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ENDPOINT ESTIMATES FOR COMMUTATORS OF INTRINSIC SQUARE FUNCTIONS IN MORREY TYPE SPACES

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Abstract. In this paper, the boundedness properties of commutators generated by b and intrinsic square functions in the endpoint case are discussed, where $b \in BMO(\mathbb{R}^n)$. We first establish the weighted weak $L\log L$-type estimates for these commutator operators. Furthermore, we will prove endpoint estimates of commutators generated by $BMO(\mathbb{R}^n)$ functions and intrinsic square functions in Morrey type spaces. In particular, we can obtain endpoint estimates of these commutators in the weighted Morrey spaces $L^{1,\kappa}(w)$ for $0<\kappa<1$ and $w \in A_1$, and in the generalized Morrey spaces $L^{1,\Theta}$, where Θ is a growth function on $(0,+\infty)$ satisfying the doubling condition.

1. Introduction and main results

The intrinsic square functions were first introduced by Wilson in [28, 29]; they are defined as follows. For $0<\alpha \leqslant 1$, let $\mathscr{C}_\alpha$ be the family of functions φ defined on $\mathbb{R}^n$ such that φ has support containing in $\{x \in \mathbb{R}^n : |x| \leqslant 1\}$, $\int_{\mathbb{R}^n} \varphi(x)\,dx = 0$, and for all $x, x' \in \mathbb{R}^n$,

$$\big|\varphi(x)-\varphi(x')\big| \leqslant \big|x-x'\big|^{\alpha}.$$

For $(y,t) \in \mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,+\infty)$ and $f \in L^1_{loc}(\mathbb{R}^n)$, we set

$$A_\alpha(f)(y,t) = \sup_{\varphi \in \mathscr{C}_\alpha} \big|f * \varphi_t(y)\big| = \sup_{\varphi \in \mathscr{C}_\alpha} \left| \int_{\mathbb{R}^n} \varphi_t(y-z) f(z)\,dz \right|, \tag{1.1}$$

where $\varphi_t(x) = t^{-n}\varphi(x/t)$. Then we define the intrinsic square function of f (of order α) by the formula

$$\mathscr{S}_\alpha(f)(x) = \left(\iint_{\Gamma(x)} \Big(A_\alpha(f)(y,t)\Big)^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \tag{1.2}$$

where $\Gamma(x)$ denotes the usual cone of aperture one:

$$\Gamma(x) = \Big\{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\Big\}.$$

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Similarly, we can define a cone of aperture β for any $\beta > 0$:

$$\Gamma_\beta(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < \beta \cdot t\},$$

and the corresponding square function

$$\mathscr{S}_{\alpha,\beta}(f)(x) = \left(\iint_{\Gamma_\beta(x)} \Big(A_\alpha(f)(y,t)\Big)^2 \frac{dydt}{t^{n+1}} \right)^{1/2}. \tag{1.3}$$

The intrinsic Littlewood–Paley $\mathscr{G}$-function and the intrinsic $\mathscr{G}^*_\lambda$-function will be given respectively by

$$\mathscr{G}_\alpha(f)(x) = \left(\int_0^\infty \Big(A_\alpha(f)(x,t)\Big)^2 \frac{dt}{t} \right)^{1/2} \tag{1.4}$$

and

$$\mathscr{G}^*_{\lambda,\alpha}(f)(x) = \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \Big(A_\alpha(f)(y,t)\Big)^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \quad \lambda > 1. \tag{1.5}$$

Let b be a locally integrable function on $\mathbb{R}^n$, in this paper, we will consider the commutators generated by b and intrinsic square functions, which are defined respectively by the following expressions in [23].

$$\big[b,\mathscr{S}_\alpha\big](f)(x) = \left(\iint_{\Gamma(x)} \sup_{\varphi\in\mathscr{C}_\alpha} \left| \int_{\mathbb{R}^n} \big[b(x)-b(z)\big]\varphi_t(y-z)f(z)\,dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \tag{1.6}$$

$$\big[b,\mathscr{G}_\alpha\big](f)(x) = \left(\int_0^\infty \sup_{\varphi\in\mathscr{C}_\alpha} \left| \int_{\mathbb{R}^n} \big[b(x)-b(y)\big]\varphi_t(x-y)f(y)\,dy \right|^2 \frac{dt}{t} \right)^{1/2}, \tag{1.7}$$

and

$$\begin{aligned}&\big[b,\mathscr{G}^*_{\lambda,\alpha}\big](f)(x)\\ &= \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \sup_{\varphi\in\mathscr{C}_\alpha} \left| \int_{\mathbb{R}^n} \big[b(x)-b(z)\big]\varphi_t(y-z)f(z)\,dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \lambda > 1.\end{aligned} \tag{1.8}$$

On the other hand, the classical Morrey spaces $\mathscr{L}^{p,\lambda}$ were originally introduced by Morrey in [14] to study the local behavior of solutions to second order elliptic partial differential equations. Since then, these spaces play an important role in studying the regularity of solutions to partial differential equations. For the boundedness of the Hardy–Littlewood maximal operator, the fractional integral operator and the Calderón–Zygmund singular integral operator on these spaces, we refer the reader to [1, 3, 17]. In [13], Mizuhara introduced the generalized Morrey space $L^{p,\Theta}$ which was later extended and studied by many authors (see [7, 8, 9, 12, 15]). In [11], Komori and Shirai defined the weighted Morrey space $L^{p,\kappa}(w)$ which could be viewed as an extension of weighted Lebesgue space, and then discussed the boundedness of the above classical operators in Harmonic Analysis on the weighted space $L^{p,\kappa}(w)$. Recently, in [23, 24, 25, 26], we have established the strong type and weak type estimates for intrinsic square functions and their commutators on $L^{p,\Theta}$ and $L^{p,\kappa}(w)$ with $1 \leqslant p < \infty$.

In order to simplify the notations, for any given $\sigma > 0$, we set

$$\Phi\left(\frac{|f(x)|}{\sigma}\right) = \frac{|f(x)|}{\sigma} \cdot \left(1 + \log^+ \frac{|f(x)|}{\sigma}\right)$$

when $\Phi(t) = t \cdot (1 + \log^+ t)$. The main results of this paper can be stated as follows. For the endpoint estimates for these commutator operators $\big[b, \mathscr{S}_\alpha\big]$, $\big[b, \mathscr{G}_\alpha\big]$ and $\big[b, \mathscr{G}^*_{\lambda,\alpha}\big]$ in the weighted Lebesgue space $L^1_w(\mathbb{R}^n)$, when $b \in BMO(\mathbb{R}^n)$ and $w \in A_1$, we will show

THEOREM 1.1. *Let $0 < \alpha \leqslant 1$, $w \in A_1$ and $b \in BMO(\mathbb{R}^n)$. Then for any given $\sigma > 0$, there exists a constant $C > 0$ independent of f and σ such that*

$$w\big(\big\{x \in \mathbb{R}^n : \big|[b, \mathscr{S}_\alpha](f)(x)\big| > \sigma\big\}\big) \leqslant C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\sigma}\right) \cdot w(x)\,dx,$$

where $\Phi(t) = t(1 + \log^+ t)$.

THEOREM 1.2. *Let $0 < \alpha \leqslant 1$, $w \in A_1$ and $b \in BMO(\mathbb{R}^n)$. Then for any given $\sigma > 0$, there exists a constant $C > 0$ independent of f and σ such that*

$$w\big(\big\{x \in \mathbb{R}^n : \big|[b, \mathscr{G}_\alpha](f)(x)\big| > \sigma\big\}\big) \leqslant C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\sigma}\right) \cdot w(x)\,dx,$$

where $\Phi(t) = t(1 + \log^+ t)$.

THEOREM 1.3. *Let $0 < \alpha \leqslant 1$, $w \in A_1$ and $b \in BMO(\mathbb{R}^n)$. If $\lambda > (3n + 2\alpha)/n$, then for any given $\sigma > 0$, there exists a constant $C > 0$ independent of f and σ such that*

$$w\big(\big\{x \in \mathbb{R}^n : \big|[b, \mathscr{G}^*_{\lambda,\alpha}](f)(x)\big| > \sigma\big\}\big) \leqslant C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\sigma}\right) \cdot w(x)\,dx,$$

where $\Phi(t) = t(1 + \log^+ t)$.

Let $0 \leqslant \kappa < 1$. Assume that θ is a positive increasing function defined in $(0, +\infty)$ and satisfies the following $\mathscr{D}_\kappa$ condition:

$$\frac{\theta(\xi)}{\xi^\kappa} \leqslant C \frac{\theta(\zeta)}{\zeta^\kappa}, \quad \text{for any } 0 < \zeta < \xi < +\infty, \tag{1.9}$$

where $C > 0$ is a constant independent of ξ and ζ. For the endpoint estimates of commutators generated by $BMO(\mathbb{R}^n)$ functions and intrinsic square functions in the Morrey type spaces associated to θ, we will prove

THEOREM 1.4. *Let $0<\alpha\leqslant 1$, $w\in A_1$ and $b\in BMO(\mathbb{R}^n)$. Assume that θ satisfies the $\mathscr{D}_\kappa$ condition (1.9) with $0\leqslant\kappa<1$, then for any given $\sigma>0$ and any ball B, there exists a constant $C>0$ independent of f, B and σ such that*

$$\frac{1}{\theta(w(B))}w\big(\big\{x\in B:\big|[b,\mathscr{S}_\alpha](f)(x)\big|>\sigma\big\}\big)\leqslant C\sup_B\left\{\frac{1}{\theta(w(B))}\int_B\Phi\left(\frac{|f(x)|}{\sigma}\right)w(x)\,dx\right\},$$

where $\Phi(t)=t(1+\log^+t)$.

THEOREM 1.5. *Let $0<\alpha\leqslant 1$, $w\in A_1$ and $b\in BMO(\mathbb{R}^n)$. Assume that θ satisfies the $\mathscr{D}_\kappa$ condition (1.9) with $0\leqslant\kappa<1$, then for any given $\sigma>0$ and any ball B, there exists a constant $C>0$ independent of f, B and σ such that*

$$\frac{1}{\theta(w(B))}w\big(\big\{x\in B:\big|[b,\mathscr{G}_\alpha](f)(x)\big|>\sigma\big\}\big)\leqslant C\sup_B\left\{\frac{1}{\theta(w(B))}\int_B\Phi\left(\frac{|f(x)|}{\sigma}\right)w(x)\,dx\right\},$$

where $\Phi(t)=t(1+\log^+t)$.

THEOREM 1.6. *Let $0<\alpha\leqslant 1$, $w\in A_1$ and $b\in BMO(\mathbb{R}^n)$. Assume that θ satisfies the $\mathscr{D}_\kappa$ condition (1.9) with $0\leqslant\kappa<1$ and $\lambda>(3n+2\alpha)/n$, then for any given $\sigma>0$ and any ball B, there exists a constant $C>0$ independent of f, B and σ such that*

$$\frac{1}{\theta(w(B))}w\big(\big\{x\in B:\big|[b,\mathscr{G}^*_{\lambda,\alpha}](f)(x)\big|>\sigma\big\}\big)\leqslant C\sup_B\left\{\frac{1}{\theta(w(B))}\int_B\Phi\left(\frac{|f(x)|}{\sigma}\right)w(x)\,dx\right\},$$

where $\Phi(t)=t(1+\log^+t)$.

