_m, b_{m+1}) \in \mathbf{R}_+^{m+1}$, we introduce $\mathbf{a} = \mathbf{b}\mathbf{A}^T$. Clearly, $\mathbf{a} = (a_1, a_2, \ldots, a_m)$, where

$$a_i = \frac{b_{i-1} + b_i}{2} + \frac{b_{m+1}}{m}, \quad i = 1, 2, \ldots, m, \quad b_0 = b_m.$$

Then for positive semidefinite matrices $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m \in \mathbf{L}^k$,

$$\sum_{j=1}^{m} b_j f\left(\frac{\mathbf{x}_j + \mathbf{x}_{j+1}}{2}\right) + b_{m+1} f\left(\frac{\mathbf{x}_1 + \mathbf{x}_2 + \ldots + \mathbf{x}_m}{m}\right) \leqslant \sum_{i=1}^{m} \left(\frac{b_{i-1} + b_i}{2} + \frac{b_{m+1}}{m}\right) f(\mathbf{x}_i). \tag{13}$$

This inequality is an extension of that given in [2, Theorem 1.2].

In fact, to show (13) take $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)$ and $\mathbf{Y} = \mathbf{XA} = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_m, \mathbf{y}_{m+1})$. Then we have

$$\mathbf{y}_j = \frac{\mathbf{x}_j + \mathbf{x}_{j+1}}{2}, \quad j = 1, 2, \ldots, m, \quad \mathbf{x}_{m+1} = \mathbf{x}_1,$$

$$\mathbf{y}_{m+1} = \frac{\mathbf{x}_1 + \mathbf{x}_2 + \ldots + \mathbf{x}_m}{m}.$$

Now, inequality (13) is a direct corollary to (12) by Theorem 2.1.

As a consequence of Theorem 2.1 we have the following Hardy-Littlewood-Pólya-Karamata type result for n-tuples of vectors.

THEOREM 2.3. *Let $f : \mathscr{I} \to \mathscr{W}$ be a $\leqslant_{\mathscr{C}}$-convex function defined on a convex set $\mathscr{I} \subset \mathscr{V}$. Let $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) \in \mathscr{I}^n$ and $\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n) \in \mathscr{I}^n$.*

Then

$$\mathbf{Y} \prec \mathbf{X} \quad \textit{implies} \quad \sum_{i=1}^{n} f(\mathbf{y}_i) \leqslant_{\mathscr{C}} \sum_{i=1}^{n} f(\mathbf{x}_i). \tag{14}$$

Proof. We put $m = n$ and $\mathbf{b} = \mathbf{e} = (1, \ldots, 1) \in \mathbf{R}^n$ in Theorem 2.1. Because $\mathbf{Y} \prec \mathbf{X}$, we see that $\mathbf{Y} = \mathbf{XA}$ with some doubly stochastic matrix $\mathbf{A}$ (cf. (10)). By setting $\mathbf{a} = \mathbf{bA}^T$ as in (11), we obtain $\mathbf{a} = \mathbf{eA}^T = \mathbf{e}$. Now, the required inequality (14) is due to (12). □

REMARK 2.4. In the case $\mathscr{V} = \mathscr{W} = \mathbf{R}$ and $\mathscr{C} = [0, \infty)$, Theorem 2.3 becomes the classical *Majorization Theorem* (see [10, pp. 92–93]).

EXAMPLE 2.5. Consider the simplest majorization relation

$$(\bar{\mathbf{x}}, \bar{\mathbf{x}}, \ldots, \bar{\mathbf{x}}) \prec (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n),$$

in the sense

$$(\bar{\mathbf{x}}, \bar{\mathbf{x}}, \ldots, \bar{\mathbf{x}}) = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)\mathbf{A},$$

where $\bar{\mathbf{x}} = \frac{\mathbf{x}_1 + \mathbf{x}_2 + \ldots + \mathbf{x}_n}{n}$ and $\mathbf{A}$ is the $n \times n$ matrix of the form

$$\mathbf{A} = \frac{1}{n} \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ldots & \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix}.$$

Then we obtain the Jensen type inequality

$$f\left(\frac{\mathbf{x}_1 + \mathbf{x}_2 + \ldots + \mathbf{x}_n}{n}\right) \leqslant_{\mathscr{C}} \frac{f(\mathbf{x}_1) + f(\mathbf{x}_2) + \ldots + f(\mathbf{x}_n)}{n},$$

as a particular case of (14) in Theorem 2.3.

A function $F:\mathscr{I}^n\to\mathscr{W}$ with a convex set $\mathscr{I}\subset\mathscr{V}$ is said to be $\leqslant_{\mathscr{C}}$*-Schur-convex* on $\mathscr{I}^n$ if for $\mathbf{X},\mathbf{Y}\in\mathscr{I}^n$,

$$\mathbf{Y}\prec\mathbf{X} \text{ implies } F(\mathbf{Y})\leqslant_{\mathscr{C}} F(\mathbf{X}).$$

A function $F:\mathscr{I}^n\to\mathscr{W}$ with a convex set $\mathscr{I}\subset\mathscr{V}$ is said to be $\leqslant_{\mathscr{C}}$*-Schur-concave* on $\mathscr{I}^n$, if $-F$ is $\leqslant_{\mathscr{C}}$*-Schur-convex* on $\mathscr{I}^n$.

Notice that Theorem 2.3 says that the function

$$F(\mathbf{x}_1,\ldots,\mathbf{x}_n)=\sum_{i=1}^n f(\mathbf{x}_i) \quad \text{for } (\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)\in\mathscr{I}^n$$

is $\leqslant_{\mathscr{C}}$-Schur-convex on $\mathscr{I}^n$, whenever f is $\leqslant_{\mathscr{C}}$-convex on $\mathscr{I}$.

A function $F:\mathscr{I}^n\to\mathscr{W}$ with a set $\mathscr{I}\subset\mathscr{V}$ is said to be *permutation-invariant* on $\mathscr{I}^n$ if for $\mathbf{X}\in\mathscr{I}^n$ and $\mathbf{P}\in\mathscr{P}_n$,

$$F(\mathbf{X}\mathbf{P})=F(\mathbf{X}),$$

where $\mathscr{P}_n$ denotes the set of all $n\times n$ permutation matrices.

