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A WEIGHTED HARDY–TYPE INEQUALITY FOR $0 < p < 1$ WITH SHARP CONSTANT

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Abstract. Hardy-type inequalities with sharp costants for $0 < p < 1$ for power weight functions were established in [10], [5]. In this work, we give an extension of these inequalities for general weight functions, prove the existence of extremal functions and write them out explicitly.

1. Introduction

It is well known that for L_p-spaces with $0 < p < 1$ the Hardy inequality is not satisfied for arbitrary non-negative measurable functions, but is satisfied for non-negative non-increasing functions. Moreover in [3], pp. 90–91, the sharp constant in the Hardy type inequality for non-negative non-increasing functions was found (see [4] for more details). Namely the following statement was proved there.

THEOREM 1. *Let* $0 < p < 1$ *and* $-\frac{1}{p} < \alpha < 1 - \frac{1}{p}$. *Then for all functions* f *non-negative and non-increasing on* $(0,\infty)$

$$\left\| x^{\alpha-1} \int_0^x f(y)dy \right\|_{L_p(0,\infty)} \leqslant \left(1 - \frac{1}{p} - \alpha\right)^{-\frac{1}{p}} \|x^\alpha f(x)\|_{L_p(0,\infty)}, \tag{1}$$

and the constant $\left(1 - \frac{1}{p} - \alpha\right)^{-\frac{1}{p}}$ *is sharp.*

Moreover, equality in [3] holds if, and only if, for some $a > 0$, $A \geqslant 0$, $f(x) = A\chi_{(0,a)}(x)$ for $x \in (0,\infty)$, $x \neq a$ and $0 \leqslant f(a) \leqslant A$, where $\chi_{(0,a)}$ is the characteristic function of the interval $(0,a)$.

REMARK 1. The assumption $\alpha > -\frac{1}{p}$ is natural because if $\alpha \leqslant -\frac{1}{p}$ then for all functions $f \not\equiv 0$ non-negative and non-increasing on $(0,\infty) \|x^\alpha f(x)\|_{L_p(0,\infty)} = \infty$. If $\alpha \geqslant 1 - \frac{1}{p}$ then for any $A > 0$ the inequality

$$\left\| x^{\alpha-1} \int_0^x f(y)dy \right\|_{L_p(0,\infty)} \leqslant A \|x^\alpha f(x)\|_{L_p(0,\infty)}$$

does not hold on the set of all functions non-negative and non-increasing on $(0,\infty)$.

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The last statement of Theorem 1 is not formulated in [3]. However, it can be easily verified if one traces the proof of inequality (1) in [3]. It is explicitly stated and proved in [1], Theorem 3.1.

The following much more general result was obtained in [9].

THEOREM 2. *Let* $0 < p \leqslant q < \infty$, $0 < p \leqslant 1$ *and* u, v, w *be non-negative measurable functions on* $(0,\infty)$. *Then for all function* f *non-negative and non-increasing on* $(0,\infty)$

$$\Big(\int_0^\infty \Big(\int_0^x f(y)u(y)\,dy\Big)^q w(x)\,dx\Big)^{\frac{1}{q}} \leqslant C\Big(\int_0^\infty [f(x)]^p v(x)\,dx\Big)^{\frac{1}{p}}$$

where

$$C = \sup_{t>0}\Big(\int_0^t v(x)\,dx\Big)^{-\frac{1}{p}}\Big(\int_0^\infty \Big(\int_0^{\min\{s,t\}} u(y)\,dy\Big)^q w(x)\,dx\Big)^{\frac{1}{q}}$$

and the constant C *is sharp.*

We also note the recent survey paper [6] dedicated to weighted integral inequalities on the cone of monotone functions and books [8] and [7] containg a lot of information about weighted inequalities of Hardy type.

Inequalities of type (1) were proved in [1], [2] for non-negative quasi-decreasing functions, also with sharp constants.

In [10], [5] a Hardy type inequality for $0 < p < 1$ was proved under weaker assumptions on f but still of monotonicity type. The result was proved for the n-dimensional variant of the Hardy operator, namely for the operator H defined for all functions $f \in L_1^{loc}(\mathbb{R}^n)$ by

$$(Hf)(r) = \frac{1}{v_n r^n}\int_{B_r} f(x)dx, \qquad 0 < r < \infty, \tag{2}$$

where B_r is the open ball centered at the origin of radius r and v_n is the volume of the unit ball in $\mathbb{R}^n$.

Note that for $n = 1$ inequality (1) is equivalent to the inequality

$$\|r^\alpha (Hf)(r)\|_{L_p(0,\infty)} \leqslant \Big(2\Big(1-\frac{1}{p}-\alpha\Big)\Big)^{-\frac{1}{p}} \||x|^\alpha f(x)\|_{L_p(-\infty,\infty)} \tag{3}$$

for all non-negative even functions on $(-\infty,\infty)$ non-increasing on $(0,\infty)$.