2. Notations and preliminaries

A weight w will always mean a positive function which is locally integrable on $\mathbb{R}^n$, $B=B(x_0,r_B)=\{x\in\mathbb{R}^n:|x-x_0|<r_B\}$ denotes the open ball centered at x_0 and with radius $r_B>0$. For $1<p<\infty$, a weight function w is said to belong to the Muckenhoupt's class A_p, if there is a constant $C>0$ such that for every ball $B\subseteq\mathbb{R}^n$ (see [6, 16]),

$$\left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-1/(p-1)}\,dx\right)^{p-1}\leqslant C.$$

For the case $p=1$, $w\in A_1$, if there is a constant $C>0$ such that for every ball $B\subseteq\mathbb{R}^n$,

$$\frac{1}{|B|}\int_B w(x)\,dx\leqslant C\cdot\operatorname*{ess\,inf}_{x\in B}w(x).$$

We also define $A_\infty=\cup_{1\leqslant p<\infty}A_p$. It is well known that if $w\in A_p$ with $1\leqslant p<\infty$, then for any ball B, there exists an absolute constant $C>0$ such that

$$w(2B)\leqslant C\,w(B). \tag{2.1}$$

In general, for $w \in A_1$ and any $j \in \mathbb{Z}_+$, there exists an absolute constant $C > 0$ such that (see [6])

$$w(2^j B) \leqslant C \cdot 2^{jn} w(B). \tag{2.2}$$

Moreover, if $w \in A_\infty$, then for all balls B and all measurable subsets E of B, there exists a number $\delta > 0$ independent of E and B such that (see [6])

$$\frac{w(E)}{w(B)} \leqslant C \left(\frac{|E|}{|B|} \right)^{\delta}. \tag{2.3}$$

A weight function w is said to belong to the reverse Hölder class RH_r, if there exist two constants $r > 1$ and $C > 0$ such that the following reverse Hölder inequality holds for every ball $B \subseteq \mathbb{R}^n$.

$$\left(\frac{1}{|B|} \int_B w(x)^r \, dx \right)^{1/r} \leqslant C \left(\frac{1}{|B|} \int_B w(x) \, dx \right).$$

Given a ball B and $\lambda > 0$, λB denotes the ball with the same center as B whose radius is λ times that of B. For a given weight function w and a measurable set E, we also denote the Lebesgue measure of E by $|E|$ and the weighted measure of E by $w(E)$, where $w(E) = \int_E w(x)\, dx$. Equivalently, we could define the above notions with cubes instead of balls. Hence we shall use these two different definitions appropriate to calculations.

Given a weight function w on $\mathbb{R}^n$, for $1 \leqslant p < \infty$, the weighted Lebesgue space $L^p_w(\mathbb{R}^n)$ is defined as the set of all functions f such that

$$\|f\|_{L^p_w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{1/p} < \infty.$$

Let $0 < \kappa < 1$ and w be a weight function on $\mathbb{R}^n$. Then the weighted Morrey space $L^{1,\kappa}(w)$ is defined by (see [11])

$$L^{1,\kappa}(w) = \left\{ f \in L^1_{loc}(w) : \|f\|_{L^{1,\kappa}(w)} = \sup_B \frac{1}{w(B)^\kappa} \int_B |f(x)| w(x) \, dx < \infty \right\},$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

Let $\Theta = \Theta(r)$, $r > 0$, be a growth function, that is, a positive increasing function in $(0, +\infty)$ and satisfy the following doubling condition:

$$\Theta(2r) \leqslant D \cdot \Theta(r), \quad \text{for all } r > 0, \tag{2.4}$$

where $D = D(\Theta) \geqslant 1$ is a doubling constant independent of r. The generalized Morrey space $L^{1,\Theta}(\mathbb{R}^n)$ is defined as the set of all locally integrable functions f for which (see [13])

$$\sup_{r>0; B(x_0,r)} \frac{1}{\Theta(r)} \int_{B(x_0,r)} |f(x)| \, dx < \infty,$$

where the supremum is taken over all balls $B(x_0,r)$ in $\mathbb{R}^n$.

We next recall some basic definitions and facts about Orlicz spaces needed for the proof of the main results. For more information on the subject, one can see [21]. A function Φ is called a Young function if it is continuous, nonnegative, convex and strictly increasing on $[0,+\infty)$ with $\Phi(0)=0$ and $\Phi(t)\to+\infty$ as $t\to+\infty$. We define the Φ-average of a function f over a ball B by means of the following Luxemburg norm:

$$\|f\|_{\Phi,B}=\inf\left\{\sigma>0:\frac{1}{|B|}\int_B\Phi\left(\frac{|f(x)|}{\sigma}\right)dx\leqslant 1\right\}.$$

An equivalent norm that is often useful in calculations is as follows (see [18, 21]):

$$\|f\|_{\Phi,B}\leqslant\inf_{\eta>0}\left\{\eta+\frac{\eta}{|B|}\int_B\Phi\left(\frac{|f(x)|}{\eta}\right)dx\right\}\leqslant 2\|f\|_{\Phi,B}. \tag{2.5}$$

Given a Young function Φ, we use $\bar{\Phi}$ to denote the complementary Young function associated to Φ. Then the following generalized Hölder's inequality holds for any given ball B (see [18, 19]).

$$\frac{1}{|B|}\int_B|f(x)\cdot g(x)|dx\leqslant 2\|f\|_{\Phi,B}\|g\|_{\bar{\Phi},B}.$$

In order to deal with the weighted case, for $w\in A_\infty$, we need to define the weighted Φ-average of a function f over a ball B by means of the weighted Luxemburg norm:

$$\|f\|_{\Phi(w),B}=\inf\left\{\sigma>0:\frac{1}{w(B)}\int_B\Phi\left(\frac{|f(x)|}{\sigma}\right)w(x)\,dx\leqslant 1\right\}.$$

It can be shown that for $w\in A_\infty$ (see [21, 30]),

$$\|f\|_{\Phi(w),B}\approx\inf_{\eta>0}\left\{\eta+\frac{\eta}{w(B)}\int_B\Phi\left(\frac{|f(x)|}{\eta}\right)w(x)\,dx\right\}, \tag{2.6}$$

and

$$\frac{1}{w(B)}\int_B|f(x)\cdot g(x)|w(x)\,dx\leqslant C\|f\|_{\Phi(w),B}\|g\|_{\bar{\Phi}(w),B}.$$

The young function that we are going to use is $\Phi(t)=t(1+\log^+t)$ with its complementary Young function $\bar{\Phi}(t)\approx\exp(t)$. Here by $A\approx B$, we mean that there exists a constant $C>1$ such that $\frac{1}{C}\leqslant\frac{A}{B}\leqslant C$. In the present situation, we denote

$$\|f\|_{L\log L(w),B}=\|f\|_{\Phi(w),B},\qquad \|g\|_{\exp L(w),B}=\|g\|_{\bar{\Phi}(w),B}.$$

By the generalized Hölder's inequality with weight, we have (see [18, 30])

$$\frac{1}{w(B)}\int_B|f(x)\cdot g(x)|w(x)\,dx\leqslant C\|f\|_{L\log L(w),B}\|g\|_{\exp L(w),B}. \tag{2.7}$$

Let us now recall the definition of the space of $BMO(\mathbb{R}^n)$ (Bounded Mean Oscillation) (see [5, 10]). A locally integrable function b is said to be in $BMO(\mathbb{R}^n)$, if

$$\|b\|_*=\sup_B\frac{1}{|B|}\int_B|b(x)-b_B|\,dx<\infty,$$

where b_B stands for the average of b on B, i.e., $b_B = \frac{1}{|B|}\int_B b(y)\,dy$ and the supremum is taken over all balls B in $\mathbb{R}^n$. Modulo constants, the space $BMO(\mathbb{R}^n)$ is a Banach space with respect to the norm $\|\cdot\|_*$. By the John–Nirenberg's inequality, it is not difficult to see that for any $w \in A_\infty$ and any given ball B (see [30]),

$$\big\|b - b_B\big\|_{\exp L(w),B} \leqslant C\|b\|_*. \tag{2.8}$$

Throughout this paper, the letter C always denotes a positive constant independent of the main parameters involved, but it may be different from line to line.

3. Proofs of Theorems 1.1 and 1.2

Given a real-valued function $b \in BMO(\mathbb{R}^n)$, we shall follow the idea developed in [2, 4] and denote $F(\xi) = e^{\xi[b(x)-b(z)]}$, $\xi \in \mathbb{C}$. Then by the analyticity of $F(\xi)$ on $\mathbb{C}$ and the Cauchy integral formula, we get

$$\begin{aligned} b(x) - b(z) = F'(0) &= \frac{1}{2\pi i}\int_{|\xi|=1}\frac{F(\xi)}{\xi^2}\,d\xi \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{e^{i\theta}[b(x)-b(z)]}\cdot e^{-i\theta}\,d\theta. \end{aligned}$$

Thus, for any $\varphi \in \mathscr{C}_\alpha$, $0 < \alpha \leqslant 1$, we obtain

$$\begin{aligned} &\left|\int_{\mathbb{R}^n}\big[b(x) - b(z)\big]\varphi_t(y-z)f(z)\,dz\right| \\ =&\left|\frac{1}{2\pi}\int_0^{2\pi}\left(\int_{\mathbb{R}^n}\varphi_t(y-z)e^{-e^{i\theta}b(z)}f(z)\,dz\right)e^{e^{i\theta}b(x)}\cdot e^{-i\theta}\,d\theta\right| \\ \leqslant&\frac{1}{2\pi}\int_0^{2\pi}\sup_{\varphi\in\mathscr{C}_\alpha}\left|\int_{\mathbb{R}^n}\varphi_t(y-z)e^{-e^{i\theta}b(z)}f(z)\,dz\right|e^{\cos\theta\cdot b(x)}\,d\theta \\ \leqslant&\frac{1}{2\pi}\int_0^{2\pi}A_\alpha\big(e^{-e^{i\theta}b}\cdot f\big)(y,t)\cdot e^{\cos\theta\cdot b(x)}\,d\theta. \end{aligned}$$

So we have

$$\big|\big[b,\mathscr{S}_\alpha\big](f)(x)\big| \leqslant \frac{1}{2\pi}\int_0^{2\pi}\mathscr{S}_\alpha\big(e^{-e^{i\theta}b}\cdot f\big)(x)\cdot e^{\cos\theta\cdot b(x)}\,d\theta,$$

and

$$\big|\big[b,\mathscr{G}_\alpha\big](f)(x)\big| \leqslant \frac{1}{2\pi}\int_0^{2\pi}\mathscr{G}_\alpha\big(e^{-e^{i\theta}b}\cdot f\big)(x)\cdot e^{\cos\theta\cdot b(x)}\,d\theta.$$

Then, by the L^p_w-boundedness of intrinsic square functions (see [29]), and using the same arguments as in [4], we can also show the following:

THEOREM 3.1. *Let $0 < \alpha \leqslant 1$, $1 < p < \infty$ and $w \in A_p$. Then the commutators $\big[b,\mathscr{S}_\alpha\big]$ and $\big[b,\mathscr{G}_\alpha\big]$ are all bounded from $L^p_w(\mathbb{R}^n)$ into itself whenever $b \in BMO(\mathbb{R}^n)$.*

We are now ready to give the proofs of Theorems 1.1 and 1.2, which are based on the Calderón–Zygmund decomposition.