A function $F:\mathscr{I}^n\to\mathscr{W}$ with a convex set $\mathscr{I}\subset\mathscr{V}$ is said to be $\leqslant_{\mathscr{C}}$*-convex* on $\mathscr{I}^n$ if for $\mathbf{X}\in\mathscr{I}^n$, $\mathbf{Y}\in\mathscr{I}^n$ and $\alpha\in[0,1]$,

$$F(\alpha\mathbf{X}+(1-\alpha)\mathbf{Y})\leqslant_{\mathscr{C}}\alpha F(\mathbf{X})+(1-\alpha)F(\mathbf{Y}).$$

A function $F:\mathscr{I}^n\to\mathscr{W}$ with a convex set $\mathscr{I}\subset\mathscr{V}$ is said to be $\leqslant_{\mathscr{C}}$*-concave* on $\mathscr{I}^n$, if $-F$ is $\leqslant_{\mathscr{C}}$*-convex* on $\mathscr{I}^n$.

THEOREM 2.6. *Let $F:\mathscr{I}^n\to\mathscr{W}$ be a permutation-invariant $\leqslant_{\mathscr{C}}$-convex function on $\mathscr{I}^n$ with a convex set $\mathscr{I}\subset\mathscr{V}$. Let $\mathbf{X}=(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)\in\mathscr{I}^n$ and $\mathbf{Y}=(\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_n)\in\mathscr{I}^n$.*

Then

$$\mathbf{Y}\prec\mathbf{X} \ \textit{implies} \ F(\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_n)\leqslant_{\mathscr{C}} F(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n). \tag{15}$$

Proof. We assume $\mathbf{Y}\prec\mathbf{X}$, that is, $\mathbf{Y}=\mathbf{X}\mathbf{A}$ for some doubly stochastic matrix $\mathbf{A}$. By virtue of Birkhoff's Theorem [10, p. 30],

$$\mathbf{A}=\sum_{i=1}^{n!}\lambda_i\mathbf{P}_i$$

for some $n\times n$ permutation matrices $\mathbf{P}_i$ with $\sum\limits_{i=1}^{n!}\lambda_i=1$ and $\lambda_k\geqslant 0$, $i=1,2,\ldots,n!$.

For this reason, we deduce that

$$\begin{aligned}
F(\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_n)&=F((\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)\mathbf{A})=F((\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)\sum_{i=1}^{n!}\lambda_i\mathbf{P}_i)\\
&=F\left(\sum_{i=1}^{n!}\lambda_i(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)\mathbf{P}_i\right)\leqslant_{\mathscr{C}}\sum_{i=1}^{n!}\lambda_iF((\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)\mathbf{P}_i)\\
&=\sum_{i=1}^{n!}\lambda_iF(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)=F(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n).
\end{aligned}$$

The above inequality is a consequence of the $\leqslant_{\mathscr{C}}$-convexity of F. The equality before the last one follows from the permutation-invariance of F. $\square$

3. Applications for the Jensen functional

Throughout this section $\mathscr{V} = \mathbf{R}^k$, $\mathscr{I} = \mathscr{P}_k^0$ and $\mathscr{W}$ is a linear space with a convex cone $\mathscr{C} \subset \mathscr{W}$ inducing preordering $\leqslant_{\mathscr{C}}$ on $\mathscr{W}$.

Proof of the next result will be simplified if we first prove a lemma.

LEMMA 3.1. *Let $f : I \to \mathscr{W}$ be a $\leqslant_{\mathscr{C}}$-convex function on an interval $I = [a,b] \subset \mathbf{R}$. Let $\mathbf{P} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_l) \in \mathscr{I}^l$, where $\mathscr{I} = \mathscr{P}_k^0$.*

Then for any fixed $\mathbf{x} \in I^k$,

$$\langle \sum_{i=1}^{l} \mathbf{p}_i, \mathbf{e} \rangle f \left(\frac{\langle \sum_{i=1}^{l} \mathbf{p}_i, \mathbf{x} \rangle}{\langle \sum_{i=1}^{l} \mathbf{p}_i, \mathbf{e} \rangle} \right) \leqslant_{\mathscr{C}} \sum_{i=1}^{l} \langle \mathbf{p}_i, \mathbf{e} \rangle f \left(\frac{\langle \mathbf{p}_i, \mathbf{x} \rangle}{\langle \mathbf{p}_i, \mathbf{e} \rangle} \right). \tag{16}$$

If f is $\leqslant_{\mathscr{C}}$-concave, then the inequality (16) is reversed.

Proof. The following identity holds:

$$\frac{\langle \sum_{i=1}^{l} \mathbf{p}_i, \mathbf{x} \rangle}{\langle \sum_{i=1}^{l} \mathbf{p}_i, \mathbf{e} \rangle} = \sum_{i=1}^{l} \alpha_i \frac{\langle \mathbf{p}_i, \mathbf{x} \rangle}{\langle \mathbf{p}_i, \mathbf{e} \rangle}, \tag{17}$$

where

$$\alpha_i = \frac{\langle \mathbf{p}_i, \mathbf{e} \rangle}{\sum_{j=1}^{l} \langle \mathbf{p}_j, \mathbf{e} \rangle} \quad \text{for } i = 1, 2, \ldots, l. \tag{18}$$

Since $f : I \to \mathscr{W}$ is $\leqslant_{\mathscr{C}}$-convex on I, and $\sum_{i=1}^{l} \alpha_i = 1$ with $\alpha_i \geqslant 0$ for $i = 1, 2, \ldots, l$, from (17)–(18) one has

$$\begin{aligned} f \left(\frac{\langle \sum_{i=1}^{l} \mathbf{p}_i, \mathbf{x} \rangle}{\langle \sum_{i=1}^{l} \mathbf{p}_i, \mathbf{e} \rangle} \right) &= f \left(\sum_{i=1}^{l} \alpha_i \frac{\langle \mathbf{p}_i, \mathbf{x} \rangle}{\langle \mathbf{p}_i, \mathbf{e} \rangle} \right) \leqslant_{\mathscr{C}} \sum_{i=1}^{l} \alpha_i f \left(\frac{\langle \mathbf{p}_i, \mathbf{x} \rangle}{\langle \mathbf{p}_i, \mathbf{e} \rangle} \right) \\ &= \sum_{i=1}^{l} \frac{\langle \mathbf{p}_i, \mathbf{e} \rangle}{\sum_{j=1}^{l} \langle \mathbf{p}_j, \mathbf{e} \rangle} f \left(\frac{\langle \mathbf{p}_i, \mathbf{x} \rangle}{\langle \mathbf{p}_i, \mathbf{e} \rangle} \right) = \frac{1}{\sum_{j=1}^{l} \langle \mathbf{p}_j, \mathbf{e} \rangle} \sum_{i=1}^{l} \langle \mathbf{p}_i, \mathbf{e} \rangle f \left(\frac{\langle \mathbf{p}_i, \mathbf{x} \rangle}{\langle \mathbf{p}_i, \mathbf{e} \rangle} \right). \end{aligned} \tag{19}$$

Now, it is not hard to see that (19) implies (16). $\square$

The next result is a Sherman like inequality (21) for the function (22).