THEOREM 3. ([10], [5]) *Let* $0 < p < 1$, $\alpha < n - \frac{1}{p}$ *and* $M > 0$. *Moreover, let* f *be a function non-negative and Lebesgue measurable on* $\mathbb{R}^n$ *such that for all* $r > 0$ $\|f(x)|x|^{\frac{n}{p'}}\|_{L_p(B_r)} < \infty$ *where* $p' = \frac{p}{p-1}$. *If for almost all* $x \in \mathbb{R}^n$

$$|f(x)| \leqslant M|x|^{-n}\|f(y)|y|^{\frac{n}{p'}}\|_{L_p(B_{|x|})}, \tag{4}$$

then

$$\|r^\alpha (Hf)(r)\|_{L_p(0,\infty)} \leqslant N\|f(x)|x|^{\alpha-\frac{n-1}{p}}\|_{L_p(\mathbb{R}^n)}, \tag{5}$$

where

$$N = pM^{1-p}v_n^{-1}\Big((n-\alpha)p-1\Big)^{-\frac{1}{p}}, \tag{6}$$

and the constant N is sharp.

REMARK 2. If $f(x) = g(|x|)$ where g is a non-negative non-increasing function, then inequality (4) is satisfied with $M = \left(pv_n^{-1}\right)^{\frac{1}{p}}$, hence for such functions inequality (5) holds with the sharp constant

$$N = \left(v_n\left(n - \frac{1}{p} - \alpha\right)\right)^{-1} \tag{7}$$

which for $n = 1$ coincides with the constant in inequality (3). The aim of this paper is to obtain conditions ensuring the validity of an analogue of inequality (5) for the weighted Lebesgue spaces $L_{p,u}(\mathbb{R}^n)$ and $L_{p,v}(0,\infty)$ for the operator H defined by (2).

2. Main results

Let Ω be a Lebesgue measurable set in $\mathbb{R}^n$, u be a non-negative Lebesgue measurable function on Ω (weight function), and $0 < p < \infty$. We denote by $L_{p,u}(\Omega)$ the space of all Lebesgue measurable functions f on Ω for which

$$\|f\|_{L_{p,u}(\Omega)} = \left(\int_\Omega |f(x)|^p\, u(x)\, dx\right)^{\frac{1}{p}} < \infty.$$

THEOREM 4. *Let $C_1 > 0$, $0 < p < 1$ and u, v be weight functions on $\mathbb{R}^n$, $(0,\infty)$ respectively. Suppose that*

$$\int_{B_r} u^{\frac{1}{1-p}}(x)dx = \infty \qquad \text{for some} \quad r > 0 \tag{8}$$

and

$$V(r) := \int_r^\infty v(\rho)\rho^{-np}d\rho < \infty \qquad \text{for all} \quad r > 0. \tag{9}$$

Consider the set of all Lebesgue measurable functions f on $\mathbb{R}^n$ satisfying the inequality

$$|f(x)| \leqslant C_1 u^{\frac{1}{1-p}}(x)\|f\|_{L_{p,u}(B_{|x|})} \tag{10}$$

for almost all $x \in \mathbb{R}^n$. Then for all functions f in this set

$$\|Hf\|_{L_{p,v}(0,\infty)} \leqslant C_2\|f\|_{L_{p,w}(\mathbb{R}^n)}, \tag{11}$$

where

$$w(x) = u(x)V(|x|), \quad x \in \mathbb{R}^n,$$

and

$$C_2 = v_n^{-1}pC_1^{1-p}.$$

If, in addition,

$$\int\limits_{B_{r_2}\setminus B_{r_1}} u^{\frac{1}{1-p}}(x)dx < \infty \qquad \text{for all} \quad 0 < r_1 < r_2 < \infty \tag{12}$$

and

$$\int_0^1 \exp\Big(-C_1^p \int_{B_1\setminus B_{|x|}} u^{\frac{1}{1-p}}(y)\,dy\Big)\, v(r) r^{-np}\,dr < \infty, \tag{13}$$

then the constant C_2 is sharp and there exists a function $f \in L_{p,w}(\mathbb{R}^n)$ not equivalent to 0, satisfying inequality (10) and such that there is equality in inequality (11).

The proof of Theorem 4 is based on the following statements.

LEMMA 1. *Assume that $0 < p < 1$, u is a weight function on $\mathbb{R}^n$ and inequality (10) is satisfied with some $C_1 > 0$ for almost all $x \in \mathbb{R}^n$.*

1. *If $\int_{B_r} u^{\frac{1}{1-p}}(x)dx < \infty$ for all $r > 0$, then f is equivalent to 0 on $\mathbb{R}^n$.*
2. *If $\int_{B_r} u^{\frac{1}{1-p}}(x)dx = \infty$ for some $r > 0$ and*

$$\rho = \inf\Big\{r > 0 : \int_{B_r} u^{\frac{1}{1-p}}(x)dx = \infty\Big\} > 0,$$

then f is equivalent to 0 on B_ρ.