Proofs of Theorems 1.1 *and* 1.2. We will only give the proof of Theorem 1.1 here, since the proof of Theorem 1.2 is similar and easier. Inspired by the work in [19, 20, 30], for any fixed $\sigma > 0$, we apply the Calderón–Zygmund decomposition of f at height σ to obtain a sequence of disjoint non-overlapping dyadic cubes $\{Q_i\}$ such that the following property holds (see [22])

$$\sigma < \frac{1}{|Q_i|}\int_{Q_i} |f(y)|\,dy \leqslant 2^n\cdot\sigma, \tag{3.1}$$

where $Q_i = Q(c_i,\ell_i)$ denotes the cube centered at c_i with side length ℓ_i and all cubes are assumed to have their sides parallel to the coordinate axes. Setting $E = \bigcup_i Q_i$. Now we define two functions g and h as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x \in E^c, \\ \frac{1}{|Q_i|}\int_{Q_i}|f(y)|\,dy & \text{if } x \in Q_i, \end{cases}$$

and

$$h(x) = f(x) - g(x) = \sum_i h_i(x),$$

where $h_i(x) = h(x)\chi_{Q_i}(x)$. Then we have

$$|g(x)| \leqslant C\cdot\sigma, \quad \text{a.e. } x \in \mathbb{R}^n, \tag{3.2}$$

and

$$f(x) = g(x) + h(x). \tag{3.3}$$

Obviously, $\operatorname{supp} h_i \subseteq Q_i$, $\int_{Q_i} h_i(x)\,dx = 0$ and $\|h_i\|_{L^1} \leqslant 2\int_{Q_i}|f(x)|\,dx$ by the above decomposition. Since $\big|[b,\mathscr{S}_\alpha](f)(x)\big| \leqslant \big|[b,\mathscr{S}_\alpha](g)(x)\big| + \big|[b,\mathscr{S}_\alpha](h)(x)\big|$ by (3.3), then we can write

$$\begin{aligned} & w\big(\{x\in\mathbb{R}^n : \big|[b,\mathscr{S}_\alpha](f)(x)\big| > \sigma\}\big) \\ \leqslant\, & w\big(\{x\in\mathbb{R}^n : \big|[b,\mathscr{S}_\alpha](g)(x)\big| > \sigma/2\}\big) + w\big(\{x\in\mathbb{R}^n : \big|[b,\mathscr{S}_\alpha](h)(x)\big| > \sigma/2\}\big) \\ :=\, & I_1 + I_2. \end{aligned}$$

Observe that $w \in A_1 \subset A_2$. Applying Chebyshev's inequality and Theorem 3.1, we obtain

$$I_1 \leqslant \frac{4}{\sigma^2}\cdot\Big\|\big[b,\mathscr{S}_\alpha\big](g)\Big\|^2_{L^2_w} \leqslant \frac{C}{\sigma^2}\cdot\|g\|^2_{L^2_w}.$$

Moreover, by the inequality (3.2) and the A_1 condition, we deduce that

$$\begin{aligned}
\|g\|_{L^2_w}^2 &\leqslant C\cdot\sigma\int_{\mathbb{R}^n}|g(x)|w(x)\,dx\\
&\leqslant C\cdot\sigma\left(\int_{E^c}|f(x)|w(x)\,dx+\int_{\bigcup_i Q_i}|g(x)|w(x)\,dx\right)\\
&\leqslant C\cdot\sigma\left(\int_{\mathbb{R}^n}|f(x)|w(x)\,dx+\sum_i\frac{w(Q_i)}{|Q_i|}\int_{Q_i}|f(y)|\,dy\right)\\
&\leqslant C\cdot\sigma\left(\int_{\mathbb{R}^n}|f(x)|w(x)\,dx+\sum_i\operatorname*{ess\,inf}_{y\in Q_i}w(y)\int_{Q_i}|f(y)|\,dy\right)\\
&\leqslant C\cdot\sigma\left(\int_{\mathbb{R}^n}|f(x)|w(x)\,dx+\int_{\bigcup_i Q_i}|f(y)|w(y)\,dy\right)\\
&\leqslant C\cdot\sigma\int_{\mathbb{R}^n}|f(x)|w(x)\,dx. \qquad (3.4)
\end{aligned}$$

So we have

$$I_1\leqslant C\int_{\mathbb{R}^n}\frac{|f(x)|}{\sigma}\cdot w(x)\,dx\leqslant C\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx.$$

To deal with the other term I_2, let $Q_i^*=2\sqrt{n}Q_i$ be the cube concentric with Q_i such that $\ell(Q_i^*)=(2\sqrt{n})\ell(Q_i)$. Then we can further decompose I_2 as follows.

$$\begin{aligned}
I_2 &\leqslant w\Big(\Big\{x\in\bigcup_i Q_i^*:\Big|\big[b,\mathscr{S}_\alpha\big](h)(x)\Big|>\sigma/2\Big\}\Big)\\
&\quad+w\Big(\Big\{x\notin\bigcup_i Q_i^*:\Big|\big[b,\mathscr{S}_\alpha\big](h)(x)\Big|>\sigma/2\Big\}\Big)\\
&:=I_3+I_4.
\end{aligned}$$

Since $w\in A_1$, then by the inequality (2.1), we can get

$$I_3\leqslant\sum_i w\big(Q_i^*\big)\leqslant C\sum_i w(Q_i).$$

Furthermore, it follows from the inequality (3.1) and the A_1 condition that

$$\begin{aligned}
I_3 &\leqslant C\sum_i\frac{1}{\sigma}\cdot\operatorname*{ess\,inf}_{y\in Q_i}w(y)\int_{Q_i}|f(y)|\,dy\\
&\leqslant\frac{C}{\sigma}\sum_i\int_{Q_i}|f(y)|w(y)\,dy\leqslant\frac{C}{\sigma}\int_{\bigcup_i Q_i}|f(y)|w(y)\,dy\\
&\leqslant C\int_{\mathbb{R}^n}\frac{|f(y)|}{\sigma}\cdot w(y)\,dy\leqslant C\int_{\mathbb{R}^n}\Phi\left(\frac{|f(y)|}{\sigma}\right)\cdot w(y)\,dy.
\end{aligned}$$

For any given $x \in \mathbb{R}^n$ and $(y,t) \in \Gamma(x)$, we have

$$\sup_{\varphi\in\mathscr{C}_\alpha}\left|\int_{\mathbb{R}^n}\big[b(x)-b(z)\big]\varphi_t(y-z)h_i(z)\,dz\right| \leqslant \big|b(x)-b_{Q_i}\big|\cdot\sup_{\varphi\in\mathscr{C}_\alpha}\left|\int_{\mathbb{R}^n}\varphi_t(y-z)h_i(z)\,dz\right| + \sup_{\varphi\in\mathscr{C}_\alpha}\left|\int_{\mathbb{R}^n}\big[b(z)-b_{Q_i}\big]\varphi_t(y-z)h_i(z)\,dz\right|. \tag{3.5}$$

Hence

$$\begin{aligned}\big|\big[b,\mathscr{S}_\alpha\big](h)(x)\big| &\leqslant \sum_i \big|b(x)-b_{Q_i}\big|\cdot\mathscr{S}_\alpha(h_i)(x)\\ &\quad+\left(\iint_{\Gamma(x)}\sup_{\varphi\in\mathscr{C}_\alpha}\left|\int_{\mathbb{R}^n}\big[b(z)-b_{Q_i}\big]\varphi_t(y-z)\cdot\sum_i h_i(z)\,dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\ &=\sum_i\big|b(x)-b_{Q_i}\big|\cdot\mathscr{S}_\alpha(h_i)(x)+\mathscr{S}_\alpha\left(\sum_i[b-b_{Q_i}]h_i\right)(x).\end{aligned}$$

Then we can write

$$\begin{aligned}I_4 \leqslant & w\left(\left\{x\notin\bigcup_i Q_i^*:\sum_i\big|b(x)-b_{Q_i}\big|\cdot\mathscr{S}_\alpha(h_i)(x)>\sigma/4\right\}\right)\\ &+w\left(\left\{x\notin\bigcup_i Q_i^*:\mathscr{S}_\alpha\left(\sum_i[b-b_{Q_i}]h_i\right)(x)>\sigma/4\right\}\right)\\ :=&I_5+I_6.\end{aligned}$$

It follows directly from the Chebyshev's inequality that

$$\begin{aligned}I_5 &\leqslant \frac{4}{\sigma}\int_{\mathbb{R}^n\setminus\cup_i Q_i^*}\left|\sum_i\big|b(x)-b_{Q_i}\big|\cdot\mathscr{S}_\alpha(h_i)(x)\right|w(x)\,dx\\ &\leqslant\frac{4}{\sigma}\sum_i\left(\int_{(Q_i^*)^c}\big|b(x)-b_{Q_i}\big|\cdot\mathscr{S}_\alpha(h_i)(x)w(x)\,dx\right).\end{aligned}$$

Denote the center of Q_i by c_i. For any $\varphi \in \mathscr{C}_\alpha$, $0<\alpha\leqslant 1$, by the cancellation condition of h_i, we obtain that for any $(y,t)\in\Gamma(x)$,

$$\begin{aligned}\big|(h_i*\varphi_t)(y)\big| &= \left|\int_{Q_i}\big[\varphi_t(y-z)-\varphi_t(y-c_i)\big]h_i(z)\,dz\right|\\ &\leqslant\int_{Q_i\cap\{z:|z-y|\leqslant t\}}\frac{|z-c_i|^\alpha}{t^{n+\alpha}}|h_i(z)|\,dz\\ &\leqslant C\cdot\frac{\ell(Q_i)^\alpha}{t^{n+\alpha}}\int_{Q_i\cap\{z:|z-y|\leqslant t\}}|h_i(z)|\,dz.\end{aligned} \tag{3.6}$$

In addition, for any $z\in Q_i$ and $x\in(Q_i^*)^c$, we have $|z-c_i|<\frac{|x-c_i|}{2}$. Thus, for all $(y,t)\in\Gamma(x)$ and $|z-y|\leqslant t$ with $z\in Q_i$, we can see that

$$t+t\geqslant|x-y|+|y-z|\geqslant|x-z|\geqslant|x-c_i|-|z-c_i|\geqslant\frac{|x-c_i|}{2}. \tag{3.7}$$