THEOREM 3.2. *Let* $f : I \to \mathscr{W}$ *be a* $\leqslant_{\mathscr{C}}$*-convex function on an interval* $I = [a,b] \subset \mathbf{R}$. *Let* $\mathbf{P} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m) \in \mathscr{I}^m$, $\mathbf{Q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n) \in \mathscr{I}^n$, $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \mathbf{R}^m_+$ *and* $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbf{R}^n_+$, *where* $\mathscr{I} = \mathscr{P}^0_k$.

If

$$\mathbf{Q} = \mathbf{P}\mathbf{A} \quad \textit{and} \quad \mathbf{a} = \mathbf{b}\mathbf{A}^T \tag{20}$$

for some $m \times n$ *column stochastic matrix* $\mathbf{A} = (a_{ij})$, *then for any fixed* $\mathbf{x} \in I^k$,

$$\sum_{j=1}^{n} b_j \langle \mathbf{q}_j, \mathbf{e} \rangle f\left(\frac{\langle \mathbf{q}_j, \mathbf{x} \rangle}{\langle \mathbf{q}_j, \mathbf{e} \rangle}\right) \leqslant_{\mathscr{C}} \sum_{i=1}^{m} a_i \langle \mathbf{p}_i, \mathbf{e} \rangle f\left(\frac{\langle \mathbf{p}_i, \mathbf{x} \rangle}{\langle \mathbf{p}_i, \mathbf{e} \rangle}\right). \tag{21}$$

If f *is* $\leqslant_{\mathscr{C}}$*-concave, then the inequality* (21) *is reversed.*

Proof. For any fixed $\mathbf{x} \in I^k$, the number $\frac{\langle \mathbf{p}, \mathbf{x} \rangle}{\langle \mathbf{p}, \mathbf{e} \rangle}$ lies in I. Namely, $\mathbf{x} \in I^k$ implies $x_i \in I$, i.e., $a \leqslant x_i \leqslant b$, for $i = 1, 2, \ldots, k$. Hence $a\mathbf{e} \leqslant \mathbf{x} \leqslant b\mathbf{e}$. Here $\leqslant$ stands for the componentwise ordering on $\mathbf{R}^k$. In this way, we have

$$\frac{\langle \mathbf{p}, a\mathbf{e} \rangle}{\langle \mathbf{p}, \mathbf{e} \rangle} \leqslant \frac{\langle \mathbf{p}, \mathbf{x} \rangle}{\langle \mathbf{p}, \mathbf{e} \rangle} \leqslant \frac{\langle \mathbf{p}, b\mathbf{e} \rangle}{\langle \mathbf{p}, \mathbf{e} \rangle},$$

and further

$$a \leqslant \frac{\langle \mathbf{p}, \mathbf{x} \rangle}{\langle \mathbf{p}, \mathbf{e} \rangle} \leqslant b.$$

Thus we find that $\frac{\langle \mathbf{p}, \mathbf{x} \rangle}{\langle \mathbf{p}, \mathbf{e} \rangle} \in I$, as wanted.

So, the function

$$\mathbf{p} \to \langle \mathbf{p}, \mathbf{e} \rangle f\left(\frac{\langle \mathbf{p}, \mathbf{x} \rangle}{\langle \mathbf{p}, \mathbf{e} \rangle}\right), \quad \mathbf{p} \in \mathscr{P}^0_k, \tag{22}$$

is well-defined. Consequently, both sides of the inequality (21) are well-defined.

We shall show that the function (22) is $\leqslant_{\mathscr{C}}$-convex on the convex set $\mathscr{I} = \mathscr{P}^0_k$. To see this, we utilize Lemma 3.1 for $l = 2$. Namely, fix any $\mathbf{p}, \mathbf{q} \in \mathscr{P}^0_k$ and $\alpha \in (0,1)$. Then $\alpha\mathbf{p}, (1-\alpha)\mathbf{q} \in \mathscr{P}^0_k$. We use (16) for the vectors $\alpha\mathbf{p}$ and $(1-\alpha)\mathbf{q}$ in place of $\mathbf{p}_1$ and $\mathbf{p}_2$. We obtain

$$\begin{aligned} &\langle \alpha\mathbf{p} + (1-\alpha)\mathbf{q}, \mathbf{e} \rangle f\left(\frac{\langle \alpha\mathbf{p} + (1-\alpha)\mathbf{q}, \mathbf{x} \rangle}{\langle \alpha\mathbf{p} + (1-\alpha)\mathbf{q}, \mathbf{e} \rangle}\right) \\ \leqslant_{\mathscr{C}} \;& \langle \alpha\mathbf{p}, \mathbf{e} \rangle f\left(\frac{\langle \alpha\mathbf{p}, \mathbf{x} \rangle}{\langle \alpha\mathbf{p}, \mathbf{e} \rangle}\right) + \langle (1-\alpha)\mathbf{q}, \mathbf{e} \rangle f\left(\frac{\langle (1-\alpha)\mathbf{q}, \mathbf{x} \rangle}{\langle (1-\alpha)\mathbf{q}, \mathbf{e} \rangle}\right) \\ = \;& \alpha \langle \mathbf{p}, \mathbf{e} \rangle f\left(\frac{\langle \mathbf{p}, \mathbf{x} \rangle}{\langle \mathbf{p}, \mathbf{e} \rangle}\right) + (1-\alpha) \langle \mathbf{q}, \mathbf{e} \rangle f\left(\frac{\langle \mathbf{q}, \mathbf{x} \rangle}{\langle \mathbf{q}, \mathbf{e} \rangle}\right). \end{aligned} \tag{23}$$

In the case $\alpha = 0$ or $\alpha = 1$ inequality (23) holds trivially.