Proof. Let $r > 0$ for Case 1 and $0 < r < \rho$ for Case 2. Hölder's inequality with the exponents $\frac{1}{p}$ and $\left(\frac{1}{p}\right)' = \frac{1}{1-p}$ implies that for $|x| \leqslant r$

$$\|f\|_{L_{p,u}(B_{|x|})} \leqslant \|f\|_{L_1(B_r)} \Big(\int_{B_r} u^{\frac{1}{1-p}}(y)dy\Big)^{\frac{1}{p}-1}.$$

Hence by inequality (10)

$$\|f\|_{L_1(B_r)} \leqslant C_1 \|f\|_{L_1(B_r)} \Big(\int_{B_r} u^{\frac{1}{1-p}}(y)dy\Big)^{\frac{1}{p}}.$$

Let for $0 < \gamma \leqslant 1$

$$r_0(\gamma) = \sup\Big\{r > 0 : \int_{B_r} u^{\frac{1}{1-p}}(x)dx \leqslant \gamma^p C_1^{-p}\Big\}.$$

Note that

$$int_{B_{r_0(\gamma)}} u^{\frac{1}{1-p}}(x)dx = \lim_{r\to r_0(\gamma)^-} \int_{B_r} u^{\frac{1}{1-p}}(x)dx \leqslant \gamma^p C_1^{-p}.$$

Then for $0 < \gamma < 1$

$$\|f\|_{L_1\left(B_{r_0(\gamma)}\right)} \leqslant \gamma \|f\|_{L_1\left(B_{r_0(\gamma)}\right)},$$

which implies that f is equivalent to 0 on $B_{r_0(\gamma)}$ for any $0<\gamma<1$, hence f is equivalent to 0 on B_{r_0} where $r_0=r_0(1)$. Clearly $r_0\leqslant\infty$ in Case 1 and $r_0\leqslant\rho$ in Case 2. If in Case 1 $r_0=\infty$ or in Case 2 $r_0=\rho$, we get the required statement. Assume that $r_0<\infty$ in Case 1 and $r_0<\rho$ in Case 2. Note that

$$\int_{B_{r_0}} u^{\frac{1}{1-p}}(x)dx=C_1^{-p}.$$

Indeed, by the definition of r_0, $\int_{B_{r_0}} u^{\frac{1}{1-p}}(x)dx\leqslant C_1^{-p}$. If $\int_{B_{r_0}} u^{\frac{1}{1-p}}(x)dx<C_1^{-p}$, then, due to continuity in r of the function $\int_{B_r} u^{\frac{1}{1-p}}(x)dx$ on $[0,\infty)$ in Case 1 and on $[0,\rho)$ in Case 2, there exists $\varepsilon>0$ such that $\int_{B_{r_0+\varepsilon}} u^{\frac{1}{1-p}}(x)dx<C_1^{-p}$. Hence, by the definition of r_0, $r_0\geqslant r_0+\varepsilon$ which is impossible.

Since f is equivalent to 0 on B_{r_0} inequality (10) takes the form

$$|f(x)|\leqslant C_1 u^{\frac{1}{1-p}}(x)\|f\|_{L_{p,u}(B_{|x|}\setminus B_{r_0})}$$

for almost all $x\in\mathbb{R}^n$ with $|x|>r_0$. Therefore by Hölder's inequality for $r>r_0$

$$\|f\|_{L_1(B_r\setminus B_{r_0})}\leqslant C_1\|f\|_{L_1(B_r\setminus B_{r_0})}\Big(\int_{B_r\setminus B_{r_0}} u^{\frac{1}{1-p}}(x)dx\Big)^{\frac{1}{p}}.$$

Let

$$r_1=\sup\left\{r>r_0:\int_{B_r\setminus B_{r_0}} u^{\frac{1}{1-p}}(x)dx\leqslant C_1^{-p}\right\}.$$

Then similarly to the above f is equivalent to 0 on $B_{r_1}\setminus B_{r_0}$ hence on B_{r_1}. If $r_1=\infty$ in Case 1 and $r_0=\rho$ in Case 2, then f is equivalent to 0 on $\mathbb{R}^n$, on B_ρ respectively. Otherwise

$$\int_{B_{r_1}\setminus B_{r_0}} u^{\frac{1}{1-p}}(x)dx=C_1^{-p}.$$

Repeating this procedure we obtain that either f is equivalent to 0 on $\mathbb{R}^n$ in Case 1 or on B_ρ in Case 2, or f is equivalent to 0 on B_{r_k} for any $k\in\mathbb{N}$ where r_k are such that $r_0<r_1<\ldots<r_k<\ldots<\infty$ in Case 1 and $r_0<r_1<\ldots<r_k<\ldots<\rho$ in Case 2, and

$$\int_{B_{r_k}\setminus B_{r_{k-1}}} u^{\frac{1}{1-p}}(x)dx=C_1^{-p}.$$

Therefore

$$\int_{B_{r_k}} u^{\frac{1}{1-p}}(x)dx=\int_{B_{r_0}} u^{\frac{1}{1-p}}(x)dx+\sum_{m=1}^{k}\int_{B_{r_m}\setminus B_{r_{m-1}}} u^{\frac{1}{1-p}}(x)dx=(k+1)C_1^{-p},\quad k\in\mathbb{N},$$

hence

$$\int_{B_\sigma} u^{\frac{1}{1-p}}(x)dx=\lim_{k\to\infty}\int_{B_{r_k}} u^{\frac{1}{1-p}}(x)dx=\infty,\quad f\sim 0\quad\text{on}\quad B_\sigma=\bigcup_{k=1}^{\infty}B_{r_k},$$

where $\sigma = \lim_{k\to\infty} r_k$.

In Case 1 this is only possible if $\sigma = \infty$, hence f is equivalent to 0 on $\mathbb{R}^n$. In Case 2 all $r_k < \rho$, hence $\sigma \leqslant \rho$. Also, by the definition of ρ, $\sigma \geqslant \rho$. So $\sigma = \rho$ and f is equivalent to 0 on B_ρ. □

REMARK 3. The statement of Lemma 1 explains the assumption of inequality (8) on u in Theorem 4. Recall that in the case of inequality (4) $u(x) = |x|^{n(p-1)}$ and $\int_{B_1} u^{\frac{1}{1-p}}(x)dx = \infty$ as required in Theorem 4.