Hence, for any $x \in (Q_i^*)^c$, by using the above inequalities (3.6) and (3.7), we obtain

$$\begin{aligned}
\big|\mathscr{S}_\alpha(h_i)(x)\big| &= \left(\iint_{\Gamma(x)}\left(\sup_{\varphi\in\mathscr{C}_\alpha}\big|(\varphi_t * h_i)(y)\big|\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\
&\leqslant C\cdot\ell(Q_i)^\alpha\left(\int_{Q_i}|h_i(z)|\,dz\right)\left(\int_{\frac{|x-c_i|}{4}}^\infty\int_{|y-x|<t}\frac{dydt}{t^{2(n+\alpha)+n+1}}\right)^{1/2}\\
&\leqslant C\cdot\ell(Q_i)^\alpha\left(\int_{Q_i}|h_i(z)|\,dz\right)\left(\int_{\frac{|x-c_i|}{4}}^\infty\frac{dt}{t^{2(n+\alpha)+1}}\right)^{1/2}\\
&\leqslant C\cdot\frac{\ell(Q_i)^\alpha}{|x-c_i|^{n+\alpha}}\left(\int_{Q_i}|f(z)|\,dz\right).
\end{aligned}$$

Since $Q_i^* = 2\sqrt{n}Q_i \supset 2Q_i$, then $(Q_i^*)^c \subset (2Q_i)^c$. This fact together with the above pointwise estimate yields

$$\begin{aligned}
I_5 &\leqslant \frac{C}{\sigma}\sum_i\left(\ell(Q_i)^\alpha\int_{Q_i}|f(z)|\,dz\int_{(Q_i^*)^c}\big|b(x)-b_{Q_i}\big|\cdot\frac{w(x)}{|x-c_i|^{n+\alpha}}dx\right)\\
&\leqslant \frac{C}{\sigma}\sum_i\left(\ell(Q_i)^\alpha\int_{Q_i}|f(z)|\,dz\int_{(2Q_i)^c}\big|b(x)-b_{Q_i}\big|\cdot\frac{w(x)}{|x-c_i|^{n+\alpha}}dx\right)\\
&\leqslant \frac{C}{\sigma}\sum_i\left(\ell(Q_i)^\alpha\int_{Q_i}|f(z)|\,dz\sum_{j=1}^\infty\int_{2^{j+1}Q_i\backslash 2^jQ_i}\big|b(x)-b_{2^{j+1}Q_i}\big|\cdot\frac{w(x)}{|x-c_i|^{n+\alpha}}dx\right)\\
&\quad+\frac{C}{\sigma}\sum_i\left(\ell(Q_i)^\alpha\int_{Q_i}|f(z)|\,dz\sum_{j=1}^\infty\int_{2^{j+1}Q_i\backslash 2^jQ_i}\big|b_{2^{j+1}Q_i}-b_{Q_i}\big|\cdot\frac{w(x)}{|x-c_i|^{n+\alpha}}dx\right)\\
&:= I+II.
\end{aligned}$$

For the term I,

$$I \leqslant \frac{C}{\sigma}\sum_i\left(\ell(Q_i)^\alpha\int_{Q_i}|f(z)|\,dz\sum_{j=1}^\infty\frac{1}{[2^{j-1}\ell(Q_i)]^{n+\alpha}}\int_{2^{j+1}Q_i\backslash 2^jQ_i}\big|b(x)-b_{2^{j+1}Q_i}\big|w(x)\,dx\right).$$

Since $w \in A_1$, we know that there exists a number $r > 1$ such that $w \in RH_r$. It then follows from Hölder's inequality, the John–Nirenberg's inequality ([10]) and (2.2) that

$$\begin{aligned}
\int_{2^{j+1}Q_i}\big|b(x)-b_{2^{j+1}Q_i}\big|w(x)\,dx &\leqslant \left(\int_{2^{j+1}Q_i}\big|b(x)-b_{2^{j+1}Q_i}\big|^{r'}dx\right)^{1/r'}\left(\int_{2^{j+1}Q_i}w(x)^r\,dx\right)^{1/r}\\
&\leqslant C\|b\|_*\cdot w\big(2^{j+1}Q_i\big)\\
&\leqslant C\|b\|_*\cdot(2^{j+1})^n w\big(Q_i\big). \qquad (3.8)
\end{aligned}$$

Hence

$$
\begin{aligned}
I &\leqslant \frac{C\cdot\|b\|_*}{\sigma}\sum_i\left(\int_{Q_i}|f(z)|\,dz\sum_{j=1}^{\infty}\frac{(2^{j+1})^n w(Q_i)}{(2^{j-1})^{n+\alpha}|Q_i|}\right)\\
&\leqslant \frac{C}{\sigma}\sum_i\left(\frac{w(Q_i)}{|Q_i|}\cdot\int_{Q_i}|f(z)|\,dz\sum_{j=1}^{\infty}\frac{1}{2^{j\alpha}}\right)\\
&\leqslant \frac{C}{\sigma}\sum_i \operatorname*{ess\,inf}_{z\in Q_i} w(z)\int_{Q_i}|f(z)|\,dz\\
&\leqslant \frac{C}{\sigma}\int_{\bigcup_i Q_i}|f(z)|w(z)\,dz \leqslant C\int_{\mathbb{R}^n}\frac{|f(z)|}{\sigma}\cdot w(z)\,dz\\
&\leqslant C\int_{\mathbb{R}^n}\Phi\left(\frac{|f(z)|}{\sigma}\right)\cdot w(z)\,dz.
\end{aligned}
$$

For the term II, since $b\in BMO(\mathbb{R}^n)$, then a simple calculation shows that

$$
\left|b_{2^{j+1}Q_i}-b_{Q_i}\right|\leqslant C\cdot(j+1)\|b\|_*. \tag{3.9}
$$

This estimate (3.9) together with the inequality (2.2) implies that

$$
\begin{aligned}
II &\leqslant \frac{C\cdot\|b\|_*}{\sigma}\sum_i\left(\ell(Q_i)^{\alpha}\int_{Q_i}|f(z)|\,dz\sum_{j=1}^{\infty}(j+1)\cdot\frac{w(2^{j+1}Q_i)}{[2^{j-1}\ell(Q_i)]^{n+\alpha}}\right)\\
&\leqslant \frac{C\cdot\|b\|_*}{\sigma}\sum_i\left(\int_{Q_i}|f(z)|\,dz\sum_{j=1}^{\infty}(j+1)\cdot\frac{(2^{j+1})^n w(Q_i)}{(2^{j-1})^{n+\alpha}|Q_i|}\right)\\
&\leqslant \frac{C}{\sigma}\sum_i\left(\frac{w(Q_i)}{|Q_i|}\cdot\int_{Q_i}|f(z)|\,dz\sum_{j=1}^{\infty}\frac{(j+1)}{2^{j\alpha}}\right)\\
&\leqslant \frac{C}{\sigma}\sum_i\left(\frac{w(Q_i)}{|Q_i|}\cdot\int_{Q_i}|f(z)|\,dz\right)\leqslant C\int_{\mathbb{R}^n}\Phi\left(\frac{|f(z)|}{\sigma}\right)\cdot w(z)\,dz.
\end{aligned}
$$

On the other hand, by using the weighted weak-type (1,1) estimate of intrinsic square functions (see [29]), we have

$$
\begin{aligned}
I_6 &\leqslant \frac{C}{\sigma}\int_{\mathbb{R}^n}\sum_i\left|b(x)-b_{Q_i}\right|\cdot|h_i(x)|\,w(x)\,dx\\
&= \frac{C}{\sigma}\sum_i\int_{Q_i}\left|b(x)-b_{Q_i}\right|\cdot|h_i(x)|\,w(x)\,dx\\
&\leqslant \frac{C}{\sigma}\sum_i\int_{Q_i}\left|b(x)-b_{Q_i}\right|\cdot|f(x)|\,w(x)\,dx\\
&\quad+\frac{C}{\sigma}\sum_i\frac{1}{|Q_i|}\int_{Q_i}|f(y)|\,dy\int_{Q_i}\left|b(x)-b_{Q_i}\right|w(x)\,dx\\
&:= III+IV.
\end{aligned}
$$

By the generalized Hölder's inequality with weight (2.7), (2.8) and (2.6), we can deduce that

$$\begin{aligned}
III &\leqslant \frac{C}{\sigma}\sum_i w(Q_i)\cdot\frac{1}{w(Q_i)}\int_{Q_i}\big|b(x)-b_{Q_i}\big|\cdot|f(x)|\,w(x)\,dx \\
&\leqslant \frac{C}{\sigma}\sum_i w(Q_i)\cdot\big\|b-b_{Q_i}\big\|_{\exp L(w),Q_i}\big\|f\big\|_{L\log L(w),Q_i} \\
&\leqslant \frac{C\cdot\|b\|_*}{\sigma}\sum_i w(Q_i)\cdot\big\|f\big\|_{L\log L(w),Q_i} \\
&\leqslant \frac{C\cdot\|b\|_*}{\sigma}\sum_i w(Q_i)\cdot\inf_{\eta>0}\left\{\eta+\frac{\eta}{w(Q_i)}\int_{Q_i}\Phi\left(\frac{|f(y)|}{\eta}\right)w(y)\,dy\right\} \\
&\leqslant \frac{C\cdot\|b\|_*}{\sigma}\sum_i w(Q_i)\cdot\left\{\sigma+\frac{\sigma}{w(Q_i)}\int_{Q_i}\Phi\left(\frac{|f(y)|}{\sigma}\right)w(y)\,dy\right\} \\
&\leqslant C\left\{\sum_i w(Q_i)+\sum_i\int_{Q_i}\Phi\left(\frac{|f(y)|}{\sigma}\right)w(y)\,dy\right\} \\
&\leqslant C\int_{\mathbb{R}^n}\Phi\left(\frac{|f(y)|}{\sigma}\right)\cdot w(y)\,dy.
\end{aligned}$$

Arguing as in the proof of (3.8), we find that

$$\begin{aligned}
\int_{Q_i}\big|b(x)-b_{Q_i}\big|w(x)\,dx &\leqslant \left(\int_{Q_i}\big|b(x)-b_{Q_i}\big|^{r'}dx\right)^{1/r'}\left(\int_{Q_i}w(x)^r dx\right)^{1/r} \\
&\leqslant C\|b\|_*\cdot w(Q_i).
\end{aligned}$$

Therefore

$$\begin{aligned}
IV &\leqslant \frac{C}{\sigma}\sum_i\frac{w(Q_i)}{|Q_i|}\int_{Q_i}|f(y)|\,dy \leqslant \frac{C}{\sigma}\sum_i \operatorname*{ess\,inf}_{y\in Q_i} w(y)\int_{Q_i}|f(y)|\,dy \\
&\leqslant \frac{C}{\sigma}\int_{\bigcup_i Q_i}|f(y)|w(y)\,dy \leqslant C\int_{\mathbb{R}^n}\frac{|f(y)|}{\sigma}\cdot w(y)\,dy \\
&\leqslant C\int_{\mathbb{R}^n}\Phi\left(\frac{|f(y)|}{\sigma}\right)\cdot w(y)\,dy.
\end{aligned}$$

Summing up all the above estimates, we get the desired result. □

4. Proof of Theorem 1.3

In order to prove the main theorem of this section, we will need the following estimates which were established by the author in [27].