Now, it follows from Theorem 2.1 applied to the $\leqslant_{\mathscr{C}}$-convex function (22) that the inequality (21) is satisfied. □

As a special case of the previous theorem, we have a Hardy-Littlewood-Pólya-Karamata type result as follows. The right-hand side of inequality (24) can be viewed as the f-divergence [7] of $\langle \mathbf{P}, \mathbf{x} \rangle = (\langle \mathbf{p}_1, \mathbf{x} \rangle, \ldots, \langle \mathbf{p}_n, \mathbf{x} \rangle)$ and $\langle \mathbf{P}, \mathbf{e} \rangle = (\langle \mathbf{p}_1, \mathbf{e} \rangle, \ldots, \langle \mathbf{p}_n, \mathbf{e} \rangle)$, respectively. Similarly for the left-hand side of inequality (24).

THEOREM 3.3. *Let $f : I \to \mathscr{W}$ be a $\leqslant_{\mathscr{C}}$-convex function on a closed interval $I \subset \mathbf{R}$. Let $\mathbf{P} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n) \in \mathscr{I}^n$ and $\mathbf{Q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n) \in \mathscr{I}^n$, where $\mathscr{I} = \mathscr{P}_k^0$. Then for any fixed $\mathbf{x} \in I^k$,*

$$\mathbf{Q} \prec \mathbf{P} \quad \textit{implies} \quad \sum_{i=1}^{n} \langle \mathbf{q}_i, \mathbf{e} \rangle f\left(\frac{\langle \mathbf{q}_i, \mathbf{x} \rangle}{\langle \mathbf{q}_i, \mathbf{e} \rangle}\right) \leqslant_{\mathscr{C}} \sum_{i=1}^{n} \langle \mathbf{p}_i, \mathbf{e} \rangle f\left(\frac{\langle \mathbf{p}_i, \mathbf{x} \rangle}{\langle \mathbf{p}_i, \mathbf{e} \rangle}\right). \tag{24}$$

If f is $\leqslant_{\mathscr{C}}$-concave, then the inequality (24) is reversed.

Proof. We set $m = n$ and $\mathbf{b} = \mathbf{e} = (1, \dots, 1) \in \mathbf{R}^n$. By virtue of the vector majorization $(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n) \prec (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n)$, we infer that $(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n) = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n)\mathbf{A}$ for some doubly stochastic matrix $\mathbf{A}$.

We define $\mathbf{a} = \mathbf{b}\mathbf{A}^T$. Then $\mathbf{a} = \mathbf{e}\mathbf{A}^T = \mathbf{e}$.

Finally, by applying Theorem 3.2, Eq. (21), we establish the inequality (24), as desired. □

It is obvious that the statement (24) asserts that for any fixed $\mathbf{x} \in I^k$, the function

$$F(\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n) = \sum_{i=1}^{n} \langle \mathbf{p}_i, \mathbf{e} \rangle f\left(\frac{\langle \mathbf{p}_i, \mathbf{x} \rangle}{\langle \mathbf{p}_i, \mathbf{e} \rangle}\right), \quad \mathbf{p}_i \in \mathscr{I},$$

is $\leqslant_{\mathscr{C}}$-Schur-convex on $\mathscr{I}^n$.

For a $\mathscr{W}$-valued function $f : I \to \mathscr{W}$, the *Jensen functional* is defined as follows. Let $\mathbf{p} = (p_1, p_2, \dots, p_k) \in \mathbf{R}_+^k$, $\mathbf{x} = (x_1, x_2, \dots, x_k) \in I^k$. Then we set

$$J(f, \mathbf{x}, \mathbf{p}) = [\mathbf{p}, f(\mathbf{x})] - \langle \mathbf{p}, \mathbf{e} \rangle f\left(\frac{\langle \mathbf{p}, \mathbf{x} \rangle}{\langle \mathbf{p}, \mathbf{e} \rangle}\right), \tag{25}$$

where $f(\mathbf{x}) = (f(x_1), f(x_2), \dots, f(x_k))$, $\mathbf{e} = (1, 1, \dots, 1) \in \mathbf{R}^k$, and $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbf{R}^k$, and $[\mathbf{p}, f(\mathbf{x})] = \sum_{i=1}^{k} p_i f(x_i) \in \mathscr{W}$.

We are now in a position to interpret the previous results in terms of the Jensen functional.

COROLLARY 3.4. *Let $f : I \to \mathscr{W}$ be a $\leqslant_{\mathscr{C}}$-convex function on a closed interval $I \subset \mathbf{R}$. Let $\mathbf{P} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_m) \in \mathscr{I}^m$, $\mathbf{Q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n) \in \mathscr{I}^n$, $\mathbf{a} = (a_1, a_2, \dots, a_m) \in \mathbf{R}_+^m$ and $\mathbf{b} = (b_1, b_2, \dots, b_n) \in \mathbf{R}_+^n$, where $\mathscr{I} = \mathscr{P}_k^0$.*

If

$$\mathbf{Q} = \mathbf{P}\mathbf{A} \quad \textit{and} \quad \mathbf{a} = \mathbf{b}\mathbf{A}^T \tag{26}$$

for some $m \times n$ column stochastic matrix $\mathbf{A} = (a_{ij})$, then for any fixed $\mathbf{x} \in I^k$,

$$\sum_{j=1}^{n} b_j J(f, \mathbf{x}, \mathbf{q}_j) \geqslant_{\mathscr{C}} \sum_{i=1}^{m} a_i J(f, \mathbf{x}, \mathbf{p}_i). \tag{27}$$

If f is $\leqslant_{\mathscr{C}}$-concave, then the inequality (27) is reversed.

Proof. From (25) we have

$$\sum_{j=1}^{n} b_j J(f,\mathbf{x},\mathbf{q}_j) = \sum_{j=1}^{n} b_j \left([\mathbf{q}_j, f(\mathbf{x})] - \langle \mathbf{q}_j, \mathbf{e}\rangle f\left(\frac{\langle \mathbf{q}_j,\mathbf{x}\rangle}{\langle \mathbf{q}_j,\mathbf{e}\rangle}\right)\right)$$
$$= \left[\sum_{j=1}^{n} b_j \mathbf{q}_j, f(\mathbf{x})\right] - \sum_{j=1}^{n} b_j \langle \mathbf{q}_j,\mathbf{e}\rangle f\left(\frac{\langle \mathbf{q}_j,\mathbf{x}\rangle}{\langle \mathbf{q}_j,\mathbf{e}\rangle}\right). \tag{28}$$