Given $r_1, r_2 > 0$, let $B_{r_1} \triangle B_{r_2}$ denote the symmetric difference of the balls B_{r_1} and B_{r_2}. Clearly $B_{r_1} \triangle B_{r_2} = B_{r_1} \backslash B_{r_2}$ if $r_1 \geqslant r_2$ and $B_{r_1} \triangle B_{r_2} = B_{r_2} \backslash B_{r_1}$ if $r_1 < r_2$.

For non-negative functions g measurable on $\mathbb{R}^n$ we introduce the following notation

$$\int^*_{B_{r_1} \triangle B_{r_2}} g(x)dx = sgn(r_1 - r_2) \int_{B_{r_1} \triangle B_{r_2}} g(x)dx.$$

Note that

$$\int^*_{B_{r_1} \triangle B_{r_2}} g(x)dx = \int_{B_{r_1}} g(x)dx - \int_{B_{r_2}} g(x)dx = \begin{cases} \int_{B_{r_1} \backslash B_{r_2}} g(x)dx, & 0 < r_2 \leqslant r_1 \leqslant \infty, \\ -\int_{B_{r_2} \backslash B_{r_1}} g(x)dx, & 0 < r_1 \leqslant r_2 \leqslant \infty. \end{cases}$$

By taking the spherical coordinates it follows that for any $0 < r_1, r_2 \leqslant \infty$

$$\int^*_{B_{r_1} \triangle B_{r_2}} g(x)dx = \int_{r_2}^{r_1} \Big(\int_{S_{n-1}} g(\rho\sigma)d\sigma \Big) \rho^{n-1} d\rho,$$

where S_{n-1} is the unit sphere in $\mathbb{R}^n$. If g is radial, i.e. $g(x) = \overline{g}(|x|)$, $x \in \mathbb{R}^n$, then

$$\int^*_{B_{r_1} \triangle B_{r_2}} g(x)dx = n v_n \int_{r_2}^{r_1} \overline{g}(\rho)\rho^{n-1} d\rho.$$

LEMMA 2. *Let $0 < p < 1$, $C_1 > 0$, u be a weight function on $\mathbb{R}^n$ satisfying conditions* (8) *and* (12), *and*

$$f(x) = K u^{\frac{1}{1-p}}(x) \exp\left(\frac{C_1^p}{p} \int^*_{B_{|x|} \triangle B_1} u^{\frac{1}{1-p}}(y)dy \right), \quad x \in \mathbb{R}^n, \tag{14}$$

where $K \geqslant 0$. Then $f \in L_{p,u}(B_r)$ for all $r > 0$ and

$$f(x) = C_1 u^{\frac{1}{1-p}}(x) \|f\|_{L_{p,u}(B_{|x|})}, \quad x \in \mathbb{R}^n. \tag{15}$$

Moreover, if u is a positive radial function satisfying (8) *and* (12), *then any radial function $f \in L_{p,u}(B_r)$ for all $r > 0$ satisfying* (15) *has form* (14).

Proof. 1. Let f be given by formula (14). By taking the spherical coordinates we get that for any $r>0$

$$\begin{aligned}
\|f\|_{L_{p,u}(B_r)} &= K\Big(\int_{B_r} u^{\frac{p}{1-p}}(x)\exp\Big(C_1^p\int^*_{B_{|x|}\triangle B_1} u^{\frac{1}{1-p}}(y)dy\Big)u(x)dx\Big)^{\frac{1}{p}}\\
&= K\Big(\int_0^r\Big(\int_{S_{n-1}} u^{\frac{1}{1-p}}(\rho\sigma)d\sigma\Big)\exp\Big(C_1^p\int_1^\rho\Big(\int_{S_{n-1}} u^{\frac{1}{1-p}}(t\omega)d\omega\Big)t^{n-1}dt\Big)\rho^{n-1}d\rho\Big)^{\frac{1}{p}}\\
&= KC_1^{-1}\Big(\int_0^r\Big(\exp\Big(C_1^p\int_1^\rho\Big(\int_{S_{n-1}} u^{\frac{1}{1-p}}(t\omega)d\omega\Big)t^{n-1}dt\Big)\Big)'_\rho d\rho\Big)^{\frac{1}{p}}\\
&= KC_1^{-1}\Big(\int_0^r\Big(\exp\Big(C_1^p\int^*_{B_\rho\triangle B_1} u^{\frac{1}{1-p}}(y)dy\Big)\Big)'_\rho d\rho\Big)^{\frac{1}{p}}\\
&= KC_1^{-1}\Big(\exp\Big(C_1^p\int^*_{B_r\triangle B_1} u^{\frac{1}{1-p}}(y)dy\Big)-\lim_{\tau\to 0^+}\exp\Big(-C_1^p\int_{B_1\setminus B_\tau} u^{\frac{1}{1-p}}(y)dy\Big)\Big)^{\frac{1}{p}}\\
&= KC_1^{-1}\exp\Big(\frac{C_1^p}{p}\int^*_{B_r\triangle B_1} u^{\frac{1}{1-p}}(y)dy\Big)<\infty
\end{aligned} \tag{16}$$

due to conditions (8) and (12). Therefore for all $x\in\mathbb{R}^n$

$$\exp\Big(\frac{C_1^p}{p}\int^*_{B_{|x|}\triangle B_1} u^{\frac{1}{1-p}}(y)dy\Big)=K^{-1}C_1\|f\|_{L_{p,u}(B_{|x|})} \tag{17}$$

and equality (14) implies equality (15).