PROPOSITION 4.1. *Let $w\in A_1$ and $0<\alpha\leqslant 1$. Then for any $j\in\mathbb{Z}_+$, we have*

$$\big\|\mathscr{S}_{\alpha,2^j}(f)\big\|_{L^2_w}\leqslant C\cdot 2^{jn/2}\big\|\mathscr{S}_\alpha(f)\big\|_{L^2_w}.$$

PROPOSITION 4.2. *Let $w \in A_1$, $0 < \alpha \leqslant 1$ and $2 < q < \infty$. Then for any $j \in \mathbb{Z}_+$, we have*

$$\big\|\mathscr{S}_{\alpha,2^j}(f)\big\|_{L^q_w} \leqslant C \cdot 2^{jn/2}\big\|\mathscr{S}_\alpha(f)\big\|_{L^q_w}.$$

PROPOSITION 4.3. *Let $w \in A_1$, $0 < \alpha \leqslant 1$ and $1 < q < 2$. Then for any $j \in \mathbb{Z}_+$, we have*

$$\big\|\mathscr{S}_{\alpha,2^j}(f)\big\|_{L^q_w} \leqslant C \cdot 2^{jn/q}\big\|\mathscr{S}_\alpha(f)\big\|_{L^q_w}.$$

Moreover, from the definition of $\mathscr{G}^*_{\lambda,\alpha}$ ($\lambda > 3$), we readily see that

$$\begin{aligned}
\Big|\mathscr{G}^*_{\lambda,\alpha}(f)(x)\Big|^2 &= \iint_{\mathbb{R}^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\Big(A_\alpha(f)(y,t)\Big)^2\frac{dydt}{t^{n+1}}\\
&= \int_0^\infty\int_{|x-y|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\Big(A_\alpha(f)(y,t)\Big)^2\frac{dydt}{t^{n+1}}\\
&\quad+\sum_{j=1}^\infty\int_0^\infty\int_{2^{j-1}t\leqslant|x-y|<2^jt}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\Big(A_\alpha(f)(y,t)\Big)^2\frac{dydt}{t^{n+1}}\\
&\leqslant C\left[\mathscr{S}_\alpha(f)(x)^2+\sum_{j=1}^\infty 2^{-j\lambda n}\mathscr{S}_{\alpha,2^j}(f)(x)^2\right]. \qquad (4.1)
\end{aligned}$$

Thus, by applying Propositions 4.1–4.3, the L^q_w-boundedness of $\mathscr{S}_\alpha$ (see [29]) and the above inequality (4.1), we obtain that for $1 < q < \infty$ and $w \in A_1$,

$$\begin{aligned}
\big\|\mathscr{G}^*_{\lambda,\alpha}(f)\big\|_{L^q_w} &\leqslant C\left(\big\|\mathscr{S}_\alpha(f)\big\|_{L^q_w}+\sum_{j=1}^\infty 2^{-\frac{j\lambda n}{2}}\big\|\mathscr{S}_{\alpha,2^j}(f)\big\|_{L^q_w}\right)\\
&\leqslant C\left(\big\|\mathscr{S}_\alpha(f)\big\|_{L^q_w}+\sum_{j=1}^\infty 2^{-\frac{j\lambda n}{2}}\cdot\big[2^{\frac{jn}{2}}+2^{\frac{jn}{q}}\big]\big\|\mathscr{S}_\alpha(f)\big\|_{L^q_w}\right)\\
&\leqslant C\big\|f\big\|_{L^q_w}\left(1+\sum_{j=1}^\infty 2^{-\frac{j\lambda n}{2}}\cdot\big[2^{\frac{jn}{2}}+2^{\frac{jn}{q}}\big]\right)\\
&\leqslant C\big\|f\big\|_{L^q_w}, \qquad (4.2)
\end{aligned}$$

where the last inequality holds under the assumption $\lambda > 3 > \max\{1, 2/q\}$ when $1 < q < \infty$. In addition, for a given real-valued function $b \in BMO(\mathbb{R}^n)$, as before, we can also prove that

$$\big|\big[b,\mathscr{G}^*_{\lambda,\alpha}\big](f)(x)\big| \leqslant \frac{1}{2\pi}\int_0^{2\pi}\mathscr{G}^*_{\lambda,\alpha}\big(e^{-e^{i\theta}b}\cdot f\big)(x)\cdot e^{\cos\theta\cdot b(x)}\,d\theta. \qquad (4.3)$$

Taking into account the inequalities (4.2) and (4.3), and following along the same arguments used in [4], we can also show the following:

THEOREM 4.1. *Let $0 < \alpha \leqslant 1$, $1 < q < \infty$ and $w \in A_1$. Suppose that $\lambda > 3$, then the commutator $\big[b,\mathscr{G}^*_{\lambda,\alpha}\big]$ is bounded from $L^q_w(\mathbb{R}^n)$ into itself whenever $b \in BMO(\mathbb{R}^n)$.*

In [24], we have established the weighted weak-type (1,1) estimate of $\mathscr{G}^*_{\lambda,\alpha}$ on $L^1_w(\mathbb{R}^n)$. More specifically, we obtained

THEOREM 4.2. *Let* $0<\alpha\leqslant 1$*,* $w\in A_1$ *and* $\lambda>(3n+2\alpha)/n$*. Then for any given* $\sigma>0$*, there exists a constant* $C>0$ *independent of* f *and* σ *such that*

$$w\Big(\Big\{x\in\mathbb{R}^n:\big|\mathscr{G}^*_{\lambda,\alpha}(f)(x)\big|>\sigma\Big\}\Big)\leqslant\frac{C}{\sigma}\int_{\mathbb{R}^n}|f(x)|w(x)\,dx.$$

Proof of Theorem 1.3. For any fixed $\sigma>0$, as before, we again perform the Calderón–Zygmund decomposition of f at the level σ to obtain a sequence of disjoint non-overlapping dyadic cubes $\{Q_i\}$ such that the following property holds (see [22])

$$\sigma<\frac{1}{|Q_i|}\int_{Q_i}|f(y)|\,dy\leqslant 2^n\sigma. \tag{4.4}$$

Setting $E=\bigcup_i Q_i$. Now we decompose $f(x)=g(x)+h(x)$, where $g(x)=f(x)$ when $x\in E^c$, and $g(x)=\frac{1}{|Q_i|}\int_{Q_i}|f(y)|\,dy$ when $x\in Q_i$. Then

$$h(x)=f(x)-g(x)=\sum_i h_i(x),$$

with $h_i(x)=h(x)\chi_{Q_i}(x)$. Clearly, by the above decomposition, we get $\operatorname{supp} h_i\subseteq Q_i$, $\int_{Q_i}h_i(x)\,dx=0$ and $\|h_i\|_{L^1}\leqslant 2\int_{Q_i}|f(x)|\,dx$. Note that

$$\big|\big[b,\mathscr{G}^*_{\lambda,\alpha}\big](f)(x)\big|\leqslant\big|\big[b,\mathscr{G}^*_{\lambda,\alpha}\big](g)(x)\big|+\big|\big[b,\mathscr{G}^*_{\lambda,\alpha}\big](h)(x)\big|,$$

then we have

$$\begin{aligned}
&w\big(\big\{x\in\mathbb{R}^n:\big|[b,\mathscr{G}^*_{\lambda,\alpha}](f)(x)\big|>\sigma\big\}\big)\\
\leqslant& w\big(\big\{x\in\mathbb{R}^n:\big|[b,\mathscr{G}^*_{\lambda,\alpha}](g)(x)\big|>\sigma/2\big\}\big)+w\big(\big\{x\in\mathbb{R}^n:\big|[b,\mathscr{G}^*_{\lambda,\alpha}](h)(x)\big|>\sigma/2\big\}\big)\\
:=&J_1+J_2.
\end{aligned}$$

Let us start with the term J_1. By using Chebyshev's inequality, Theorem 4.1 and the inequality (3.4), we obtain

$$\begin{aligned}
J_1&\leqslant\frac{4}{\sigma^2}\cdot\Big\|\big[b,\mathscr{G}^*_{\lambda,\alpha}\big](g)\Big\|^2_{L^2_w}\leqslant\frac{C}{\sigma^2}\cdot\|g\|^2_{L^2_w}\\
&\leqslant\frac{C}{\sigma^2}\cdot\sigma\int_{\mathbb{R}^n}|f(x)|w(x)\,dx\\
&\leqslant C\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx.
\end{aligned}$$

To estimate the other term J_2, as before, we also let $Q_i^*=2\sqrt{n}Q_i$ be the cube concentric with Q_i such that $\ell(Q_i^*)=(2\sqrt{n})\ell(Q_i)$. Then we can further split J_2 into two parts as follows.

$$
\begin{aligned}
J_2 \leqslant & w\Big(\Big\{x\in \bigcup_i Q_i^*: \Big|\big[b,\mathscr{G}^*_{\lambda,\alpha}\big](h)(x)\Big| > \sigma/2\Big\}\Big) \\
& + w\Big(\Big\{x\notin \bigcup_i Q_i^*: \Big|\big[b,\mathscr{G}^*_{\lambda,\alpha}\big](h)(x)\Big| > \sigma/2\Big\}\Big) \\
:= & J_3 + J_4.
\end{aligned}
$$

The part of the argument involving J_3 proceeds as in Theorem 1.1,

$$
J_3 \leqslant \sum_i w\big(Q_i^*\big) \leqslant C\sum_i w(Q_i) \leqslant C\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx.
$$

By the previous estimate (3.5), we thus obtain

$$
\begin{aligned}
\big|\big[b,\mathscr{G}^*_{\lambda,\alpha}\big](h)(x)\big| \leqslant & \sum_i \big|b(x)-b_{Q_i}\big|\cdot \mathscr{G}^*_{\lambda,\alpha}(h_i)(x) \\
& + \left(\iint_{\Gamma(x)} \sup_{\varphi\in\mathscr{C}_\alpha}\left|\int_{\mathbb{R}^n}\big[b(z)-b_{Q_i}\big]\varphi_t(y-z)\cdot\sum_i h_i(z)\,dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\
= & \sum_i \big|b(x)-b_{Q_i}\big|\cdot \mathscr{G}^*_{\lambda,\alpha}(h_i)(x) + \mathscr{G}^*_{\lambda,\alpha}\left(\sum_i [b-b_{Q_i}]h_i\right)(x).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
J_4 \leqslant & w\left(\left\{x\notin \bigcup_i Q_i^*: \sum_i \big|b(x)-b_{Q_i}\big|\cdot\mathscr{G}^*_{\lambda,\alpha}(h_i)(x) > \sigma/4\right\}\right) \\
& + w\left(\left\{x\notin \bigcup_i Q_i^*: \mathscr{G}^*_{\lambda,\alpha}\left(\sum_i [b-b_{Q_i}]h_i\right)(x) > \sigma/4\right\}\right) \\
:= & J_5 + J_6.
\end{aligned}
$$

