A UNIFIED TREATMENT OF HILBERT–TYPE
INEQUALITIES INVOLVING THE HARDY OPERATOR

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Abstract. In this paper we shall provide a unified treatment of Hilbert-type inequalities involving the Hardy operator with a general homogeneous kernel. Furthermore, we establish some new Hilbert-type inequalities involving the integral operator. We shall show that the constants in our results are best possible.

1. Introduction

One of the earliest variants of the classical Hilbert's inequality, that holds for all non-negative functions $0 < \int_{\mathbb{R}_+} f^p(x)dx < \infty$ and $0 < \int_{\mathbb{R}_+} g^q(x)dx < \infty$, is

$$\int_{\mathbb{R}_+^2} \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left(\int_{\mathbb{R}_+} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}_+} g^q(x)dx\right)^{\frac{1}{q}}, \qquad (1)$$

where p and q are conjugate exponents, that is, $1/p + 1/q = 1$, $p > 1$. The constant $\pi/\sin(\pi/p)$ is the best possible in the sense that it can not be replaced with a smaller constant (see [8]).

The Hilbert inequality is classical but very important in mathematical analysis and its applications and it is still a field of interest of numerous mathematicians. During decades, it was generalized in many different directions, such as different choices of kernels, sets of integration etc. The resulting inequalities are usually called the Hilbert-type inequalities. For more details about the Hilbert inequality the reader is referred to [5, 7, 8, 10, 13, 15, 16, 17, 18].

The starting point in this article is a recent result of Adiyasuren and Batbold [2], who derived a pair of Hilbert-type inequalities with a homogeneous kernel, involving the operator $\mathscr{H} : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$, $p > 1$, defined by $(\mathscr{H}f)(x) = \frac{1}{x}\int_0^x f(t)dt$. Namely, they obtained inequalities

$$\int_{\mathbb{R}_+^2} K_\lambda(x,y)x^{r-\frac{1}{q}}y^{s-\frac{1}{p}}(\mathscr{H}f)(x)(\mathscr{H}g)(y)dxdy < pqc_\lambda(s)\|f\|_p\|g\|_q \qquad (2)$$

and

$$\left[\int_{\mathbb{R}_+} y^{ps-1} \left(\int_{\mathbb{R}_+} K_\lambda(x,y) x^{r-\frac{1}{q}} (\mathscr{H}f)(x) dx \right)^p dy \right]^{\frac{1}{p}} < q c_\lambda(s) \|f\|_p, \tag{3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, $r, s > 0$, $\lambda = r + s$, $0 < \|f\|_p, \|g\|_q < \infty$, and $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a homogeneous function of degree $-\lambda$ satisfying $0 < c_\lambda(s) = \int_0^\infty K_\lambda(1,t) t^{s-1} dt < \infty$. Observe that the operator $\mathscr{H}$ is well defined due to the famous Hardy inequality $\|\mathscr{H}f\|_{L^p(\mathbb{R}_+)} < q\|f\|_{L^p(\mathbb{R}_+)}$ that holds for all measurable functions f satisfying $0 < \|f\|_{L^p(\mathbb{R}_+)} < \infty$. In fact, this operator is known in the literature as the Hardy operator (for more details see [11]).

Let p_i be the real parameters satisfying

$$\sum_{i=1}^n \frac{1}{p_i} = 1, \quad p_i > 1, \ i = 1, 2, \dots, n. \tag{4}$$

The parameters q_i are defined as associated conjugates, that is,

$$\frac{1}{p_i} + \frac{1}{q_i} = 1, \quad i = 1, 2, \dots, n. \tag{5}$$

Recall that the function $K : \mathbb{R}_+^n \to \mathbb{R}$ is said to be homogeneous of degree $-\lambda$, $\lambda > 0$, if $K(t\mathbf{x}) = t^{-\lambda} K(\mathbf{x})$ for all $t > 0$ and $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$. Furthermore, if $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, we define

$$k_i(\mathbf{a}) = \int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^i) \prod_{j=1, j\neq i}^n u_j^{a_j} \hat{d}^i \mathbf{u}, \quad i = 1, 2, \dots, n, \tag{6}$$

where $\hat{\mathbf{u}}^i = (u_1, \dots, u_{i-1}, 1, u_{i+1}, \dots, u_n)$, $\hat{d}^i \mathbf{u} = du_1 \dots du_{i-1} du_{i+1} \dots du_n$, and provided that the above integral converges. Further, in the sequel $d\mathbf{u}$ is an abbreviation for $du_1 du_2 \dots du_n$.

In 2012, Krnić [9] generalized inequalities (2) and (3) with homogeneous kernels in multidimensional setting as follows:

Let $K : \mathbb{R}_+^n \to \mathbb{R}$ be a non-negative measurable homogeneous function of degree $-\lambda$, $\lambda > 0$, such that for every $i = 2, 3, \dots, n$

$$K(1, t_2, \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_n) \leqslant C_K K(1, t_2, \dots t_{i-1}, 0, t_{i+1}, \dots, t_n), \ 0 \leqslant t_i \leqslant 1,$$

where C_K is a positive constant. Denote by $\mathbf{1}$ all-1s vector; $\mathbf{1} = (1, 1, \dots, 1)$. Further, let $\mu_i \in (1/p_i, 1]$, $i = 1, 2, \dots, n$, and let the parameters $s_i > 0$, $i = 2, \dots, n$ satisfy $\sum_{i=1}^n s_i = \lambda$. Then

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{s_i - \frac{1}{q_i}} (\mathscr{H}f_i)^{\mu_i}(x_i) d\mathbf{x} \leqslant \bar{m}_n^s(\mathbf{p}, \mathbf{s}, \mathbf{m}) \prod_{i=1}^n \|f_i^{\mu_i}\|_{p_i}, \tag{7}$$

$$\left[\int_{\mathbb{R}_+} x_n^{s_n q_n - 1} \left(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{s_i - \frac{1}{q_i}} (\mathscr{H}f_i)^{\mu_i}(x_i) \hat{d}^n \mathbf{x} \right)^{q_n} dx_n \right]^{1/q_n} \tag{8}$$

$$\leqslant \bar{m}_{n-1}^s(\mathbf{p}, \mathbf{s}, \mathbf{m}) \prod_{i=1}^{n-1} \|f_i^{\mu_i}\|_{p_i},$$

where the constants $\bar{m}_n^s(\mathbf{p},\mathbf{s},\mathbf{m}) = k_1(s-1)\prod_{i=1}^n \left(\frac{p_i\mu_i}{p_i\mu_i-1}\right)^{\mu_i}$ and $\bar{m}_{n-1}^s(\mathbf{p},\mathbf{s},\mathbf{m}) = k_1(s-1)\prod_{i=1}^{n-1}\left(\frac{p_i\mu_i}{p_i\mu_i-1}\right)^{\mu_i}$ are the best possible.