2. Next assume that $f\in L_{p,u}(B_r)$ for all $r>0$ and satisfies equality (15). Then for all $x\in\mathbb{R}^n$ which we represent in the form $x=r\sigma$, $\sigma\in S^{n-1}$ we get

$$f(r\sigma)^p=C_1^p u^{\frac{p}{1-p}}(r\sigma)\int_0^r\Big(\int_{S_{n-1}} f(\rho\omega)^p u(\rho\omega)d\omega\Big)\rho^{n-1}d\rho.$$

Multiplying by $u(r\sigma)$ and integrating over S_{n-1} yields

$$\int_{S_{n-1}} f(r\sigma)^p u(r\sigma)d\sigma=C_1^p\Big(\int_{S_{n-1}} u^{\frac{1}{1-p}}(r\sigma)d\sigma\Big)\int_0^r\Big(\int_{S_{n-1}} f(\rho\omega)^p u(\rho\omega)d\omega\Big)\rho^{n-1}d\rho \tag{18}$$

Let for $r>0$

$$F(r)=\int_0^r\Big(\int_{S_{n-1}} f(\rho\omega)^p u(\rho\omega)d\omega\Big)\rho^{n-1}d\rho=\|f\|^p_{L_{p,u}(B_r)}$$

and

$$U(r)=\int_{S_{n-1}} u^{\frac{1}{1-p}}(r\sigma)d\sigma.$$

Then the function F is absolutely continuous on $[0,a]$ for any $a>0$ and for almost all $r>0$ we have

$$F'(r)=\Big(\int_{S_{n-1}} f(r\omega)^p u(r\omega)d\omega\Big)r^{n-1} \tag{19}$$

and equality (18) takes the form

$$F'(r) = C_1^p U(r) r^{n-1} F(r). \tag{20}$$

If f is equivalent to 0 on $\mathbb{R}^n$, then equality (14) is trivial (with $K = 0$). Assume that f is not equivalent to 0 on $\mathbb{R}^n$. Then $r_0 = \inf\{r > 0 : F(r) > 0\} < \infty$. Hence, for $r > r_0$ $F(r) > 0$ and for almost all $r > 0$

$$(\ln F(r))' = \frac{F'(r)}{F(r)} = C_1^p U(r) r^{n-1}.$$

Therefore, by using the Newton-Leibnitz formula for the derivative of the absolutely continuous function $\ln F$ for the interval $[r_0+1, r]$, we get

$$F(r) = F(r_0+1)\exp\Big(C_1^p \int_{r_0+1}^{r} U(\rho)\rho^{n-1} d\rho\Big). \tag{21}$$

By passing to the limit as $r \to r_0^+$ we get that $F(r_0) > 0$ which is impossible if $r_0 > 0$ (because, by the definition of r_0, $F(r_0) = 0$). So $r_0 = 0$ and equalities (18)–(21) imply that for almost all $r > 0$

$$\begin{aligned}
\int_{S_{n-1}} f(r\sigma)^p u(r\sigma) d\sigma &= F'(r) r^{1-n} = C_1^p U(r) F(r) \\
&= C_1^p U(r) F(1) \exp\Big(C_1^p \int_1^r U(\rho)\rho^{n-1} d\rho\Big) \\
&= K^p \Big(\int_{S_{n-1}} u^{\frac{1}{1-p}}(r\sigma) d\sigma\Big) \exp\Big(C_1^p \int_1^r U(\rho)\rho^{n-1} d\rho\Big) \\
&= K^p \Big(\int_{S_{n-1}} u^{\frac{1}{1-p}}(r\sigma) d\sigma\Big) \exp\Big(C_1^p \int_{B_r \triangle B_1}^{*} u^{\frac{1}{1-p}}(y) dy\Big),
\end{aligned}$$

where

$$K = C_1 F(1)^{\frac{1}{p}} = C_1 \|f\|_{L_{p,u}(B_1)}.$$

Assume now that the functions u and f are radial: $u(x) = \bar{u}(|x|)$, $f(x) = \bar{f}(|x|)$. Then by (21) we get that for almost all $r > 0$

$$n v_n \bar{f}(r)^p \bar{u}(r) = K^p n v_n \bar{u}^{\frac{1}{1-p}}(r) \exp\Big(C_1^p \int_{B_r \triangle B_1}^{*} u^{\frac{1}{1-p}}(y) dy\Big)$$

which implies equality (14). □

EXAMPLE 1. In the case $u(x) = |x|^\mu$ with $\mu > n(p-1)$ equality (14) takes the form

$$f(x) = K_1 |x|^{\frac{\mu}{1-p}} \exp\Big(\frac{C_1^p n v_n}{p}\Big(n + \frac{\mu}{1-p}\Big)^{-1} |x|^{n+\frac{\mu}{1-p}}\Big)$$

where $K_1 = K \exp\Big(-\frac{C_1^p n v_n}{p}\Big(n + \frac{\mu}{1-p}\Big)^{-1}\Big)$.