It follows directly from the Chebyshev's inequality that

$$
\begin{aligned}
J_5 & \leqslant \frac{4}{\sigma}\int_{\mathbb{R}^n\setminus\bigcup_i Q_i^*}\left|\sum_i \big|b(x)-b_{Q_i}\big|\cdot\mathscr{G}^*_{\lambda,\alpha}(h_i)(x)\right| w(x)\,dx \\
& \leqslant \frac{4}{\sigma}\sum_i\left(\int_{(Q_i^*)^c}\big|b(x)-b_{Q_i}\big|\cdot\mathscr{G}^*_{\lambda,\alpha}(h_i)(x)\,w(x)\,dx\right).
\end{aligned}
$$

We also denote the center of Q_i by c_i. In the proof of Theorem 1.1, we have already shown that

$$
\big|\mathscr{S}_\alpha(h_i)(x)\big| \leqslant C\cdot\frac{\ell(Q_i)^\alpha}{|x-c_i|^{n+\alpha}}\left(\int_{Q_i}|f(z)|\,dz\right). \tag{4.5}
$$

Below we will give the pointwise estimates of $\big|\mathscr{S}_{\alpha,2^j}(h_i)(x)\big|$ for $j=1,2,\dots$. Notice that for any $z\in Q_i$ and $x\in(Q_i^*)^c$, we get $|z-c_i|<\frac{|x-c_i|}{2}$. Thus, for all $(y,t)\in\Gamma_{2^j}(x)$ and $|z-y|\leqslant t$ with $z\in Q_i$, we can deduce that

$$
t+2^jt \geqslant |x-y|+|y-z| \geqslant |x-z| \geqslant |x-c_i|-|z-c_i| \geqslant \frac{|x-c_i|}{2}. \tag{4.6}
$$

Hence, for any $x \in (Q_i^*)^c$, by the inequalities (3.6) and (4.6), we obtain that for $j = 1,2,\dots,$

$$\begin{aligned}\left|\mathscr{S}_{\alpha,2^j}(h_i)(x)\right| &= \left(\iint_{\Gamma_{2^j}(x)}\left(\sup_{\varphi\in\mathscr{C}_\alpha}\left|(\varphi_t * h_i)(y)\right|\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\ &\leqslant C\cdot\ell(Q_i)^\alpha\left(\int_{Q_i}|h_i(z)|\,dz\right)\left(\int_{\frac{|x-c_i|}{2^{j+2}}}^\infty\int_{|y-x|<2^jt}\frac{dydt}{t^{2(n+\alpha)+n+1}}\right)^{1/2}\\ &\leqslant C\cdot 2^{jn/2}\ell(Q_i)^\alpha\left(\int_{Q_i}|h_i(z)|\,dz\right)\left(\int_{\frac{|x-c_i|}{2^{j+2}}}^\infty\frac{dt}{t^{2(n+\alpha)+1}}\right)^{1/2}\\ &\leqslant C\cdot 2^{j(3n+2\alpha)/2}\frac{\ell(Q_i)^\alpha}{|x-c_i|^{n+\alpha}}\left(\int_{Q_i}|f(z)|\,dz\right).\end{aligned}$$

Therefore, by using the pointwise estimate we just derived above and the inequality (4.1),

$$\begin{aligned}\left|\mathscr{G}^*_{\lambda,\alpha}(h_i)(x)\right| &\leqslant C\left[\left|\mathscr{S}_\alpha(h_i)(x)\right| + \sum_{j=1}^\infty 2^{-j\lambda n/2}\left|\mathscr{S}_{\alpha,2^j}(h_i)(x)\right|\right]\\ &\leqslant C\cdot\frac{\ell(Q_i)^\alpha}{|x-c_i|^{n+\alpha}}\left(\int_{Q_i}|f(z)|\,dz\right)\left(1+\sum_{j=1}^\infty 2^{-j\lambda n/2}\cdot 2^{j(3n+2\alpha)/2}\right)\\ &\leqslant C\cdot\frac{\ell(Q_i)^\alpha}{|x-c_i|^{n+\alpha}}\left(\int_{Q_i}|f(z)|\,dz\right),\end{aligned}$$

where the last inequality is due to our assumption $\lambda > (3n+2\alpha)/n$. Consequently,

$$J_5 \leqslant \frac{C}{\sigma}\sum_i\left(\ell(Q_i)^\alpha\int_{Q_i}|f(z)|\,dz\int_{(Q_i^*)^c}\left|b(x)-b_{Q_i}\right|\cdot\frac{w(x)}{|x-c_i|^{n+\alpha}}dx\right).$$

Following along the same lines as in Theorem 1.1, we can also show

$$J_5 \leqslant C\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx.$$

On the other hand, by using the weighted weak-type (1,1) estimate of $\mathscr{G}^*_{\lambda,\alpha}$ (see Theorem 4.2), we have

$$J_6 \leqslant \frac{C}{\sigma}\int_{\mathbb{R}^n}\sum_i\left|b(x)-b_{Q_i}\right|\cdot|h_i(x)|\,w(x)\,dx.$$

The rest of the proof is exactly the same as that of Theorem 1.1, and we finally obtain

$$J_6 \leqslant C\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx.$$

Collecting all these estimates, we get the desired estimate. □

5. Proofs of Theorems 1.4, 1.5 and 1.6

Proofs of Theorems 1.4 *and* 1.5. We will only give the proof of Theorem 1.4 here, because the proof of Theorem 1.5 is essentially the same. Fix a ball $B=B(x_0,r_B)\subseteq\mathbb{R}^n$ and decompose $f=f_1+f_2$, where $f_1=f\cdot\chi_{2B}$, χ_{2B} denotes the characteristic function of $2B=B(x_0,2r_B)$. For any $0\leqslant\kappa<1$, $w\in A_1$ and any given $\sigma>0$, we then write

$$\begin{aligned}
&\frac{1}{\theta(w(B))}\cdot w\big(\big\{x\in B:\big|[b,\mathscr{S}_\alpha](f)(x)\big|>\sigma\big\}\big)\\
\leqslant&\frac{1}{\theta(w(B))}\cdot w\big(\big\{x\in B:\big|[b,\mathscr{S}_\alpha](f_1)(x)\big|>\sigma/2\big\}\big)\\
&+\frac{1}{\theta(w(B))}\cdot w\big(\big\{x\in B:\big|[b,\mathscr{S}_\alpha](f_2)(x)\big|>\sigma/2\big\}\big)\\
:=&I_1+I_2.
\end{aligned}$$

By using Theorem 1.1, we get

$$\begin{aligned}
I_1&\leqslant C\cdot\frac{1}{\theta(w(B))}\int_{\mathbb{R}^n}\Phi\left(\frac{|f_1(x)|}{\sigma}\right)\cdot w(x)\,dx\\
&=C\cdot\frac{1}{\theta(w(B))}\int_{2B}\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx\\
&=C\cdot\frac{\theta(w(2B))}{\theta(w(B))}\cdot\frac{1}{\theta(w(2B))}\int_{2B}\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx.
\end{aligned}$$

Moreover, since $0<w(B)<w(2B)<+\infty$ when $w\in A_1$, then by the $\mathscr{D}_\kappa$ condition (1.9) of θ and the inequality (2.1), we have

$$\begin{aligned}
I_1&\leqslant C\cdot\frac{w(2B)^\kappa}{w(B)^\kappa}\cdot\frac{1}{\theta(w(2B))}\int_{2B}\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx\\
&\leqslant C\cdot\sup_B\left\{\frac{1}{\theta(w(B))}\int_B\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx\right\}.
\end{aligned}$$

For any $x\in B$, we can easily check that

$$\big|\big[b,\mathscr{S}_\alpha\big](f_2)(x)\big|\leqslant\big|b(x)-b_B\big|\cdot\mathscr{S}_\alpha(f_2)(x)+\mathscr{S}_\alpha\Big([b-b_B]f_2\Big)(x).$$

So we have

$$\begin{aligned}
I_2\leqslant&\frac{1}{\theta(w(B))}\cdot w\big(\big\{x\in B:\big|b(x)-b_B\big|\cdot\mathscr{S}_\alpha(f_2)(x)>\sigma/4\big\}\big)\\
&+\frac{1}{\theta(w(B))}\cdot w\big(\big\{x\in B:\big|\mathscr{S}_\alpha\Big([b-b_B]f_2\Big)(x)\big|>\sigma/4\big\}\big)\\
:=&I_3+I_4.
\end{aligned}$$

For the term I_3, for all $0<\alpha\leqslant 1$ and $x\in B$, it was proved by the author [23] that

$$\big|\mathscr{S}_\alpha(f_2)(x)\big| \leqslant C\sum_{j=1}^\infty \frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f(z)|\,dz. \tag{5.1}$$

Since $w\in A_1$, then there exists a number $r>1$ such that $w\in RH_r$. Hence, by using the above pointwise estimate (5.1), Chebyshev's inequality together with Hölder's inequality and John–Nirenberg's inequality (see [10]), we conclude that

$$\begin{aligned}
I_3 &\leqslant \frac{1}{\theta(w(B))}\cdot\frac{4}{\sigma}\int_B\big|b(x)-b_B\big|\cdot\mathscr{S}_\alpha(f_2)(x)w(x)\,dx\\
&\leqslant C\sum_{j=1}^\infty\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\frac{|f(z)|}{\sigma}\,dz\\
&\quad\times\frac{1}{\theta(w(B))}\cdot\left(\int_B\big|b(x)-b_B\big|^{r'}dx\right)^{1/r'}\left(\int_B w(x)^r\,dx\right)^{1/r}\\
&\leqslant C\sum_{j=1}^\infty\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\frac{|f(z)|}{\sigma}\,dz\frac{w(B)}{\theta(w(B))}.
\end{aligned}$$

It then follows from the A_1 condition that

$$\begin{aligned}
I_3 &\leqslant C\sum_{j=1}^\infty\frac{1}{w(2^{j+1}B)}\int_{2^{j+1}B}\frac{|f(z)|}{\sigma}\cdot w(z)\,dz\frac{w(B)}{\theta(w(B))}\\
&\leqslant C\cdot\sup_B\left\{\frac{1}{\theta(w(B))}\int_B\Phi\left(\frac{|f(z)|}{\sigma}\right)\cdot w(z)\,dz\right\}\sum_{j=1}^\infty\frac{\theta(w(2^{j+1}B))}{\theta(w(B))}\cdot\frac{w(B)}{w(2^{j+1}B)}.
\end{aligned}$$

Note that $w\in A_1\subset A_\infty$, by using the $\mathscr{D}_\kappa$ condition (1.9) of θ again, the inequality (2.3) and the fact that $0\leqslant\kappa<1$, we find that

$$\begin{aligned}
\sum_{j=1}^\infty\frac{\theta(w(2^{j+1}B))}{\theta(w(B))}\cdot\frac{w(B)}{w(2^{j+1}B)} &\leqslant C\sum_{j=1}^\infty\frac{w(B)^{1-\kappa}}{w(2^{j+1}B)^{1-\kappa}}\\
&\leqslant C\sum_{j=1}^\infty\left(\frac{|B|}{|2^{j+1}B|}\right)^{\delta(1-\kappa)}\\
&\leqslant C\sum_{j=1}^\infty\left(\frac{1}{2^{jn}}\right)^{\delta(1-\kappa)}\leqslant C,
\end{aligned}\tag{5.2}$$