Recently, Moazzen and Lashkaripour [14] established the following two inequalities involving the Riemann-Liouville integral operator of order α;

$$\left(\mathscr{H}_\alpha f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt, \alpha > 0$$

and the Weyl integral operator of order α;

$$\left(\mathscr{H}'_\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}f(t)dt, \alpha > 0.$$

Let $f,g \geqslant 0, p,q,\lambda > 1, r > 0$ and $\frac{1}{p}+\frac{1}{q}=1$. If $k(x,y)$ is non-negative homogeneous function of degree -2λ, then

$$\int_{\mathbb{R}_+^2} k(x,y)x^{\lambda-\frac{1}{q}}y^{\lambda-\frac{1}{p}}\left(\mathscr{H}_r f\right)^{\lambda-\frac{1}{q}}(x)\left(\mathscr{H}_r g\right)^{\lambda-\frac{1}{p}}(y)dxdy \tag{9}$$

$$< C_\lambda \left(\frac{\Gamma(1-\frac{1}{p(\lambda-1)+1})}{\Gamma(r+1-\frac{1}{p(\lambda-1)+1})}\right)^{\lambda-\frac{1}{q}}\left(\frac{\Gamma(1-\frac{1}{q(\lambda-1)+1})}{\Gamma(r+1-\frac{1}{q(\lambda-1)+1})}\right)^{\lambda-\frac{1}{p}}\|f^{\lambda-\frac{1}{q}}\|_p\|g^{\lambda-\frac{1}{p}}\|_q$$

and

$$\int_{\mathbb{R}_+^2} k(x,y)x^{\lambda-\frac{1}{q}}y^{\lambda-\frac{1}{p}}\left(\mathscr{H}'_r f\right)^{\lambda-\frac{1}{q}}(x)\left(\mathscr{H}'_r g\right)^{\lambda-\frac{1}{p}}(y)dxdy \tag{10}$$

$$< C_\lambda \left(\frac{\Gamma(\frac{1}{p(\lambda-1)+1})}{\Gamma(r+\frac{1}{p(\lambda-1)+1})}\right)^{\lambda-\frac{1}{q}}\left(\frac{\Gamma(\frac{1}{q(\lambda-1)+1})}{\Gamma(r+\frac{1}{q(\lambda-1)+1})}\right)^{\lambda-\frac{1}{p}}\|f^{\lambda-\frac{1}{q}}\|_p\|g^{\lambda-\frac{1}{p}}\|_q,$$

where $C_\lambda = \left(\int_0^\infty t^{\lambda-1}k(1,t)dt\right)^{\frac{1}{p}}\left(\int_0^\infty t^{\lambda-1}k(t,1)dt\right)^{\frac{1}{q}}$.

They also proved three-dimensional cases of the above two inequalities. But Moazzen and Lashkaripour did not prove that the constant factors in the new inequalities are the best possible. We recall that the same inequalities with $r = 2$ were also studied in the paper [1]. For more details about the Hilbert-type inequalities involving some operators, the reader is referred to [3, 4, 6, 12].

The main objective of this paper is a unified treatment of the above Hilbert-type inequalities involving the Hardy, the Riemann-Liouville and the Weyl integral operators. Furthermore, we establish some new Hilbert-type inequalities involving the integral operator similar to the Riemann-Liouville integral operator. We shall show that the constants in our results are best possible.

2. Preliminary Lemmas

In order to prove main results, we need the following lemmas.

LEMMA 1. *Let p_i, q_i, $i = 1,\ldots,n$ satisfy* (4) *and* (5) *with $q_i = p_i'$, and let λ and $s_i > 0$, $i = 1,\ldots,n$ be such that $\sum_{i=1}^n s_i = \lambda$, $\lambda > 0$. If $K : \mathbb{R}_+^n \to \mathbb{R}$ is a non-negative measurable homogeneous function of degree $-\lambda$, then*

$$\int_{\mathbb{R}_+^{n-1}} K(\mathbf{u}) u_i^{s_i} \prod_{j=1,j\neq i}^n u_j^{s_j-1} \hat{d}^i \mathbf{u} = k_1(\mathbf{s}-\mathbf{1}), \quad i = 1,\ldots,n.$$

Proof. We freeze u_1. Utilizing the change of variables $u_2 = t_2 u_1,\ldots,u_n = t_n u_1$, which provides the Jacobian of the transformation

$$\left| \frac{\partial(u_2,\ldots,u_n)}{\partial(t_2,\ldots,t_n)} \right| = u_1^{n-1},$$

we have

$$\int_{\mathbb{R}_+^{n-1}} K(\mathbf{u}) u_1^{s_1} \prod_{j=2}^n u_j^{s_j-1} \hat{d}^1 \mathbf{u}$$

$$= \int_{\mathbb{R}_+^{n-1}} K(1,t_2,t_3,\ldots,t_n) u_1^{\lambda+n-1} u_1^{\sum_{j=1}^n s_j-n+1} \prod_{j=2}^n t_j^{s_j-1} \hat{d}^1 \mathbf{t}$$

$$= k_1(\mathbf{s}-\mathbf{1}).$$

Next, we freeze $i = 2,3,\ldots,n$. If we set $u_1 = \frac{t_1}{t_i},\ldots,u_{i-1} = \frac{t_{i-1}}{t_i}$, $u_{i+1} = \frac{t_{i+1}}{t_i},\ldots,$ $u_n = \frac{t_n}{t_i}$, $i = 2,\ldots,n$ whose Jacobian is equal to t_i^{-n+1}, we have

$$\int_{\mathbb{R}_+^{n-1}} K(\mathbf{u}) u_i^{s_i} \prod_{j=1,j\neq i}^n u_j^{s_j-1} \hat{d}^i \mathbf{u}$$

$$= \int_{\mathbb{R}_+^{n-1}} K\left(\frac{u_1}{u_i},\ldots,\frac{u_{i-1}}{u_i},1,\frac{u_{i+1}}{u_i},\ldots,\frac{u_n}{u_i} \right) u_i^{1-n} \prod_{j=1,j\neq i}^n \left(\frac{u_j}{u_i} \right)^{s_j-1} \hat{d}^i \mathbf{u}$$

$$= \int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{t}}^i) \prod_{j=1,j\neq i}^n t_j^{s_j-1} \hat{d}^i \mathbf{t} = k_i(\mathbf{s}-\mathbf{1}).$$

A similar change of variables yields $k_i(\mathbf{s}-\mathbf{1}) = k_1(\mathbf{s}-\mathbf{1})$ and the proof is complete. $\square$

We will need the following inequality for later considerations.