Likewise,

$$\sum_{i=1}^{m} a_i J(f,\mathbf{x},\mathbf{p}_i) = \sum_{i=1}^{m} a_i \left([\mathbf{p}_i, f(\mathbf{x})] - \langle \mathbf{p}_i, \mathbf{e}\rangle f\left(\frac{\langle \mathbf{p}_i,\mathbf{x}\rangle}{\langle \mathbf{p}_i,\mathbf{e}\rangle}\right)\right)$$
$$= \left[\sum_{i=1}^{m} a_i \mathbf{p}_i, f(\mathbf{x})\right] - \sum_{i=1}^{m} a_i \langle \mathbf{p}_i,\mathbf{e}\rangle f\left(\frac{\langle \mathbf{p}_i,\mathbf{x}\rangle}{\langle \mathbf{p}_i,\mathbf{e}\rangle}\right). \tag{29}$$

By (26) via the standard algebra it can be shown that

$$\sum_{j=1}^{n} b_j \mathbf{q}_j = \sum_{i=1}^{m} a_i \mathbf{p}_i.$$

So, by comparing (28) and (29) and taking into account Theorem 3.2, we conclude that inequality (27) is valid. This completes the proof. □

By setting $\mathbf{a}$ and $\mathbf{b}$ to be the all ones vector $\mathbf{e}$, with the aid of a doubly stochastic matrix $\mathbf{A}$, one obtains the following result from Corollary 3.4.

COROLLARY 3.5. *Let $f : I \to \mathscr{W}$ be a $\leqslant_{\mathscr{C}}$-convex function on a closed interval $I \subset \mathbf{R}$. Let $\mathbf{P} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n) \in \mathscr{I}^n$ and $\mathbf{Q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n) \in \mathscr{I}^n$, where $\mathscr{I} = \mathscr{P}_k^0$.*

Then for any fixed $\mathbf{x} \in I^k$,

$$\mathbf{Q} \prec \mathbf{P} \quad \textit{implies} \quad \sum_{i=1}^{n} J(f,\mathbf{x},\mathbf{q}_i) \geqslant_{\mathscr{C}} \sum_{i=1}^{n} J(f,\mathbf{x},\mathbf{p}_i). \tag{30}$$

If f is $\leqslant_{\mathscr{C}}$-concave, then the inequality (30) is reversed.

Proof. It suffices to apply Corollary 3.4 with $m = n$ and $\mathbf{b} = \mathbf{e} = (1,\dots,1) \in \mathbf{R}^n$. Indeed, from $\mathbf{Q} \prec \mathbf{P}$ we get $\mathbf{Q} = \mathbf{PA}$ for some doubly stochastic matrix $\mathbf{A}$ (cf. (10)). By setting $\mathbf{a} = \mathbf{bA}^T$ as in (26) we obtain $\mathbf{a} = \mathbf{eA}^T = \mathbf{e}$. So, the required inequality (30) follows from (27). □

Notice that the above result (30)) says that for any fixed $\mathbf{x} \in I^k$, the function

$$J(\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n) = \sum_{i=1}^{n} J(f,\mathbf{x},\mathbf{p}_i), \quad \mathbf{p}_i \in I,$$

is $\leqslant_{\mathscr{C}}$-Schur-concave on $\mathscr{I}^n$.

EXAMPLE 3.6. Let $f : I \to \mathscr{W}$ be a $\leqslant_{\mathscr{C}}$-convex function on a closed interval $I \subset \mathbf{R}$. In our consideration we use the following symmetric, tridiagonal, doubly stochastic matrix $\mathbf{A} = \mathbf{A}^T$ of size $n \times n$ generated by the scalars $c_1, c_2, \dots, c_{n-1}$. That is,

$$\mathbf{A} = \begin{pmatrix} 1-c_1 & c_1 & 0 & \dots & 0 & 0 & 0 \\ c_1 & 1-c_1-c_2 & c_2 & \dots & 0 & 0 & 0 \\ 0 & c_2 & 1-c_2-c_3 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \ddots & 1-c_{n-3}-c_{n-2} & c_{n-2} & 0 \\ 0 & 0 & \dots & \dots & c_{n-2} & 1-c_{n-2}-c_{n-1} & c_{n-1} \\ 0 & 0 & \dots & \dots & 0 & c_{n-1} & 1-c_{n-1} \end{pmatrix}.$$

For any positive vector $\mathbf{b} = (b_1, b_2, \dots, b_n) \in \mathbf{R}^n_+$ and $\mathbf{P} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n) \in \mathscr{I}^n$, we define $\mathbf{a} = (a_1, a_2, \dots, a_n)$ by $\mathbf{a} = \mathbf{b}\mathbf{A}^T$, and $\mathbf{Q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n) \in \mathscr{I}^n$ by $\mathbf{Q} = \mathbf{P}\mathbf{A}$, where $\mathscr{I} = \mathscr{P}^0_k$.

Thus we derive

$$a_i = c_{i-1}b_{i-1} + (1-c_{i-1}-c_i)b_i + c_i b_{i+1}, \quad i = 1, 2, \dots, n, \quad b_0 = c_0 = 0, \quad b_{n+1} = c_n = 0,$$

$$\mathbf{q}_i = c_{i-1}\mathbf{p}_{i-1} + (1-c_{i-1}-c_i)\mathbf{p}_i + c_i\mathbf{p}_{i+1}, \quad i = 1, 2, \dots, n, \quad \mathbf{p}_0 = 0, \quad \mathbf{p}_{n+1} = 0.$$

For this reason, for any fixed $\mathbf{x} \in I^k$, the following inequality (31) can be established from (21):

$$\begin{aligned} &\sum_{j=1}^{n} b_j \langle c_{j-1}\mathbf{p}_{j-1} + (1-c_{j-1}-c_j)\mathbf{p}_j + c_j\mathbf{p}_{j+1}, \mathbf{e}\rangle \\ &\times \quad f\left(\frac{\langle c_{j-1}\mathbf{p}_{j-1} + (1-c_{j-1}-c_j)\mathbf{p}_j + c_j\mathbf{p}_{j+1}, \mathbf{x}\rangle}{\langle c_{j-1}\mathbf{p}_{j-1} + (1-c_{j-1}-c_j)\mathbf{p}_j + c_j\mathbf{p}_{j+1}, \mathbf{e}\rangle}\right) \\ \leqslant_{\mathscr{C}} &\sum_{i=1}^{m} [c_{i-1}b_{i-1} + (1-c_{i-1}-c_i)b_i + c_i b_{i+1}] \langle \mathbf{p}_i, \mathbf{e}\rangle f\left(\frac{\langle \mathbf{p}_i, \mathbf{x}\rangle}{\langle \mathbf{p}_i, \mathbf{e}\rangle}\right). \end{aligned} \tag{31}$$

Furthermore, it follows from (27) that

$$\begin{aligned} &\sum_{j=1}^{n} b_j J(f, \mathbf{x}, c_{j-1}\mathbf{p}_{j-1} + (1-c_{j-1}-c_j)\mathbf{p}_j + c_j\mathbf{p}_{j+1}) \\ \geqslant_{\mathscr{C}} &\sum_{i=1}^{m} [c_{i-1}b_{i-1} + (1-c_{i-1}-c_i)b_i + c_i b_{i+1}] J(f, \mathbf{x}, \mathbf{p}_i). \end{aligned}$$

This result can be compared to [2, Theorem 1.2].