Substituting the above inequality (5.2) into the term I_3, we thus obtain

$$I_3\leqslant C\cdot\sup_B\left\{\frac{1}{\theta(w(B))}\int_B\Phi\left(\frac{|f(z)|}{\sigma}\right)\cdot w(z)\,dz\right\}.$$

Similar to the proof of (5.1), for all $0<\alpha\leqslant 1$ and all $x\in B$, we can show the following pointwise estimate as well.

$$\Big|\mathscr{S}_\alpha\Big([b-b_B]f_2\Big)(x)\Big|\leqslant C\sum_{j=1}^\infty\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\big|b(z)-b_B\big|\cdot\big|f(z)\big|\,dz. \tag{5.3}$$

Applying the above pointwise estimate (5.3) and Chebyshev's inequality, we have

$$\begin{aligned}
I_4 &\leqslant \frac{1}{\theta(w(B))}\cdot\frac{4}{\sigma}\int_B\Big|\mathscr{S}_\alpha\Big([b-b_B]f_2\Big)(x)\Big|w(x)\,dx\\
&\leqslant \frac{w(B)}{\theta(w(B))}\cdot\frac{C}{\sigma}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\big|b(z)-b_B\big|\cdot\big|f(z)\big|\,dz\\
&\leqslant \frac{w(B)}{\theta(w(B))}\cdot\frac{C}{\sigma}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\big|b(z)-b_{2^{j+1}B}\big|\cdot\big|f(z)\big|\,dz\\
&\quad+\frac{w(B)}{\theta(w(B))}\cdot\frac{C}{\sigma}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\big|b_{2^{j+1}B}-b_B\big|\cdot\big|f(z)\big|\,dz\\
&:=I_5+I_6.
\end{aligned}$$

For the term I_5, observe that for any $a,b>0$, $\Phi(a\cdot b)\leqslant\Phi(a)\cdot\Phi(b)$ when $\Phi(t)=t(1+\log^+t)$. We then use the generalized Hölder's inequality with weight (2.7), (2.8) and (2.6) together with (5.2) to obtain

$$\begin{aligned}
I_5 &\leqslant \frac{C}{\sigma}\cdot\frac{w(B)}{\theta(w(B))}\sum_{j=1}^{\infty}\frac{1}{w(2^{j+1}B)}\int_{2^{j+1}B}\big|b(z)-b_{2^{j+1}B}\big|\cdot\big|f(z)\big|w(z)\,dz\\
&\leqslant \frac{C}{\sigma}\cdot\frac{w(B)}{\theta(w(B))}\sum_{j=1}^{\infty}\big\|b-b_{2^{j+1}B}\big\|_{\exp L(w),2^{j+1}B}\big\|f\big\|_{L\log L(w),2^{j+1}B}\\
&\leqslant \frac{C\|b\|_*}{\sigma}\cdot\frac{w(B)}{\theta(w(B))}\sum_{j=1}^{\infty}\inf_{\eta>0}\left\{\eta+\frac{\eta}{w(2^{j+1}B)}\int_{2^{j+1}B}\Phi\left(\frac{|f(z)|}{\eta}\right)w(z)\,dz\right\}\\
&\leqslant \frac{C\|b\|_*}{\sigma}\cdot\frac{w(B)}{\theta(w(B))}\sum_{j=1}^{\infty}\left\{\sigma\cdot\frac{\theta(w(2^{j+1}B))}{w(2^{j+1}B)}+\frac{\sigma}{w(2^{j+1}B)}\int_{2^{j+1}B}\Phi\left(\frac{|f(z)|}{\sigma}\right)w(z)\,dz\right\}\\
&\leqslant C\|b\|_*\cdot\left[1+\sup_B\left\{\frac{1}{\theta(w(B))}\int_B\Phi\left(\frac{|f(z)|}{\sigma}\right)w(z)\,dz\right\}\right]\sum_{j=1}^{\infty}\frac{\theta(w(2^{j+1}B))}{\theta(w(B))}\cdot\frac{w(B)}{w(2^{j+1}B)}\\
&\leqslant C\cdot\sup_B\left\{\frac{1}{\theta(w(B))}\int_B\Phi\left(\frac{|f(z)|}{\sigma}\right)w(z)\,dz\right\}.
\end{aligned}$$

For the last term I_6 we proceed as follows. An application of the inequality (3.9) leads to that

$$\begin{aligned}
I_6 &\leqslant C\cdot\frac{w(B)}{\theta(w(B))}\sum_{j=1}^{\infty}(j+1)\|b\|_*\cdot\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\frac{|f(z)|}{\sigma}\,dz\\
&\leqslant C\cdot\frac{w(B)}{\theta(w(B))}\sum_{j=1}^{\infty}(j+1)\|b\|_*\cdot\frac{1}{w(2^{j+1}B)}\int_{2^{j+1}B}\frac{|f(z)|}{\sigma}\cdot w(z)\,dz\\
&\leqslant C\cdot\sup_B\left\{\frac{1}{\theta(w(B))}\int_B\Phi\left(\frac{|f(z)|}{\sigma}\right)\cdot w(z)\,dz\right\}\\
&\quad\times\sum_{j=1}^{\infty}(j+1)\cdot\frac{\theta(w(2^{j+1}B))}{\theta(w(B))}\cdot\frac{w(B)}{w(2^{j+1}B)}.
\end{aligned}$$

Notice that $w \in A_1 \subset A_\infty$, by using the $\mathscr{D}_\kappa$ condition (1.9) of θ and the inequality (2.3) again together with the fact that $0 \leqslant \kappa < 1$, we thus have

$$\begin{aligned}\sum_{j=1}^{\infty}(j+1)\cdot\frac{\theta(w(2^{j+1}B))}{\theta(w(B))}\cdot\frac{w(B)}{w(2^{j+1}B)} &\leqslant C\sum_{j=1}^{\infty}(j+1)\cdot\frac{w(B)^{1-\kappa}}{w(2^{j+1}B)^{1-\kappa}}\\ &\leqslant C\sum_{j=1}^{\infty}(j+1)\cdot\left(\frac{|B|}{|2^{j+1}B|}\right)^{\delta(1-\kappa)}\\ &\leqslant C\sum_{j=1}^{\infty}(j+1)\cdot\left(\frac{1}{2^{(j+1)n}}\right)^{\delta(1-\kappa)}\leqslant C, \quad (5.4)\end{aligned}$$

which in turn gives that

$$I_6 \leqslant C\cdot\sup_B\left\{\frac{1}{\theta(w(B))}\int_B\Phi\left(\frac{|f(z)|}{\sigma}\right)\cdot w(z)\,dz\right\}.$$

Summarizing the above discussions, we obtain the conclusion of the main theorem. □

Proof of Theorem 1.6. For any ball $B = B(x_0, r_B) \subseteq \mathbb{R}^n$ with $x_0 \in \mathbb{R}^n$ and $r_B > 0$, we set $f = f_1 + f_2$, where $f_1 = f\cdot\chi_{2B}$. Then for any $0 \leqslant \kappa < 1$, $w \in A_1$ and for each fixed $\sigma > 0$, we have

$$\begin{aligned}&\frac{1}{\theta(w(B))}\cdot w\big(\big\{x\in B: \big|[b,\mathscr{G}^*_{\lambda,\alpha}](f)(x)\big|>\sigma\big\}\big)\\ \leqslant&\frac{1}{\theta(w(B))}\cdot w\big(\big\{x\in B: \big|[b,\mathscr{G}^*_{\lambda,\alpha}](f_1)(x)\big|>\sigma/2\big\}\big)\\ &+\frac{1}{\theta(w(B))}\cdot w\big(\big\{x\in B: \big|[b,\mathscr{G}^*_{\lambda,\alpha}](f_2)(x)\big|>\sigma/2\big\}\big)\\ :=&J_1+J_2.\end{aligned}$$

We consider the term J_1 first. Applying Theorem 1.3, the $\mathscr{D}_\kappa$ condition (1.9) of θ and the inequality (2.1), we conclude that

$$\begin{aligned}J_1 &\leqslant C\cdot\frac{1}{\theta(w(B))}\int_{\mathbb{R}^n}\Phi\left(\frac{|f_1(x)|}{\sigma}\right)\cdot w(x)\,dx\\ &= C\cdot\frac{1}{\theta(w(B))}\int_{2B}\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx\\ &= C\cdot\frac{\theta(w(2B))}{\theta(w(B))}\cdot\frac{1}{\theta(w(2B))}\int_{2B}\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx\\ &\leqslant C\cdot\frac{w(2B)^\kappa}{w(B)^\kappa}\cdot\frac{1}{\theta(w(2B))}\int_{2B}\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx\\ &\leqslant C\cdot\sup_B\left\{\frac{1}{\theta(w(B))}\int_B\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx\right\}.\end{aligned}$$

We now turn our attention to the estimate of J_2. For any $x\in B$, we are able to verify that

$$\big|[b,\mathscr{G}^*_{\lambda,\alpha}](f_2)(x)\big| \leqslant \big|b(x)-b_B\big|\cdot\mathscr{G}^*_{\lambda,\alpha}(f_2)(x)+\mathscr{G}^*_{\lambda,\alpha}\Big([b-b_B]f_2\Big)(x).$$

So we have

$$\begin{aligned}J_2\leqslant&\frac{1}{\theta(w(B))}\cdot w\big(\big\{x\in B:\big|b(x)-b_B\big|\cdot\mathscr{G}^*_{\lambda,\alpha}(f_2)(x)>\sigma/4\big\}\big)\\&+\frac{1}{\theta(w(B))}\cdot w\Big(\Big\{x\in B:\Big|\mathscr{G}^*_{\lambda,\alpha}\Big([b-b_B]f_2\Big)(x)\Big|>\sigma/4\Big\}\Big)\\:=&J_3+J_4.\end{aligned}$$

For the term J_3, for all $0<\alpha\leqslant 1$, $x\in B$ and $j\in\mathbb{Z}_+$, it was also shown by the author [23] that

$$\big|\mathscr{S}_{\alpha,2^j}(f_2)(x)\big|\leqslant C\cdot 2^{3jn/2}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f(z)|\,dz. \tag{5.5}$$