LEMMA 2. (Hardy et al. [8]) *If $p > 1$, $\alpha > 0$, then*

$$\int_{\mathbb{R}_+} (\mathscr{H}_\alpha f)^p(x) dx < \left[\frac{\Gamma(1-\frac{1}{p})}{\Gamma(\alpha+1-\frac{1}{p})} \right]^p \int_{\mathbb{R}_+} f^p(x) dx, \tag{11}$$

and

$$\int_{\mathbb{R}_+} \left(\mathscr{H}'_\alpha f\right)^p (x) dx < \left[\frac{\Gamma(\frac{1}{p})}{\Gamma(\alpha + \frac{1}{p})}\right]^p \int_{\mathbb{R}_+} x^{\alpha p} f^p (x) dx, \tag{12}$$

unless $f \equiv 0$. *In each case the constant is the best possible.*

Inequalities (11) and (12) may be rewritten as $\|\mathscr{H}_\alpha f\|_p < \frac{\Gamma(1-\frac{1}{p})}{\Gamma(\alpha+1-\frac{1}{p})}\|f\|_p$ and $\|\mathscr{H}'_\alpha f\|_p < \frac{\Gamma(\frac{1}{p})}{\Gamma(\alpha+\frac{1}{p})}\|x^\alpha f\|_p$, respectively.

In order to define the integral operator similar to the Riemann-Liouville integral operator, we first cite the following Minkowski's integral inequality from [8]. If $p > 1$, then

$$\left[\int_{\mathbb{R}_+} \left|\int_{\mathbb{R}_+} F(x,y)\frac{dy}{y}\right|^p \frac{dx}{x}\right]^{\frac{1}{p}} \leqslant \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} F(x,y)^p \frac{dx}{x}\right)^{\frac{1}{p}} \frac{dy}{y} \tag{13}$$

for all positive measurable functions $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$.

LEMMA 3. *Let* α *and* β *be real numbers such that* $\alpha > \beta > 0$ *and* f *be a non-negative measurable function. If* $0 < \int_{\mathbb{R}_+} x^{-1} f^p (x) dx < \infty$, *then*

$$\left[\int_{\mathbb{R}_+} x^{-1} \left(x^{-\alpha} \int_0^x y^{\alpha-\beta-1}(x^\beta - y^\beta) f(y) dy\right)^p dx\right]^{\frac{1}{p}} \tag{14}$$

$$< \frac{\beta}{\alpha(\alpha-\beta)} \left(\int_{\mathbb{R}_+} x^{-1} f^p (x) dx\right)^{\frac{1}{p}},$$

where the constant $\frac{\beta}{\alpha(\alpha-\beta)}$ *is the best possible.*

Proof. Setting $F(x,y) = \chi_{(1,\infty)}(y) y^{-\alpha}(y^\beta - 1) f\left(\frac{x}{y}\right)$, inequality (13) reduces to

$$\left[\int_{\mathbb{R}_+} \left(\int_1^\infty y^{-\alpha}(y^\beta - 1) f\left(\frac{x}{y}\right) \frac{dy}{y}\right)^p \frac{dx}{x}\right]^{\frac{1}{p}}$$

$$\leqslant \int_{\mathbb{R}_+} \chi_{(1,\infty)}(y) y^{-\alpha}(y^\beta - 1) \left(\int_{\mathbb{R}_+} f^p(x) \frac{dx}{x}\right)^{\frac{1}{p}} \frac{dy}{y}$$

$$= \frac{1}{\beta} \left(\int_{\mathbb{R}_+} x^{-1} f^p(x) dx\right)^{\frac{1}{p}} \int_1^\infty y^{-\frac{\alpha}{\beta}-1}(y-1) dy$$

$$= \frac{B(\alpha/\beta - 1, 2)}{\beta} \left(\int_{\mathbb{R}_+} x^{-1} f^p(x) dx\right)^{\frac{1}{p}}.$$

Thus,

$$\left[\int_{\mathbb{R}_+} x^{-1} \left(x^{-\alpha} \int_0^x y^{\alpha-\beta-1}(x^\beta - y^\beta) f(y) dy \right)^p dx \right]^{\frac{1}{p}}$$

$$\leqslant \frac{\beta}{\alpha(\alpha-\beta)} \left(\int_{\mathbb{R}_+} x^{-1} f^p(x) dx \right)^{\frac{1}{p}}.$$

In order to prove that (14) includes the best possible constant, we suppose that there exists a positive L, smaller than $\frac{\beta}{\alpha(\alpha-\beta)}$, such that the inequality

$$\left[\int_{\mathbb{R}_+} x^{-1} \left(x^{-\alpha} \int_0^x y^{\alpha-\beta-1}(x^\beta - y^\beta) f(y) dy \right)^p dx \right]^{\frac{1}{p}} \leqslant L \left(\int_{\mathbb{R}_+} x^{-1} f^p(x) dx \right)^{\frac{1}{p}}$$

holds for all non-negative functions $f : \mathbb{R}_+ \to \mathbb{R}$, provided $\int_{\mathbb{R}_+} x^{-1} f^p(x) dx < \infty$. Considering the function $\widetilde{f}(x) = \chi_{(0,1)}(x) x^\varepsilon$ where $\varepsilon > 0$ is sufficiently small number, we have

$$L \left(\int_0^1 x^{p\varepsilon-1} dx \right)^{\frac{1}{p}} \geqslant \left[\int_{\mathbb{R}_+} x^{-1} \left(x^{-\alpha} \int_0^x y^{\alpha-\beta-1}(x^\beta - y^\beta) \widetilde{f}(y) dy \right)^p dx \right]^{\frac{1}{p}}$$

$$\geqslant \left[\int_0^1 x^{-1} \left(x^{-\alpha} \int_0^x y^{\alpha-\beta-1+\varepsilon}(x^\beta - y^\beta) dy \right)^p dx \right]^{\frac{1}{p}}$$

$$= \frac{B(\alpha/\beta - 1 + \varepsilon/\beta, 2)}{\beta} \left(\int_0^1 x^{p\varepsilon-1} dx \right)^{\frac{1}{p}}$$

$$= \frac{\beta}{(\alpha+\varepsilon)(\alpha-\beta+\varepsilon)} \left(\int_0^1 x^{p\varepsilon-1} dx \right)^{\frac{1}{p}}.$$

The above relation yields $\frac{\beta}{(\alpha+\varepsilon)(\alpha-\beta+\varepsilon)} \leqslant L$, and for $\varepsilon \to 0^+$, it follows that $\frac{\beta}{\alpha(\alpha-\beta)} \leqslant L$. This contradicts to $L < \frac{\beta}{\alpha(\alpha-\beta)}$, which means that $\frac{\beta}{\alpha(\alpha-\beta)}$ is the best possible constant in (14).