In the case of inequality (4) $u(x) = |x|^{n(p-1)}$ and equality (14) takes the form

$$f(x) = K|x|^{\left(\frac{C_1^p v_n}{p} - 1\right)n}.$$

If, as in Remark 2, $C_1 = (p v_n^{-1})^{\frac{1}{p}}$ then $f(x) = K$.

LEMMA 3. *Let $0 < p < 1$ and u be a weight function on $\mathbb{R}^n$ satisfying the condition (8). If a measurable function f is such that for some $C_1 > 0$ condition (10) is satisfied for almost all $x \in \mathbb{R}^n$, then for all $r > 0$*

$$\|f\|_{L_1(B_r)} \leqslant C_3 \|f\|_{L_{p,u}(B_r)} \tag{22}$$

where

$$C_3 = pC_1^{1-p}.$$

If u also satisfies condition (12), then the constant C_3 in (22) is sharp and equality is attained for any function f defined by equality (14).

Moreover, if u is a positive radial function satisfying (8) and (12), then any positive radial function $f \in L_{p,u}(B_r)$ for all $r > 0$ for which there is equality in (22) has the form (14).

Proof. 1. Since $1 - p > 0$ by inequality (10)

$$|f(x)|^{1-p} \leqslant C_1^{1-p} \Big(\int_{B_{|x|}} |f(y)|^p \, u(y) \, dy \Big)^{\frac{1-p}{p}} u(x)$$

which implies that

$$|f(x)| \leqslant C_1^{1-p} \Big(\int_{B_{|x|}} |f(y)|^p \, u(y) \, dy \Big)^{\frac{1}{p}-1} |f(x)|^p \, u(x)$$

for almost all $x \in \mathbb{R}^n$. All such x we represent in the form $x = \rho\sigma$, $\sigma \in S_{n-1}$. By taking the spherical coordinates and multiplying both sides by ρ^{n-1} we get

$$|f(\rho\sigma)|\rho^{n-1} \leqslant C_1^{1-p} \Big(\int_0^\rho \Big(\int_{S_{n-1}} |f(t\omega)|^p u(t\omega) d\omega \Big) t^{n-1} dt \Big)^{\frac{1}{p}-1} |f(\rho\sigma)|^p u(\rho\sigma) \rho^{n-1}$$

for almost all $\rho > 0$ and $\sigma \in S_{n-1}$.

By integrating over S_{n-1} we get that for almost all $\rho > 0$

$$\begin{aligned} &\Big(\int_{S_{n-1}} |f(\rho\sigma)| d\sigma \Big) \rho^{n-1} \\ &\leqslant C_1^{1-p} \Big(\int_0^\rho \Big(\int_{Sn-1} |f(t\omega)|^p u(t\omega) d\omega \Big) t^{n-1} dt \Big)^{\frac{1}{p}-1} \Big(\int_{S_{n-1}} |f(\rho\sigma)|^p u(\rho\sigma) d\sigma \Big) \rho^{n-1} \\ &= pC_1^{1-p} \Big[\Big(\int_0^\rho \Big(\int_{S_{n-1}} |f(t\omega)|^p u(t\omega) d\omega \Big) t^{n-1} dt \Big)^{\frac{1}{p}} \Big]_\rho' . \end{aligned}$$

By integrating over $(0, r)$ we get

$$\int_0^r \Big(\int_{S_{n-1}} |f(\rho\sigma)| d\sigma \Big) \rho^{n-1} d\rho \leqslant pC_1^{1-p} \Big(\int_0^r \Big(\int_{S_{n-1}} |f(t\omega)|^p u(t\omega) d\omega \Big) t^{n-1} dt \Big)^{\frac{1}{p}}$$

which implies inequality (22).

2. Let f be defined by equality (14). By taking the spherical coordinates we get that for any $r>0$

$$\begin{aligned}\|f\|_{L_1(B_r)} &= K\int_{B_r} u^{\frac{p}{1-p}}(x)\exp\Big(\frac{C_1^p}{p}\int^*_{B_{|x|}\triangle B_1} u^{\frac{1}{1-p}}(y)dy\Big)dx\\ &= K\int_0^r\Big(\int_{S_{n-1}} u^{\frac{1}{1-p}}(\rho\sigma)d\sigma\Big)\exp\Big(\frac{C_1^p}{p}\int_1^\rho\Big(\int_{S_{n-1}} u^{\frac{1}{1-p}}(t\omega)d\omega\Big)t^{n-1}dt\Big)\rho^{n-1}d\rho\\ &= KpC_1^{-p}\int_0^r\Big(\exp\Big(\frac{C_1^p}{p}\int_1^\rho\Big(\int_{S_{n-1}} u^{\frac{1}{1-p}}(t\omega)d\omega\Big)t^{n-1}dt\Big)\Big)'_\rho d\rho\\ &= KpC_1^{-p}\int_0^r\Big(\exp\Big(\frac{C_1^p}{p}\int^*_{B_\rho\triangle B_1} u^{\frac{1}{1-p}}(y)dy\Big)\Big)'_\rho d\rho\\ &= KpC_1^{-p}\Big(\exp\Big(\frac{C_1^p}{p}\int^*_{B_r\triangle B_1} u^{\frac{1}{1-p}}(y)dy\Big)-\lim_{\varepsilon\to 0^+}\exp\Big(-\frac{C_1^p}{p}\int_{B_1\setminus B_\varepsilon} u^{\frac{1}{1-p}}(y)dy\Big)\\ &= KpC_1^{-p}\exp\Big(\frac{C_1^p}{p}\int^*_{B_r\triangle B_1} u^{\frac{1}{1-p}}(y)dy\Big) \qquad (23)\end{aligned}$$

due to condition (8). Comparing (23) and (16) we see that there is equality in (22).