Hence, it follows from the inequalities (5.5), (5.1) and (4.1) that

$$\begin{aligned}\Big|\mathscr{G}^*_{\lambda,\alpha}(f_2)(x)\Big|&\leqslant C\Big[\big|\mathscr{S}_{\alpha}(f_2)(x)\big|+\sum_{j=1}^{\infty}2^{-j\lambda n/2}\big|\mathscr{S}_{\alpha,2^j}(f_2)(x)\big|\Big]\\&\leqslant C\cdot\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f(z)|\,dz\left(1+\sum_{j=1}^{\infty}2^{-j\lambda n/2}\cdot 2^{3jn/2}\right)\\&\leqslant C\cdot\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f(z)|\,dz,\end{aligned}\tag{5.6}$$

where the last inequality is due to our assumption $\lambda>(3n+2\alpha)/n>3$. Hence, we can continue the estimate of J_3 in the same way as in Theorem 1.4, and obtain

$$\begin{aligned}J_3&\leqslant\frac{1}{\theta(w(B))}\cdot\frac{4}{\sigma}\int_B\big|b(x)-b_B\big|\cdot\mathscr{G}^*_{\lambda,\alpha}(f_2)(x)w(x)\,dx\\&\leqslant C\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\frac{|f(z)|}{\sigma}\,dz\frac{1}{\theta(w(B))}\cdot\int_B\big|b(x)-b_B\big|w(x)\,dx\\&\leqslant C\sum_{j=1}^{\infty}\frac{1}{w(2^{j+1}B)}\int_{2^{j+1}B}\frac{|f(z)|}{\sigma}\cdot w(z)\,dz\frac{w(B)}{\theta(w(B))}\\&\leqslant C\cdot\sup_B\left\{\frac{1}{\theta(w(B))}\int_B\Phi\left(\frac{|f(z)|}{\sigma}\right)\cdot w(z)\,dz\right\}\sum_{j=1}^{\infty}\frac{\theta(w(2^{j+1}B))}{\theta(w(B))}\cdot\frac{w(B)}{w(2^{j+1}B)}\\&\leqslant C\cdot\sup_B\left\{\frac{1}{\theta(w(B))}\int_B\Phi\left(\frac{|f(z)|}{\sigma}\right)\cdot w(z)\,dz\right\}.\end{aligned}$$

For the last term J_4, similar to the proof of (5.6), for all $0<\alpha\leqslant 1$, all $x\in B$ and $\lambda>3$, we can show the following pointwise estimate as well.

$$\Big|\mathscr{G}^*_{\lambda,\alpha}\Big([b-b_B]f_2\Big)(x)\Big|\leqslant C\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\big|b(z)-b_B\big|\cdot\big|f(z)\big|\,dz. \tag{5.7}$$

Following the same arguments as in the proof of Theorem 1.4 and using the pointwise estimate (5.7) and Chebyshev's inequality, we have eventually obtained

$$\begin{aligned} J_4 &\leqslant \frac{1}{\theta(w(B))}\cdot\frac{4}{\sigma}\int_B\Big|\mathscr{G}^*_{\lambda,\alpha}\Big([b-b_B]f_2\Big)(x)\Big|w(x)\,dx\\ &\leqslant \frac{w(B)}{\theta(w(B))}\cdot\frac{C}{\sigma}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\big|b(z)-b_B\big|\cdot\big|f(z)\big|\,dz\\ &\leqslant C\cdot\sup_B\left\{\frac{1}{\theta(w(B))}\int_B\Phi\left(\frac{|f(z)|}{\sigma}\right)\cdot w(z)\,dz\right\}. \end{aligned}$$

Summing up all the above estimates, we therefore conclude the proof of the main theorem. □

6. Corollaries

In particular, if we take $\theta(x)=x^\kappa$ with $0<\kappa<1$, then we immediately get the following endpoint estimates of commutators in the weighted Morrey spaces $L^{1,\kappa}(w)$ for all $0<\kappa<1$ and $w\in A_1$.

COROLLARY 6.1. *Let $0<\alpha\leqslant 1$, $0<\kappa<1$, $w\in A_1$ and $b\in BMO(\mathbb{R}^n)$. Then for any given $\sigma>0$ and any ball B, there exists a constant $C>0$ independent of f, B and σ such that*

$$\frac{1}{w(B)^\kappa}\cdot w\big(\big\{x\in B:\big|[b,\mathscr{S}_\alpha](f)(x)\big|>\sigma\big\}\big)\leqslant C\cdot\sup_B\frac{1}{w(B)^\kappa}\int_B\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx,$$

where $\Phi(t)=t(1+\log^+t)$.

COROLLARY 6.2. *Let $0<\alpha\leqslant 1$, $0<\kappa<1$, $w\in A_1$ and $b\in BMO(\mathbb{R}^n)$. Then for any given $\sigma>0$ and any ball B, there exists a constant $C>0$ independent of f, B and σ such that*

$$\frac{1}{w(B)^\kappa}\cdot w\big(\big\{x\in B:\big|[b,\mathscr{G}_\alpha](f)(x)\big|>\sigma\big\}\big)\leqslant C\cdot\sup_B\frac{1}{w(B)^\kappa}\int_B\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx,$$

where $\Phi(t)=t(1+\log^+t)$.

COROLLARY 6.3. *Let $0<\alpha\leqslant 1$, $0<\kappa<1$, $w\in A_1$ and $b\in BMO(\mathbb{R}^n)$. If $\lambda>(3n+2\alpha)/n$, then for any given $\sigma>0$ and any ball B, there exists a constant $C>0$ independent of f, B and σ such that*

$$\frac{1}{w(B)^\kappa}\cdot w\big(\big\{x\in B:\big|[b,\mathscr{G}^*_{\lambda,\alpha}](f)(x)\big|>\sigma\big\}\big)\leqslant C\cdot\sup_B\frac{1}{w(B)^\kappa}\int_B\Phi\left(\frac{|f(x)|}{\sigma}\right)\cdot w(x)\,dx,$$

where $\Phi(t)=t(1+\log^+t)$.

We can also take w to be a constant function, then we immediately get the following unweighted results.

COROLLARY 6.4. *Let $0<\alpha\leqslant 1$ and $b\in BMO(\mathbb{R}^n)$. Assume that θ satisfies the $\mathscr{D}_\kappa$ condition (1.9) with $0\leqslant\kappa<1$, then for any given $\sigma>0$ and any ball B, there exists a constant $C>0$ independent of f, B and σ such that*

$$\frac{1}{\theta(|B|)}\cdot\left|\left\{x\in B:\left|[b,\mathscr{S}_\alpha](f)(x)\right|>\sigma\right\}\right|\leqslant C\cdot\sup_B\left\{\frac{1}{\theta(|B|)}\int_B\Phi\left(\frac{|f(x)|}{\sigma}\right)dx\right\},$$

where $\Phi(t)=t(1+\log^+t)$.

COROLLARY 6.5. *Let $0<\alpha\leqslant 1$ and $b\in BMO(\mathbb{R}^n)$. Assume that θ satisfies the $\mathscr{D}_\kappa$ condition (1.9) with $0\leqslant\kappa<1$, then for any given $\sigma>0$ and any ball B, there exists a constant $C>0$ independent of f, B and σ such that*

$$\frac{1}{\theta(|B|)}\cdot\left|\left\{x\in B:\left|[b,\mathscr{G}_\alpha](f)(x)\right|>\sigma\right\}\right|\leqslant C\cdot\sup_B\left\{\frac{1}{\theta(|B|)}\int_B\Phi\left(\frac{|f(x)|}{\sigma}\right)dx\right\},$$

where $\Phi(t)=t(1+\log^+t)$.

COROLLARY 6.6. *Let $0<\alpha\leqslant 1$ and $b\in BMO(\mathbb{R}^n)$. Assume that θ satisfies the $\mathscr{D}_\kappa$ condition (1.9) with $0\leqslant\kappa<1$ and $\lambda>(3n+2\alpha)/n$, then for any given $\sigma>0$ and any ball B, there exists a constant $C>0$ independent of f, B and σ such that*

$$\frac{1}{\theta(|B|)}\cdot\left|\left\{x\in B:\left|[b,\mathscr{G}^*_{\lambda,\alpha}](f)(x)\right|>\sigma\right\}\right|\leqslant C\cdot\sup_B\left\{\frac{1}{\theta(|B|)}\int_B\Phi\left(\frac{|f(x)|}{\sigma}\right)dx\right\},$$

where $\Phi(t)=t(1+\log^+t)$.

Let $\Theta=\Theta(r)$, $r>0$, be a growth function with doubling constant $D(\Theta):1\leqslant D(\Theta)<2^n$. If for any fixed $x_0\in\mathbb{R}^n$, we set $\theta(|B(x_0,r)|)=\Theta(r)$, then

$$\theta(2^n|B(x_0,r)|)=\theta(|B(x_0,2r)|)=\Theta(2r).$$

For the doubling constant $D(\Theta)$ satisfying $1\leqslant D(\Theta)<2^n$, which means that $D(\Theta)=2^{\kappa\cdot n}$ for some $0\leqslant\kappa<1$, then we are able to verify that θ is an increasing function and satisfies the $\mathscr{D}_\kappa$ condition (1.9) with some $0\leqslant\kappa<1$. Thus, by the above unweighted results (corollaries 6.4–6.6), we can also obtain endpoint estimates of commutators in the generalized Morrey spaces $L^{1,\Theta}$ when Θ satisfies the doubling condition (2.4).

COROLLARY 6.7. *Let $0<\alpha\leqslant 1$ and $b\in BMO(\mathbb{R}^n)$. Suppose that Θ satisfies (2.4) and $1\leqslant D(\Theta)<2^n$, then for any given $\sigma>0$ and any ball $B(x_0,r)$, there exists a constant $C>0$ independent of f, $B(x_0,r)$ and σ such that*

$$\frac{1}{\Theta(r)}\cdot\left|\left\{x\in B(x_0,r):\left|[b,\mathscr{S}_\alpha](f)(x)\right|>\sigma\right\}\right|\leqslant C\cdot\sup_{r>0}\frac{1}{\Theta(r)}\int_{B(x_0,r)}\Phi\left(\frac{|f(x)|}{\sigma}\right)dx,$$

where $\Phi(t)=t(1+\log^+t)$.

COROLLARY 6.8. *Let $0<\alpha\leqslant 1$ and $b\in BMO(\mathbb{R}^n)$. Suppose that Θ satisfies (2.4) and $1\leqslant D(\Theta)<2^n$, then for any given $\sigma>0$ and any ball $B(x_0,r)$, there exists a constant $C>0$ independent of f, $B(x_0,r)$ and σ such that*

$$\frac{1}{\Theta(r)}\cdot\left|\left\{x\in B(x_0,r):\left|[b,\mathscr{G}_\alpha](f)(x)\right|>\sigma\right\}\right|\leqslant C\cdot\sup_{r>0}\frac{1}{\Theta(r)}\int_{B(x_0,r)}\Phi\left(\frac{|f(x)|}{\sigma}\right)dx,$$

where $\Phi(t)=t(1+\log^+t)$.

COROLLARY 6.9. *Let $0<\alpha\leqslant 1$ and $b\in BMO(\mathbb{R}^n)$. Suppose that Θ satisfies (2.4), $1\leqslant D(\Theta)<2^n$ and $\lambda>(3n+2\alpha)/n$, then for any given $\sigma>0$ and any ball $B(x_0,r)$, there exists a constant $C>0$ independent of f, $B(x_0,r)$ and σ such that*

$$\frac{1}{\Theta(r)}\cdot\left|\left\{x\in B(x_0,r):\left|[b,\mathscr{G}^*_{\lambda,\alpha}](f)(x)\right|>\sigma\right\}\right|\leqslant C\cdot\sup_{r>0}\frac{1}{\Theta(r)}\int_{B(x_0,r)}\Phi\left(\frac{|f(x)|}{\sigma}\right)dx,$$

where $\Phi(t)=t(1+\log^+t)$.

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