The failure of equality follows similary to the Hardy inequality. □

Motivated by Lemma 3, we define a new integral operator similar to the Riemann-Liouville integral operator, $\mathscr{H}_{\alpha,\beta}$ by

$$(\mathscr{H}_{\alpha,\beta} f)(x) = x^{-\alpha} \int_0^x t^{\alpha-\beta-1}(x^\beta - t^\beta) f(t) dt.$$

Inequality (14) may be rewritten as

$$\|x^{-1/p} \mathscr{H}_{\alpha,\beta} f\|_p \leqslant \frac{\beta}{\alpha(\alpha-\beta)} \|x^{-1/p} f\|_p, \quad p > 1.$$

We assume that all integrals exist on the respective domains of their definitions.

3. Main results

In the next theorems it is to understood that $\mathcal{H}_0 f(x) = \mathcal{H}'_0 f(x) = f(x)$ and that $\mathbf{a}$ is an abbreviation of $(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

THEOREM 1. *Suppose* p_i, q_i, $i = 1, \ldots, n$ *are as in* (4) *and* (5) *with* $q_i = p'_i$, *and* $s_i > 0$, $i = 1, \ldots, n$, $\lambda > 0$ *are real parameters satisfying* $\sum_{i=1}^{n} s_i = \lambda$. *Further let* μ_i, $i = 1, \ldots, n$ *be real parameters such that* $\mu_i p_i > 1$, *that* $\sum_{i=1}^{n} \alpha_i > 0$ *and that* $\alpha_i \geqslant 0$, $i = 1, \ldots, n$. *If* $K : \mathbb{R}_+^n \to \mathbb{R}$ *is a non-negative measurable homogeneous function of degree* $-\lambda$, *and* $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, \ldots, n$ *are non-negative measurable functions, then*

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} x_i^{s_i - \frac{1}{q_i}} (\mathcal{H}_{\alpha_i} f_i)^{\mu_i}(x_i) d\mathbf{x} < C_n(\mathbf{a}, \mathbf{p}, \mathbf{s}, \mathbf{m}) \prod_{i=1}^{n} \|f_i^{\mu_i}\|_{p_i}, \tag{15}$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{q_n s_n - 1} \left(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{s_i - \frac{1}{q_i}} (\mathcal{H}_{\alpha_i} f_i)^{\mu_i}(x_i) \hat{d}^n \mathbf{x} \right)^{q_n} dx_n \right]^{1/q_n} \tag{16}$$

$$< C_{n-1}(\mathbf{a}, \mathbf{p}, \mathbf{s}, \mathbf{m}) \prod_{i=1}^{n-1} \|f_i^{\mu_i}\|_{p_i},$$

where the constant factors

$$C_n(\mathbf{a}, \mathbf{p}, \mathbf{s}, \mathbf{m}) = k_1(\mathbf{s} - 1) \prod_{i=1}^{n} \left(\frac{\Gamma(1 - \frac{1}{\mu_i p_i})}{\Gamma(\alpha_i + 1 - \frac{1}{\mu_i p_i})} \right)^{\mu_i},$$

$$C_{n-1}(\mathbf{a}, \mathbf{p}, \mathbf{s}, \mathbf{m}) = k_1(\mathbf{s} - 1) \prod_{i=1}^{n-1} \left(\frac{\Gamma(1 - \frac{1}{\mu_i p_i})}{\Gamma(\alpha_i + 1 - \frac{1}{\mu_i p_i})} \right)^{\mu_i}$$

are the best possible.

Proof. By Hölder's inequality and Lemma 1, we have

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} x_i^{s_i - \frac{1}{q_i}} (\mathcal{H}_{\alpha_i} f_i)^{\mu_i}(x_i) d\mathbf{x}$$

$$= \int_{\mathbb{R}_+^n} K(\mathbf{x})^{\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n}} \prod_{i=1}^{n} \left(x_i^{\frac{s_i}{p_i}} \prod_{j=1, j \neq i}^{n} x_j^{\frac{s_j - 1}{p_i}} (\mathcal{H}_{\alpha_i} f_i)^{\mu_i}(x_i) \right) d\mathbf{x}$$

$$\leqslant \prod_{i=1}^{n} \left(\int_{\mathbb{R}_+^n} K(\mathbf{x}) x_i^{s_i} \prod_{j=1, j \neq i}^{n} x_j^{s_j - 1} (\mathcal{H}_{\alpha_i} f_i)^{\mu_i p_i}(x_i) d\mathbf{x} \right)^{\frac{1}{p_i}}$$

$$= k_1(\mathbf{s} - 1) \prod_{i=1}^{n} \left(\int_{\mathbb{R}_+} (\mathcal{H}_{\alpha_i} f_i)^{\mu_i p_i}(x_i) dx_i \right)^{\frac{1}{p_i}} = k_1(\mathbf{s} - 1) \prod_{i=1}^{n} \|\mathcal{H}_{\alpha_i} f_i\|_{\mu_i p_i}^{\mu_i}.$$

Now, applying inequality (11) to the right-hand side of the above inequality yields,

$$k_1(\mathbf{s}-1)\prod_{i=1}^{n}\|\mathscr{H}_{\alpha_i}f_i\|_{\mu_i p_i}^{\mu_i} < C_n(\mathbf{a},\mathbf{p},\mathbf{s},\mathbf{m})\prod_{i=1}^{n}\|f_i^{\mu_i}\|_{p_i}.$$

Then, from the above inequalities, we obtain inequality (15).

We turn to the proof of the optimality of the constant: Suppose to the contrary that there exists a positive constant $\widetilde{C}$: $0 < \widetilde{C} < C_n(\mathbf{a},\mathbf{p},\mathbf{s},\mathbf{m})$ such that inequality

$$\int_{\mathbb{R}_+^n} K(\mathbf{x})\prod_{i=1}^{n}x_i^{s_i-\frac{1}{q_i}}(\mathscr{H}_{\alpha_i}f_i)^{\mu_i}(x_i)d\mathbf{x} < \widetilde{C}\prod_{i=1}^{n}\|f_i^{\mu_i}\|_{p_i}, \tag{17}$$

holds for non-negative measurable functions $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1,\ldots,n$. Let us set

$$K_N(\mathbf{x}) = \min(N, K(\mathbf{x})) \times \chi_{(N^{-1},N)^{n-1}}\left(\frac{x_2}{x_1},\frac{x_3}{x_1},\ldots,\frac{x_n}{x_1}\right). \tag{18}$$

Considering this inequality with the function

$$\widetilde{f}_i(x_i) = x_i^{\frac{\varepsilon-1}{\mu_i p_i}}\chi_{(0,1)}(x_i), \quad i = 1,\ldots,n,$$

where ε is a positive sufficiently small number, its right-hand side becomes

$$\widetilde{C}\prod_{i=1}^{n}\|\widetilde{f}_i^{\mu_i}\|_{p_i} = \widetilde{C}\prod_{i=1}^{n}\left(\int_0^1 x_i^{\varepsilon-1}dx_i\right)^{\frac{1}{p_i}} = \frac{\widetilde{C}}{\varepsilon}.$$