4. Applications for the Jensen-Mercer functional

As previously, we assume that $\mathscr{V} = \mathbf{R}^k$, $\mathscr{I} = \mathscr{P}_k^0$ and $\mathscr{W}$ is a linear space with a convex cone $\mathscr{C} \subset \mathscr{W}$ inducing preordering $\leqslant_{\mathscr{C}}$ on $\mathscr{W}$.

Let $f : I \to \mathscr{W}$ be a $\mathscr{W}$-valued function with a (closed) interval $I = [a,b] \subset \mathbf{R}$, and $\mathbf{p} = (p_1, p_2, \dots, p_k) \in \mathscr{I}$, $\mathbf{x} = (x_1, x_2, \dots, x_k) \in I^k$. The *Jensen-Mercer functional* is defined by

$$M(f,\mathbf{x},\mathbf{p}) = [\mathbf{p}, (f(a)+f(b))\mathbf{e}] - [\mathbf{p}, f(\mathbf{x})] - \langle \mathbf{p},\mathbf{e}\rangle f\left(\frac{(a+b)\langle \mathbf{p},\mathbf{e}\rangle - \langle \mathbf{p},\mathbf{x}\rangle}{\langle \mathbf{p},\mathbf{e}\rangle}\right),$$

where $f(\mathbf{x}) = (f(x_1), f(x_2), \dots, f(x_k))$, $\mathbf{e} = (1,1,\dots,1) \in \mathbf{R}^k$, and $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbf{R}^k$, and

$$[\mathbf{p}, f(\mathbf{x})] = \sum_{i=1}^k p_i f(x_i) \in \mathscr{W},$$

$$(f(a)+f(b))\mathbf{e} = (f(a)+f(b), f(a)+f(b), \dots, f(a)+f(b)) \in \mathscr{W}^k,$$

$$[\mathbf{p}, (f(a)+f(b))\mathbf{e}] = \sum_{i=1}^k p_i (f(a)+f(b)) \in \mathscr{W}.$$

It is obvious that

$$M(f,\mathbf{x},\mathbf{p}) = [\mathbf{p}, \widetilde{f}(\mathbf{y})] - \langle \mathbf{p},\mathbf{e}\rangle f\left(\frac{\langle \mathbf{p},\mathbf{y}\rangle}{\langle \mathbf{p},\mathbf{e}\rangle}\right),$$

where $\mathbf{y} = (a+b)\mathbf{e} - \mathbf{x}$ and $\widetilde{f}(\mathbf{y}) = (f(a)+f(b))\mathbf{e} - f(\mathbf{x})$.

In Theorem 4.1 we demonstrate a Sherman like inequality (34) related to the function

$$\mathbf{p} \to \langle \mathbf{p},\mathbf{e}\rangle f\left(\frac{\langle \mathbf{p}, (a+b)\mathbf{e} - \mathbf{x}\rangle}{\langle \mathbf{p},\mathbf{e}\rangle}\right), \quad \mathbf{p} \in \mathscr{P}_k^0. \tag{32}$$

THEOREM 4.1. *Let $f : I \to \mathscr{W}$ be a $\leqslant_{\mathscr{C}}$-convex function on an interval $I = [a,b]$. Let $\mathbf{P} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_m) \in \mathscr{I}^m$, $\mathbf{Q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n) \in \mathscr{I}^n$, $\mathbf{a} = (a_1, a_2, \dots, a_m) \in \mathbf{R}_+^m$ and $\mathbf{b} = (b_1, b_2, \dots, b_n) \in \mathbf{R}_+^n$, where $\mathscr{I} = \mathscr{P}_k^0$.*

If

$$\mathbf{Q} = \mathbf{P}\mathbf{A} \quad \text{and} \quad \mathbf{a} = \mathbf{b}\mathbf{A}^T \tag{33}$$

for some $m \times n$ column stochastic matrix $\mathbf{A} = (a_{ij})$, then for any fixed $\mathbf{x} \in I^k$,

$$\sum_{j=1}^n b_j \langle \mathbf{q}_j,\mathbf{e}\rangle f\left(\frac{\langle \mathbf{q}_j, (a+b)\mathbf{e} - \mathbf{x}\rangle}{\langle \mathbf{q}_j,\mathbf{e}\rangle}\right) \leqslant_{\mathscr{C}} \sum_{i=1}^m a_i \langle \mathbf{p}_i,\mathbf{e}\rangle f\left(\frac{\langle \mathbf{p}_i, (a+b)\mathbf{e} - \mathbf{x}\rangle}{\langle \mathbf{p}_i,\mathbf{e}\rangle}\right). \tag{34}$$

If f is $\leqslant_{\mathscr{C}}$-concave, then the inequality (34) is reversed.

Proof. It is easily seen that the vector $\mathbf{y} = (a+b)\mathbf{e} - \mathbf{x}$ lies in I^k. So, the function (32)) is well-defined. Therefore both the sides of the inequality (34) are well-defined, too.