3. Finally assume that u is a positive radial function satisfying (8) and (12), $u(x)=\bar{u}(|x|)$, $x\in\mathbb{R}^n$, $f\in L_{p,u}(B_r)$ for all $r>0$ is a positive radial function, $f(x)=\bar{f}(|x|)$, $x\in\mathbb{R}^n$, and there is equality in (22) for all $r>0$. Then for any $r>0$

$$nv_n\int_0^r \bar{f}(\rho)\rho^{n-1}d\rho = C_3\Big(nv_n\int_0^r \bar{f}(\rho)^p u(\rho)\rho^{n-1}d\rho\Big)^{\frac{1}{p}}.$$

Differentiating this equality we get that for almost all $r>0$

$$(nv_n)^{\frac{p-1}{p}}\bar{f}(r)r^{n-1} = \frac{C_3}{p}\Big(\int_0^r \bar{f}(\rho)^p\bar{u}(\rho)\rho^{n-1}d\rho\Big)^{\frac{1}{p}-1}\bar{f}(r)^p\bar{u}(r)r^{n-1}$$

which implies that

$$\bar{f}(r)\bar{u}(r)^p r^{n-1} = C_1^p nv_n\bar{u}(r)^{\frac{1}{1-p}}r^{n-1}\int_0^r \bar{f}(\rho)^p\bar{u}(\rho)\rho^{n-1}d\rho.$$

If

$$F(r)=\int_0^r \bar{f}(\rho)^p\bar{u}(\rho)\rho^{n-1}d\rho$$

then it follows that

$$F'(r)=C_1^p nv_n\bar{u}^{\frac{1}{1-p}}(r)r^{n-1}F(r)$$

Similarly to the argument of Step 2 of the proof of Lemma 2 this implies that for all $r>0$

$$F(r)=F(1)\exp\Big(C_1^p nv_n\int_1^r \bar{u}^{\frac{1}{1-p}}(\rho)\rho^{n-1}d\rho\Big)$$

hence for almost all $r>0$

$$F'(r)=\bar{f}(r)^p r^{n-1}=F(1)nv_nC_1^p\exp\Big(C_1^p nv_n\int_1^r \bar{u}^{\frac{1}{1-p}}(\rho)\rho^{n-1}d\rho\Big)\bar{u}^{\frac{1}{1-p}}(r)r^{n-1}$$

and

$$\bar{f}(r) = K\bar{u}^{\frac{1}{1-p}}(r)\exp\Big(\frac{C_1^p}{p}\int^*_{B_r\triangle B_1} u^{\frac{1}{1-p}}(y)dy\Big),$$

where

$$K = C_1\|f\|_{L_{p,u}}(B_1),$$

and the statement follows. □

Proof of Theorem 4. Since

$$\begin{aligned}\|Hf\|_{L_{p,v}(0,\infty)} &= \Big(\int_0^\infty v(r)\,|(Hf)(r)|^p\,dr\Big)^{\frac{1}{p}}\\ &= \Big(\int_0^\infty v(r)\,\frac{1}{v_n^p\,r^{np}}\,\Big|\int_{B_r} f(x)\,dx\Big|^p\,dr\Big)^{\frac{1}{p}}\\ &\leqslant \frac{1}{v_n}\,\Big(\int_0^\infty v(r)\,\frac{1}{r^{np}}\,\Big(\int_{B_r}|f(x)|\,dx\Big)^p\,dr\Big)^{\frac{1}{p}},\end{aligned}$$

by Lemma 3 we obtain

$$\|Hf\|_{L_{p,v}(0,\infty)} \leqslant \frac{C_3}{v_n}\,\Big(\int_0^\infty\Big(\frac{v(r)}{r^{np}}\,\int_{B_r}|f(x)|^p\,u(x)dx\,\Big)dr\Big)^{\frac{1}{p}}.$$

Fubini's theorem yields

$$\begin{aligned}\|Hf\|_{L_{p,v}(0,\infty)} &\leqslant \frac{C_3}{v_n}\Big(\int_{\mathbb{R}^n}\,|f(x)|^p\,u(x)\,\,\Big(\int_{|x|}^\infty\frac{v(r)}{r^{np}}\,dr\Big)\,dx\Big)^{\frac{1}{p}}\\ &= C_2\Big(\int_{\mathbb{R}^n}\,|f(x)|^p\,u(x)\,V(|x|)\,dx\Big)^{\frac{1}{p}}\end{aligned}$$

and inequality (11) follows.

2. Assume that condition (12) is also satisfied. Let for all $x\in\mathbb{R}^n$

$$g(x) = u^{\frac{1}{1-p}}(x)\exp\Big(\frac{C_1^p}{p}\int^*_{B_{|x|}\triangle B_1} u^{\frac{1}{1-p}}(y)dy\Big)$$

and for $a>0$ for all $x\in\mathbb{R}^n$

$$g_a(x) = g(x)\chi_{B_a}(x).$$

Then by Lemma 2 the function g_a satisfies inequality (10), and for $x\in B_a$ there is equality in (15).