On the other hand, since

$$0 < x_i \leqslant 1, \quad \left(\mathscr{H}_{\alpha_i}\widetilde{f}_i\right)(x_i) = \frac{x_i^{-\alpha_i}}{\Gamma(\alpha_i)}\int_0^{x_i}(x_i-t)^{\alpha_i-1}\widetilde{f}_i(t)dt$$

$$= \frac{x_i^{-\alpha_i}}{\Gamma(\alpha_i)}\int_0^{x_i}(x_i-t)^{\alpha_i-1}t^{\frac{\varepsilon-1}{\mu_i p_i}}dt$$

$$= \frac{x_i^{\frac{\varepsilon-1}{\mu_i p_i}}}{\Gamma(\alpha_i)}B\left(1+\frac{\varepsilon-1}{\mu_i p_i},\alpha_i\right)$$

$$= x_i^{\frac{\varepsilon-1}{\mu_i p_i}}\frac{\Gamma(1+\frac{\varepsilon-1}{\mu_i p_i})}{\Gamma(\alpha_i+1+\frac{\varepsilon-1}{\mu_i p_i})},$$

the left-hand side of (17), can be estimated as

$$\int_{\mathbb{R}_+^n} K_N(\mathbf{x}) \prod_{i=1}^n x_i^{s_i - \frac{1}{q_i}} \left(\mathscr{H}_{\alpha_i} \widetilde{f_i} \right)^{\mu_i} (x_i) d\mathbf{x} \tag{19}$$

$$> \prod_{i=1}^n \left(\frac{\Gamma(1 + \frac{\varepsilon - 1}{\mu_i p_i})}{\Gamma(\alpha_i + 1 + \frac{\varepsilon - 1}{\mu_i p_i})} \right)^{\mu_i} \int_{(0,1]^n} K_N(\mathbf{x}) \prod_{i=1}^n x_i^{s_i + \frac{\varepsilon}{p_i} - 1} d\mathbf{x}$$

$$= \prod_{i=1}^n \left(\frac{\Gamma(1 + \frac{\varepsilon - 1}{\mu_i p_i})}{\Gamma(\alpha_i + 1 + \frac{\varepsilon - 1}{\mu_i p_i})} \right)^{\mu_i} \int_0^1 x_1^{\varepsilon - 1} \left[\int_{(0,x_1]^{n-1}} K_N(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{s_i + \frac{\varepsilon}{p_i} - 1} \hat{d}^1 \mathbf{u} \right] dx_1$$

$$\geqslant \prod_{i=1}^n \left(\frac{\Gamma(1 + \frac{\varepsilon - 1}{\mu_i p_i})}{\Gamma(\alpha_i + 1 + \frac{\varepsilon - 1}{\mu_i p_i})} \right)^{\mu_i} \left[\int_0^1 x_1^{\varepsilon - 1} \left(\int_{\mathbb{R}_+^{n-1}} K_N(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{s_i + \frac{\varepsilon}{p_i} - 1} \hat{d}^1 \mathbf{u} \right) dx_1 \right.$$

$$\left. - \int_0^1 x_1^{\varepsilon - 1} \left(\sum_{i=2}^n \int_{\mathbb{D}_i} K_N(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{s_j + \frac{\varepsilon}{p_j} - 1} \hat{d}^1 \mathbf{u} \right) dx_1 \right].$$

Let $\mathbb{D}_i = \{(u_2, \ldots, u_n) : u_i > \frac{1}{x_1}, u_j > 0, j \neq i\}$. Then we have

$$\int_{\mathbb{R}_+^n} K_N(\mathbf{x}) \prod_{i=1}^n x_i^{s_i - \frac{1}{q_i}} \left(\mathscr{H}_{\alpha_i} \widetilde{f_i} \right)^{\mu_i} (x_i) d\mathbf{x} \tag{20}$$

$$\geqslant \prod_{i=1}^n \left(\frac{\Gamma(1 + \frac{\varepsilon - 1}{\mu_i p_i})}{\Gamma(\alpha_i + 1 + \frac{\varepsilon - 1}{\mu_i p_i})} \right)^{\mu_i} \left[\frac{1}{\varepsilon} \int_{\mathbb{R}_+^{n-1}} K_N(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{s_i + \frac{\varepsilon}{p_i} - 1} \hat{d}^1 \mathbf{u} \right.$$

$$\left. - \int_0^1 x_1^{-1} \left(\sum_{i=2}^n \int_{\mathbb{D}_i} K_N(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{s_j + \frac{\varepsilon}{p_j} - 1} \hat{d}^1 \mathbf{u} \right) dx_1 \right].$$

Without loss of generality, it suffices to find the appropriate estimate for the integral $\int_{\mathbb{D}_2} K_N(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{s_j + \frac{\varepsilon}{p_j} - 1} \hat{d}^1 \mathbf{u}$: We plan to find a constant M_N independent of $\varepsilon > 0$ such that

$$\int_{\mathbb{D}_2} K_N(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{s_j + \frac{\varepsilon}{p_j} - 1} \hat{d}^1 \mathbf{u} \leqslant M_N$$

for all $0 < \varepsilon < 1$. Note that M_N depends on N.

By virtue of the Fubini theorem, we have

$$\int_0^1 x_1^{-1} \left(\int_{\mathbb{D}_2} K_N(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{s_j + \frac{\varepsilon}{p_j} - 1} \hat{d}^1 \mathbf{u} \right) dx_1 \tag{21}$$

$$= \int_0^1 x_1^{-1} \left(\int_{\mathbb{R}_+^{n-2}} \int_{1/x_1}^\infty K_N(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{s_j + \frac{\varepsilon}{p_j} - 1} \hat{d}^1 \mathbf{u} \right) dx_1.$$