Applying Theorem 3.2 for the vector $\mathbf{y} \in I^k$, we have

$$\sum_{j=1}^{n} b_j \langle \mathbf{q}_j, \mathbf{e} \rangle f\left(\frac{\langle \mathbf{q}_j, \mathbf{y} \rangle}{\langle \mathbf{q}_j, \mathbf{e} \rangle}\right) \leqslant_{\mathscr{C}} \sum_{i=1}^{m} a_i \langle \mathbf{p}_i, \mathbf{e} \rangle f\left(\frac{\langle \mathbf{p}_i, \mathbf{y} \rangle}{\langle \mathbf{p}_i, \mathbf{e} \rangle}\right),$$

whence we obtain the desired inequality (34). □

THEOREM 4.2. *Let $f : I \to \mathscr{W}$ be a $\leqslant_{\mathscr{C}}$-convex function on an interval $I = [a,b]$. Let $\mathbf{P} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n) \in \mathscr{I}^n$ and $\mathbf{Q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n) \in \mathscr{I}^n$, where $\mathscr{I} = \mathscr{P}_k^0$.*

Then for any fixed $\mathbf{x} \in I^k$,

$$\mathbf{Q} \prec \mathbf{P} \quad \textit{implies} \quad \sum_{i=1}^{n} \langle \mathbf{q}_i, \mathbf{e} \rangle f\left(\frac{\langle \mathbf{q}_i, (a+b)\mathbf{e} - \mathbf{x} \rangle}{\langle \mathbf{q}_i, \mathbf{e} \rangle}\right) \leqslant_{\mathscr{C}} \sum_{i=1}^{n} \langle \mathbf{p}_i, \mathbf{e} \rangle f\left(\frac{\langle \mathbf{p}_i, (a+b)\mathbf{e} - \mathbf{x} \rangle}{\langle \mathbf{p}_i, \mathbf{e} \rangle}\right). \tag{35}$$

Proof. Since $\mathbf{Q} = \mathbf{PA}$ for some doubly stochastic matrix $\mathbf{A}$, we have $\mathbf{e} = \mathbf{e}\mathbf{A}^T$. By setting $m = n$ and $\mathbf{a} = \mathbf{b} = \mathbf{e}$, we obtain $\mathbf{a} = \mathbf{b}\mathbf{A}^T$. Now, the required result (35) can be deduced from Theorem 4.1. □

It is worth noting that (35) guarantees $\leqslant_{\mathscr{C}}$-Schur-convexity on $\mathscr{I}^n$ of the function

$$F(\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n) = \sum_{i=1}^{n} \langle \mathbf{p}_i, \mathbf{e} \rangle f\left(\frac{\langle \mathbf{p}_i, (a+b)\mathbf{e} - \mathbf{x} \rangle}{\langle \mathbf{p}_i, \mathbf{e} \rangle}\right), \quad \mathbf{p}_i \in \mathscr{I}.$$

In the forthcoming corollaries we provide inequalities involving the Jensen-Mercer functional.

COROLLARY 4.3. *Let $f : I \to \mathscr{W}$ be a $\leqslant_{\mathscr{C}}$-convex function on an interval $I = [a,b]$. Let $\mathbf{P} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_m) \in \mathscr{I}^m$, $\mathbf{Q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n) \in \mathscr{I}^n$, $\mathbf{a} = (a_1, a_2, \dots, a_m) \in \mathbf{R}_+^m$ and $\mathbf{b} = (b_1, b_2, \dots, b_n) \in \mathbf{R}_+^n$, where $\mathscr{I} = \mathscr{P}_k^0$.*

If

$$\mathbf{Q} = \mathbf{PA} \quad \textit{and} \quad \mathbf{a} = \mathbf{b}\mathbf{A}^T \tag{36}$$

for some $m \times n$ column stochastic matrix $\mathbf{A} = (a_{ij})$, then for any fixed $\mathbf{x} \in I^k$,

$$\sum_{j=1}^{n} b_j M(f, \mathbf{x}, \mathbf{q}_j) \geqslant_{\mathscr{C}} \sum_{i=1}^{m} a_i M(f, \mathbf{x}, \mathbf{p}_i). \tag{37}$$

If f is $\leqslant_{\mathscr{C}}$-concave, then the inequality (37) is reversed.

Proof. We take $\mathbf{y} = (a+b)\mathbf{e} - \mathbf{x}$ and $\widetilde{f}(\mathbf{y}) = (f(a) + f(b))\mathbf{e} - f(\mathbf{x})$. It follows from (34) that

$$\sum_{j=1}^{n} b_j \langle \mathbf{q}_j, \mathbf{e} \rangle f\left(\frac{\langle \mathbf{q}_j, \mathbf{y} \rangle}{\langle \mathbf{q}_j, \mathbf{e} \rangle}\right) \leqslant_{\mathscr{C}} \sum_{i=1}^{m} a_i \langle \mathbf{p}_i, \mathbf{e} \rangle f\left(\frac{\langle \mathbf{p}_i, \mathbf{y} \rangle}{\langle \mathbf{p}_i, \mathbf{e} \rangle}\right). \tag{38}$$

Moreover, we have

$$\sum_{j=1}^{n} b_j M(f,\mathbf{x},\mathbf{q}_j) = \sum_{j=1}^{n} b_j \left([\mathbf{q}_j, \widetilde{f}(\mathbf{y})] - \langle \mathbf{q}_j, \mathbf{e} \rangle f \left(\frac{\langle \mathbf{q}_j, \mathbf{y} \rangle}{\langle \mathbf{q}_j, \mathbf{e} \rangle} \right) \right)$$
$$= \left[\sum_{j=1}^{n} b_j \mathbf{q}_j, \widetilde{f}(\mathbf{y}) \right] - \left\langle \sum_{j=1}^{n} b_j \mathbf{q}_j, \mathbf{e} \right\rangle f \left(\frac{\langle \mathbf{q}_j, \mathbf{y} \rangle}{\langle \mathbf{q}_j, \mathbf{e} \rangle} \right). \tag{39}$$

Similarly,

$$\sum_{i=1}^{m} a_i M(f,\mathbf{x},\mathbf{p}_i) = \sum_{i=1}^{m} a_i \left(\left[\mathbf{p}_i, \widetilde{f}(\mathbf{y}) \right] - \langle \mathbf{p}_i, \mathbf{e} \rangle f \left(\frac{\langle \mathbf{p}_i, \mathbf{y} \rangle}{\langle \mathbf{p}_i, \mathbf{e} \rangle} \right) \right)$$
$$= \left[\sum_{i=1}^{m} a_i \mathbf{p}_i, \widetilde{f}(\mathbf{y}) \right] - \left\langle \sum_{i=1}^{m} a_i \mathbf{p}_i, \mathbf{e} \right\rangle f \left(\frac{\langle \mathbf{p}_i, \mathbf{y} \rangle}{\langle \mathbf{p}_i, \mathbf{e} \rangle} \right). \tag{40}$$