Moreover, by (23) with $K=1$

$$\begin{aligned}\|Hg_a\|^p_{L_{p,v}(0,\infty)} &= \int_0^a (v_nr^n)^{-p}\|g\|^p_{L_1(B_r)}\,v(r)dr + \int_a^\infty (v_nr^n)^{-p}\|g\|^p_{L_1(B_a)}\,v(r)dr\\ &= (pC_1^{-p}v_n^{-1})^p\Big[\int_0^a \exp\Big(C_1^p\int^*_{B_r\triangle B_1}u^{\frac{1}{1-p}}(y)dy\Big)v(r)r^{-np}dr\\ &\quad+\exp\Big(C_1^p\int^*_{B_a\triangle B_1}u^{\frac{1}{1-p}}(y)dy\Big)V(a)\Big].\end{aligned} \tag{24}$$

Also

$$\begin{aligned}
\|g_a\|^p_{L_{p,w}(\mathbb{R}^n)} &= \|g\|^p_{L_{p,w}(B_a)} = \int_{B_a} u^{\frac{1}{1-p}}(x)\exp\Big(C_1^p \int^*_{B_{|x|}\triangle B_1} u^{\frac{1}{1-p}}(y)dy\Big)V(|x|)dx \\
&= \int_0^a \Big(\int_{S_{n-1}} u^{\frac{1}{1-p}}(r\sigma)d\sigma\Big)r^{n-1}\exp\Big(C_1^p\int_1^r \Big(\int_{S_{n-1}} u^{\frac{1}{1-p}}(\rho\omega)d\omega\Big)\rho^{n-1}d\rho\Big)V(r)dr \\
&= C_1^{-p}\int_0^a \Big(\exp\Big(C_1^p\int_1^r \Big(\int_{S_{n-1}} u^{\frac{1}{1-p}}(\rho\omega)d\omega\Big)\rho^{n-1}d\rho\Big)\Big)'_r V(r)dr \\
&= C_1^{-p}\lim_{\varepsilon\to 0^+}\int_\varepsilon^a \Big(\exp\Big(C_1^p\int^*_{B_r\triangle B_1} u^{\frac{1}{1-p}}(y)dy\Big)\Big)'_r V(r)dr. \qquad (25)
\end{aligned}$$

Let us put

$$I(\varepsilon,a) := \int_\varepsilon^a \Big(\exp\Big(C_1^p\int^*_{B_r\triangle B_1} u^{\frac{1}{1-p}}(y)dy\Big)\Big)'_r V(r)dr.$$

Then we have

$$\begin{aligned}
I(\varepsilon,a) &= \exp\Big(C_1^p\int^*_{B_a\triangle B_1} u^{\frac{1}{1-p}}(y)dy\Big)V(a) - \exp\Big(-C_1^p\int_{B_1\setminus B_\varepsilon} u^{\frac{1}{1-p}}(y)dy\Big)V(\varepsilon) \\
&\quad -\int_\varepsilon^a \exp\Big(C_1^p\int^*_{B_r\triangle B_1} u^{\frac{1}{1-p}}(y)dy\Big)V'(r)dr \\
&\leqslant \exp\Big(C_1^p\int^*_{B_a\triangle B_1} u^{\frac{1}{1-p}}(y)dy\Big)V(a)\,r \\
&\quad +\int_0^a \exp\Big(C_1^p\int^*_{B_r\triangle B_1} u^{\frac{1}{1-p}}(y)dy\Big)v(r)r^{-np}dr < \infty \qquad (26)
\end{aligned}$$

by condition (13). Therefore in (25) the limit as $\varepsilon\to 0^+$ exists and is finite. Hence, $g_a \in L_{p,w}(\mathbb{R}^n)$.

Moreover, equality (26) implies that also the limit

$$\lim_{\varepsilon\to 0^+}\exp\Big(-C_1^p\int_{B_1\setminus B_\varepsilon} u^{\frac{1}{1-p}}(y)dy\Big)V(\varepsilon)$$

exists and is finite. This limit cannot be positive, otherwise condition (13) will not be satisfied. Hence, it is equali to 0. By passing in (26) to the limit as $\varepsilon\to 0^+$ we get that

$$\begin{aligned}
\|g_a\|^p_{L_{p,w}(\mathbb{R}^n)} = C_1^{-p}\Big[&\int_0^a \exp\Big(C_1^p\int^*_{B_r\triangle B_1} u^{\frac{1}{1-p}}(y)dy\Big)v(r)r^{-np}dr \\
&+\exp\Big(C_1^p\int^*_{B_a\triangle B_1} u^{\frac{1}{1-p}}(y)dy\Big)V(a)\Big]. \qquad (27)
\end{aligned}$$

By comparing (24) and (27) we see that for $f = g_a$ in inequality (11) there is equality for any $a > 0$. □

REMARK 4. If in Theorem 4 we take $u(x) = |x|^{-n(1-p)}$ and $v(r) = r^{\alpha p}$ with $\alpha p - np + 1 < 0$, then we obtain inequality (5) in Theorem 3.

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