Observe $u_2^{-1}\log u_2 \leqslant e^{-1} \leqslant 1$ $(u_2 \in [1,\infty))$. By enlarging the domain of integration, we obtain

$$
\int_0^1 x_1^{-1}\left(\int_{\mathbb{D}_2} K_N(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{s_j+\frac{\varepsilon}{p_j}-1}\hat{d}^1\mathbf{u}\right)dx_1 \tag{22}
$$

$$
\leqslant \int_{(1,\infty)\times\mathbb{R}_+^{n-2}} K_N(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{s_j+\frac{\varepsilon}{p_j}-1}\left(\int_{1/u_2}^1 x_1^{-1}dx_1\right)\hat{d}^1\mathbf{u}
$$

$$
= \int_{(1,\infty)\times\mathbb{R}_+^{n-2}} K_N(\hat{\mathbf{u}}^1)u_2^{s_2+\frac{\varepsilon}{p_2}}\prod_{j=3}^n u_j^{s_j+\frac{\varepsilon}{p_j}-1}(u_2^{-1}\log u_2)\hat{d}^1\mathbf{u}
$$

$$
\leqslant \int_{(1,\infty)\times\mathbb{R}_+^{n-2}} K_N(\hat{\mathbf{u}}^1)u_2^{s_2+\frac{\varepsilon}{p_2}}\prod_{j=3}^n u_j^{s_j+\frac{\varepsilon}{p_j}-1}\hat{d}^1\mathbf{u}
$$

$$
\leqslant \int_{\mathbb{R}_+^{n-1}} K_N(\hat{\mathbf{u}}^1)u_2^{s_2+\frac{\varepsilon}{p_2}}\prod_{j=3}^n u_j^{s_j+\frac{\varepsilon}{p_j}-1}\hat{d}^1\mathbf{u} < \infty,
$$

where for the last inequality we have used the fact that K_N is given by (18). Hence, we have

$$
\prod_{i=1}^n\left(\frac{\Gamma(1+\frac{\varepsilon-1}{\mu_i p_i})}{\Gamma(\alpha_i+1+\frac{\varepsilon-1}{\mu_i p_i})}\right)^{\mu_i}\left[\int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^1)\prod_{i=2}^n u_i^{s_i+\frac{\varepsilon}{p_i}-1}\hat{d}^1\mathbf{u}-\bigcirc(1)\right] < \widetilde{C}.
$$

Obviously, if $\varepsilon \to 0^+$, then

$$
\widetilde{C} \geqslant C_n(N;\mathbf{a},\mathbf{p},\mathbf{s},\mathbf{m}) = \prod_{i=1}^n\left(\frac{\Gamma(1-\frac{1}{\mu_i p_i})}{\Gamma(\alpha_i+1-\frac{1}{\mu_i p_i})}\right)^{\mu_i}\int_{\mathbb{R}_+^{n-1}} K_N(\hat{\mathbf{u}}^i)\prod_{j=1,j\neq i}^n u_j^{s_j-1}\hat{d}^1\mathbf{u}
$$

for all $N = 1,2,\ldots$, which contradicts to our assumption $0 < \widetilde{C} < C_n(\mathbf{a},\mathbf{p},\mathbf{s},\mathbf{m})$. Hence, $C_n(\mathbf{a},\mathbf{p},\mathbf{s},\mathbf{m})$ is the best possible.

By Hölder's inequality and Lemma 1, we have

$$
L(x_n)
$$

$$
:= \int_{\mathbb{R}_+^{n-1}} K(\mathbf{x})x_n^{s_n-\frac{1}{q_n}}\prod_{i=1}^{n-1} x_i^{s_i-\frac{1}{q_i}}\,(\mathscr{H}_{\alpha_i}f_i)^{\mu_i}(x_i)\hat{d}^n\mathbf{x}
$$

$$
= \int_{\mathbb{R}_+^{n-1}} K(\mathbf{x})\left[x_n^{\frac{s_n}{p_n}}\prod_{i=1}^{n-1} x_i^{\frac{s_i-1}{p_n}}\right]\left[\prod_{i=1}^{n-1}\left(x_i^{\frac{s_i}{p_i}}\prod_{j=1,j\neq i}^n x_j^{\frac{s_j-1}{p_i}}(\mathscr{H}_{\alpha_i}f_i)^{\mu_i}(x_i)\right)\right]\hat{d}^n\mathbf{x}
$$

$$
\leqslant k_1(\mathbf{s}-\mathbf{1})^{\frac{1}{p_n}}
$$

$$
\times\left[\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x})\prod_{i=1}^{n-1}\left(x_i^{\frac{q_n s_i}{p_i}}\prod_{j=1,j\neq i}^n x_j^{\frac{q_n(s_j-1)}{p_i}}(\mathscr{H}_{\alpha_i}f_i)^{\mu_i q_n}(x_i)\right)\hat{d}^n\mathbf{x}\right]^{\frac{1}{q_n}}.
$$

Hence again applying Hölder's inequality, Lemma 1 and inequality (11), we get

$$
\left[\int_{\mathbb{R}_+} L^{q_n}(x_n)dx_n\right]^{\frac{1}{q_n}}
$$

$$
\leqslant [k_1(\mathbf{s}-\mathbf{1})]^{\frac{1}{p_n}}\left[\int_{\mathbb{R}_+^n} K(\mathbf{x})\prod_{i=1}^{n-1}\left(x_i^{\frac{q_n s_i}{p_i}}\prod_{j=1,j\neq i}^{n} x_j^{\frac{q_n(s_j-1)}{p_i}}(\mathscr{H}_{\alpha_i}f_i)^{\mu_i q_n}(x_i)\right)d\mathbf{x}\right]^{\frac{1}{q_n}}
$$

$$
\leqslant [k_1(\mathbf{s}-\mathbf{1})]^{\frac{1}{p_n}}\prod_{i=1}^{n-1}\left[\int_{\mathbb{R}_+^n} K(\mathbf{x})\left(x_i^{s_i}\prod_{j=1,j\neq i}^{n} x_j^{s_j-1}(\mathscr{H}_{\alpha_i}f_i)^{\mu_i p_i}(x_i)\right)d\mathbf{x}\right]^{\frac{1}{p_i}}
$$

$$
= k_1(\mathbf{s}-\mathbf{1})\prod_{i=1}^{n-1}\|\mathscr{H}_{\alpha_i}f_i\|_{\mu_i p_i}^{\mu_i} < C_{n-1}(\mathbf{p},\mathbf{s},\mathbf{m})\prod_{i=1}^{n-1}\|f_i^{\mu_i}\|_{p_i}.
$$

Assume that there exists a positive constant $\widehat{C}$, smaller than $C_{n-1}(\mathbf{a},\mathbf{p},\mathbf{s},\mathbf{m})$, such that inequality (16) holds when replacing $C_{n-1}(\mathbf{a},\mathbf{p},\mathbf{s},\mathbf{m})$ by $\widehat{C}$.

The left-hand side of inequality (15), denoted here by L, can be rewritten in the following form:

$$
L = \int_{\mathbb{R}_+} x_n^{s_n-\frac{1}{q_n}}\left(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x})\prod_{i=1}^{n-1} x_i^{s_i-\frac{1}{q_i}}(\mathscr{H}_{\alpha_i}f_i)^{\mu_i}(x_i)d\mathbf{x}\right)(\mathscr{H}_{\alpha_n}f_n)^{\mu_n}(x_n)dx_n.
$$

Now, applying the Hölder inequality with conjugate exponents p_n and q_n to the above expression yields inequality

$$
L \leqslant L'\|\mathscr{H}_{\alpha_n}f_n\|_{p_n}, \tag{23}
$$

where L' denotes the left-hand side of (16).