By making use of (36) it can be shown that

$$\sum_{j=1}^{n} b_j \mathbf{q}_j = \sum_{i=1}^{m} a_i \mathbf{p}_i.$$

Now, combining (38), (39) and (40) leads to (37), as claimed. □

COROLLARY 4.4. *Let $f : I \to \mathscr{W}$ be a $\leqslant_{\mathscr{C}}$-convex function on an interval $I = [a,b]$. Let $\mathbf{P} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n) \in \mathscr{I}^n$ and $\mathbf{Q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n) \in \mathscr{I}^n$, where $\mathscr{I} = \mathscr{P}_k^0$. Then for any fixed $\mathbf{x} \in I^k$,*

$$\mathbf{Q} \prec \mathbf{P} \quad \textit{implies} \quad \sum_{i=1}^{n} M(f,\mathbf{x},\mathbf{q}_i) \geqslant_{\mathscr{C}} \sum_{i=1}^{n} M(f,\mathbf{x},\mathbf{p}_i). \tag{41}$$

Proof. It is sufficient to appeal to Corollary 4.3 with $\mathbf{a} = \mathbf{b} = \mathbf{e}$, because $\mathbf{Q} = \mathbf{PA}$ and $\mathbf{a} = \mathbf{bA}^T$ for some doubly stochastic matrix $\mathbf{A}$. This yields (41), which was to be proven. □

It may be noted here that statement (41) ensures $\leqslant_{\mathscr{C}}$-Schur-concavity on $\mathscr{I}^n$ of the function

$$M(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n) = \sum_{i=1}^{n} M(f,\mathbf{x},\mathbf{p}_i), \quad \mathbf{p}_i \in \mathscr{I}.$$

EXAMPLE 4.5. The *circular matrix (circulant)* induced by a real sequence $(c_1, c_2, \ldots, c_n)$ is the $n \times n$ matrix whose first column is $(c_1, c_2, \ldots, c_n)^T$ and the other columns are obtained by succesive cyclic permutations of the first column, i.e.,

$$\mathbf{A} = \begin{pmatrix} c_1 & c_n & c_{n-1} & \ldots & c_3 & c_2 \\ c_2 & c_1 & c_n & \ldots & c_4 & c_3 \\ c_3 & c_2 & c_1 & \ddots & c_5 & c_4 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ c_{n-1} & c_{n-2} & c_{n-3} & \ldots & c_1 & c_n \\ c_n & c_{n-1} & c_{n-2} & \ldots & c_2 & c_1 \end{pmatrix} \tag{42}$$

(cf. [14]). If in addition the circular matrix (42) has nonnegative entries summing to one in each column and row, then $\mathbf{A}$ is called a *doubly stochastic circular matrix* (see [10, pp. 62–64]).

In this context, for a $\leqslant_{\mathscr{C}}$-convex function $f: I \to \mathscr{W}$, $\mathscr{I} = \mathscr{P}_k^0$, $m = n$, $\mathbf{P} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n) \in \mathscr{I}^n$, $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbf{R}_+^n$, and $\mathbf{x} \in I^k$, Theorem 4.1 gives

$$\sum_{j=1}^{n} b_j \langle \mathbf{q}_j, \mathbf{e} \rangle f\left(\frac{\langle \mathbf{q}_j, (a+b)\mathbf{e} - \mathbf{x} \rangle}{\langle \mathbf{q}_j, \mathbf{e} \rangle} \right) \leqslant_{\mathscr{C}} \sum_{i=1}^{n} a_i \langle \mathbf{p}_i, \mathbf{e} \rangle f\left(\frac{\langle \mathbf{p}_i, (a+b)\mathbf{e} - \mathbf{x} \rangle}{\langle \mathbf{p}_i, \mathbf{e} \rangle} \right), \tag{43}$$

where $(\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n) = \mathbf{Q} = \mathbf{PA}$ and $(a_1, a_2, \ldots, a_n) = \mathbf{a} = \mathbf{bA}^T$. Hence it can be deduced that for $i, j = 1, 2, \ldots, n$,

$$\begin{aligned}
\mathbf{q}_j &= c_{n-j+2}\mathbf{p}_1 + c_{n-j+3}\mathbf{p}_2 + c_{n-j+4}\mathbf{p}_3 + \ldots + c_n\mathbf{p}_{j-1} + c_1\mathbf{p}_j + c_2\mathbf{p}_{j+1} + \ldots \\
&\quad + c_{n-j}\mathbf{p}_{n-1} + c_{n-j+1}\mathbf{p}_n, \\
a_i &= c_i b_1 + c_{i-1} b_2 + c_{i-2} b_3 + \ldots \\
&\quad + c_n b_{i+1} + c_{n-1} b_{i+2} + c_{n-2} b_{i+3} + \ldots + c_{i+2} b_{n-1} + c_{i+1} b_n,
\end{aligned}$$

where $c_{n+i} = c_i$.

For instance, for

$$\mathbf{A} = \mathbf{A}^T = \frac{1}{n-1} \begin{pmatrix} 0 & 1 & 1 & \ldots & \ldots & 1 \\ 1 & 0 & 1 & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 1 & 0 \end{pmatrix}$$

and

$$\mathbf{q}_j = \mathbf{p}_1 + \ldots + \mathbf{p}_{j-1} + \mathbf{p}_{j+1} + \ldots + \mathbf{p}_n, \qquad j = 1, 2, \ldots, n,$$

$$a_i = b_1 + \ldots + b_{i-1} + b_{i+1} + \ldots + b_n, \qquad i = 1, 2, \ldots, n,$$

inequality (43) can be rewritten as

$$\sum_{j=1}^{n} b_j \langle (\mathbf{p}_1 + \ldots + \mathbf{p}_{j-1} + \mathbf{p}_{j+1} + \ldots + \mathbf{p}_n), \mathbf{e} \rangle f\left(\frac{\langle (\mathbf{p}_1 + \ldots + \mathbf{p}_{j-1} + \mathbf{p}_{j+1} + \ldots + \mathbf{p}_n), \mathbf{y} \rangle}{\langle (\mathbf{p}_1 + \ldots + \mathbf{p}_{j-1} + \mathbf{p}_{j+1} + \ldots + \mathbf{p}_n), \mathbf{e} \rangle} \right)$$

$$\leqslant_{\mathscr{C}} \sum_{i=1}^{n} (b_1 + \ldots + b_{i-1} + b_{i+1} + \ldots + b_n) \langle \mathbf{p}_i, \mathbf{e} \rangle f\left(\frac{\langle \mathbf{p}_i, \mathbf{y} \rangle}{\langle \mathbf{p}_i, \mathbf{e} \rangle} \right),$$

where $\mathbf{y} = (a+b)\mathbf{e} - \mathbf{x}$. This inequality is related to those in [2, Theorem 1.4] and [14, Corollary 3.4].

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