Moreover, $L' \leqslant \widehat{C}\prod_{i=1}^{n-1}\|f_i^{\mu_i}\|_{p_i}$, while inequality (11) yields

$$
\|\mathscr{H}_{\alpha_n}f_n\|_{\mu_n p_n}^{\mu_n} < \left(\frac{\Gamma(1-\frac{1}{\mu_n p_n})}{\Gamma(\alpha_n+1-\frac{1}{\mu_n p_n})}\right)^{\mu_n}\|f_n\|_{p_n}.
$$

Therefore relation (23) yields the inequality

$$
L \leqslant \widetilde{C}\left(\frac{\Gamma(1-\frac{1}{\mu_n p_n})}{\Gamma(\alpha_n+1-\frac{1}{\mu_n p_n})}\right)^{\mu_n}\prod_{i=1}^{n}\|f_i^{\mu_i}\|_{p_i}. \tag{24}
$$

Finally, taking into account our assumption $0 < \widetilde{C} < C_{n-1}(\mathbf{a},\mathbf{p},\mathbf{s},\mathbf{m})$, we have

$$
0 < \widetilde{C}\left(\frac{\Gamma(1-\frac{1}{\mu_n p_n})}{\Gamma(\alpha_n+1-\frac{1}{\mu_n p_n})}\right)^{\mu_n}
$$

$$
< C_{n-1}(\mathbf{a},\mathbf{p},\mathbf{s},\mathbf{m})\left(\frac{\Gamma(1-\frac{1}{\mu_n p_n})}{\Gamma(\alpha_n+1-\frac{1}{\mu_n p_n})}\right)^{\mu_n} = C_n(\mathbf{a},\mathbf{p},\mathbf{s},\mathbf{m}).
$$

Hence, relation (24) contradicts with the fact that $C_n(\mathbf{a}, \mathbf{p}, \mathbf{s}, \mathbf{m})$ is the best possible constant in inequality (15). Thus, the assumption that $C_{n-1}(\mathbf{a}, \mathbf{p}, \mathbf{s}, \mathbf{m})$ is not the best possible is false. The proof is now completed. $\square$

REMARK 1. If we set $\alpha_i = 1$, $i = 1, \ldots, n$, the relations (15) and (16) become inequalities (7) and (8) from the Introduction, with a weaker condition $\mu_i p_i > 1$, $i = 1, \ldots, n$.

THEOREM 2. *Under the assumptions of Theorem* 1,

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} x_i^{s_i - \frac{1}{q_i}} \left(\mathscr{H}_{\alpha_i}' f_i \right)^{\mu_i} (x_i) d\mathbf{x} < C_n'(\mathbf{a}, \mathbf{p}, \mathbf{s}, \mathbf{m}) \prod_{i=1}^{n} \|x_i^{\alpha_i \mu_i} f_i^{\mu_i}\|_{p_i}, \qquad (25)$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{q_n s_n - 1} \left(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{s_i - \frac{1}{q_i}} \left(\mathscr{H}_{\alpha_i}' f_i \right)^{\mu_i} (x_i) \hat{d}^n \mathbf{x} \right)^{q_n} dx_n \right]^{1/q_n} \qquad (26)$$
$$< C_{n-1}'(\mathbf{a}, \mathbf{p}, \mathbf{s}, \mathbf{m}) \prod_{i=1}^{n-1} \|x_i^{\alpha_i \mu_i} f_i^{\mu_i}\|_{p_i},$$

where the constant factors

$$C_n'(\mathbf{a}, \mathbf{p}, \mathbf{s}, \mathbf{m}) = k_1(\mathbf{s} - \mathbf{1}) \prod_{i=1}^{n} \left(\frac{\Gamma(\frac{1}{\mu_i p_i})}{\Gamma(\alpha_i + \frac{1}{\mu_i p_i})} \right)^{\mu_i},$$

$$C_{n-1}'(\mathbf{a}, \mathbf{p}, \mathbf{s}, \mathbf{m}) = k_1(\mathbf{s} - \mathbf{1}) \prod_{i=1}^{n-1} \left(\frac{\Gamma(\frac{1}{\mu_i p_i})}{\Gamma(\alpha_i + \frac{1}{\mu_i p_i})} \right)^{\mu_i}$$

are the best possible.

Proof. The proof is similar to the proof of the previous theorem, except that we use inequality (11) instead of inequality (12) and functions

$$\widetilde{f}_i(x_i) = x_i^{\frac{-\varepsilon - 1}{\mu_i p_i} - \alpha_i} \chi_{(1, \infty)}(x_i), \ \ i = 1, \ldots, n. \quad \square$$

THEOREM 3. *Suppose* p_i, q_i, $i = 1, \ldots, n$ *are as in* (4) *and* (5), *and* $s_i > 0$, $i = 1, \ldots, n$, $\lambda > 0$ *are real parameters satisfying* $\sum_{i=1}^{n} s_i = \lambda$. *Further let* μ_i, $i = 1, \ldots, n$ *be real parameters such that* $\mu_i p_i > 1$ *and* $\alpha_i > \beta_i > 0$, $i = 1, \ldots, n$. *If* $K : \mathbb{R}_+^n \to \mathbb{R}$ *is a non-negative measurable homogeneous function of degree* $-\lambda$, *and* $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, \ldots, n$ *are non-negative measurable functions, then*

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} x_i^{s_i - 1} \left(\mathscr{H}_{\alpha_i, \beta_i} f_i \right)^{\mu_i} (x_i) d\mathbf{x} \leqslant \overline{C}_n(\mathbf{a}, \mathbf{b}, \mathbf{s}, \mathbf{m}) \prod_{i=1}^{n} \|x_i^{-1/p_i} f_i^{\mu_i}\|_{p_i}, \qquad (27)$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{-q_n(s_n-1)} \left(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{s_i-1} \left(\mathcal{H}_{\alpha_i,\beta_i} f_i \right)^{\mu_i} (x_i) \hat{d}^n \mathbf{x} \right)^{q_n} dx_n \right]^{1/q_n} \quad (28)$$

$$\leqslant \overline{C}_{n-1}(\mathbf{a},\mathbf{b},\mathbf{s},\mathbf{m}) \prod_{i=1}^{n-1} \|x_i^{-1/p_i} f_i^{\mu_i}\|_{p_i},$$

where the constant factors

$$\overline{C}_n(\mathbf{a},\mathbf{b},\mathbf{s},\mathbf{m}) = k_1(\mathbf{s}-\mathbf{1}) \prod_{i=1}^{n} \left(\frac{\beta_i}{\alpha_i(\alpha_i-\beta_i)} \right)^{\mu_i},$$

$$\overline{C}_{n-1}(\mathbf{a},\mathbf{b},\mathbf{s},\mathbf{m}) = k_1(\mathbf{s}-\mathbf{1}) \prod_{i=1}^{n-1} \left(\frac{\beta_i}{\alpha_i(\alpha_i-\beta_i)} \right)^{\mu_i}$$

are the best possible.

Proof. We follow the same procedure as in the proof of Theorem 1, except that we use inequality (11) instead of inequality (14). So, we prove the first inequality; the second is left to the reader. We have

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} x_i^{s_i-1} \left(\mathcal{H}_{\alpha_i,\beta_i} f_i \right)^{\mu_i} (x_i) d\mathbf{x}$$

$$= \int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} \left(x_i^{\frac{s_i-1}{p_i}} \prod_{j=1,j\neq i}^{n} x_j^{\frac{s_j-1}{p_i}} \left(\mathcal{H}_{\alpha_i,\beta_i} f_i \right)^{\mu_i}(x_i) \right) d\mathbf{x}$$

$$\leqslant \prod_{i=1}^{n} \left(\int_{\mathbb{R}_+^n} K(\mathbf{x}) x_i^{s_i-1} \prod_{j=1,j\neq i}^{n} x_j^{s_j-1} \left(\mathcal{H}_{\alpha_i,\beta_i} f_i \right)^{\mu_i p_i} (x_i) d\mathbf{x} \right)^{\frac{1}{p_i}}$$

$$= k_1(\mathbf{s}-\mathbf{1}) \prod_{i=1}^{n} \left(\int_{\mathbb{R}_+} x_i^{-1} (\mathcal{H}_{\alpha_i,\beta_i} f_i)^{\mu_i p_i}(x_i) dx_i \right)^{\frac{1}{p_i}}$$

$$= k_1(\mathbf{s}-\mathbf{1}) \prod_{i=1}^{n} \|x_i^{-1/\mu_i p_i} \mathcal{H}_{\alpha_i,\beta_i} f_i\|_{\mu_i p_i}^{\mu_i} \leqslant \overline{C}_n(\mathbf{a},\mathbf{b},\mathbf{s},\mathbf{m}) \prod_{i=1}^{n} \|x_i^{-1/p_i} f_i^{\mu_i}\|_{p_i}.$$

We suppose that the inequality

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} x_i^{s_i-1} \left(\mathcal{H}_{\alpha_i,\beta_i} f_i \right)^{\mu_i} (x_i) d\mathbf{x} \leqslant \widetilde{C} \prod_{i=1}^{n} \|x_i^{-1/p_i} f_i^{\mu_i}\|_{p_i}, \quad (29)$$

holds with a positive constant $\widetilde{C}$, smaller than $\overline{C}_n(\mathbf{a},\mathbf{b},\mathbf{s},\mathbf{m})$. Considering this inequality with the functions

$$\widetilde{f}_i(x_i) = x_i^{\frac{\varepsilon}{\mu_i p_i}} \chi_{(0,1)}(x_i), \quad i = 1,\ldots,n,$$

where ε is sufficiently small number, its right-hand side reduces to

$$\widetilde{\overline{C}}\prod_{i=1}^{n}\|x_i^{-1/p_i}\widetilde{f_i}^{\mu_i}\|_{p_i} = \frac{\widetilde{\overline{C}}}{\varepsilon}. \tag{30}$$

Moreover, since

$$x_i > 1, \quad \left(\mathscr{H}_{\alpha_i,\beta_i}\widetilde{f_i}\right)(x_i) = x_i^{-\alpha_i}\int_0^{x_i}t^{\alpha_i-\beta_i-1}(x_i^{\beta_i}-t^{\beta_i})\widetilde{f_i}(t)dt$$

$$= x_i^{-\alpha_i}\int_0^1 t^{\alpha_i-\beta_i-1+\frac{\varepsilon}{\mu_i p_i}}(x_i^{\beta_i}-t^{\beta_i})dt > 0$$

and

$$0 < x_i \leqslant 1, \quad \left(\mathscr{H}_{\alpha_i,\beta_i}\widetilde{f_i}\right)(x_i) = x_i^{-\alpha_i}\int_0^{x_i}t^{\alpha_i-\beta_i-1}(x_i^{\beta_i}-t^{\beta_i})\widetilde{f_i}(t)dt$$

$$= x_i^{-\alpha_i}\int_0^{x_i}t^{\alpha_i-\beta_i-1+\frac{\varepsilon}{\mu_i p_i}}(x_i^{\beta_i}-t^{\beta_i})dt$$

$$= \frac{x_i^{\frac{\varepsilon}{\mu_i p_i}}}{\beta_i}B\left(\frac{\alpha_i}{\beta_i}+\frac{\varepsilon}{\mu_i p_i \beta_i}-1, 2\right)$$

$$= \frac{\beta_i x_i^{\frac{\varepsilon}{\mu_i p_i}}}{\left(\alpha_i+\frac{\varepsilon}{\mu_i p_i}\right)\left(\alpha_i-\beta_i+\frac{\varepsilon}{\mu_i p_i}\right)},$$

the left-hand side of (29), denoted here by L, reads

$$L = \int_{\mathbb{R}_+^n}K(\mathbf{x})\prod_{i=1}^{n}x_i^{s_i-1}\left(\mathscr{H}_{\alpha_i,\beta_i}\widetilde{f_i}\right)^{\mu_i}(x_i)d\mathbf{x}$$

$$= \prod_{i=1}^{n}\left(\frac{\beta_i}{\left(\alpha_i+\frac{\varepsilon}{\mu_i p_i}\right)\left(\alpha_i-\beta_i+\frac{\varepsilon}{\mu_i p_i}\right)}\right)^{\mu_i}\int_{(0,1]^n}K(\mathbf{x})\prod_{i=1}^{n}x_i^{s_i+\frac{\varepsilon}{p_i}-1}d\mathbf{x}.$$

Then, utilizing (19), (22) and (30), we conclude

$$\prod_{i=1}^{n}\left(\frac{\beta_i}{\left(\alpha_i+\frac{\varepsilon}{\mu_i p_i}\right)\left(\alpha_i-\beta_i+\frac{\varepsilon}{\mu_i p_i}\right)}\right)^{\mu_i}\int_{\mathbb{R}_+^{n-1}}K(\hat{\mathbf{u}}^1)\prod_{i=2}^{n}u_i^{s_i+\frac{\varepsilon}{p_i}-1}\hat{d}^1\mathbf{u}-\bigcirc(1) < \widetilde{\overline{C}}.$$

Hence, we have that $\overline{C}_n(\mathbf{a},\mathbf{b},\mathbf{s},\mathbf{m}) \leqslant \widetilde{\overline{C}}$ when $\varepsilon \to 0^+$, which is an obvious contradiction. This means that the constant $\overline{C}_n(\mathbf{a},\mathbf{b},\mathbf{s},\mathbf{m})$ is the best possible in (27). $\quad\square$

REMARK 2. In these theorems, if $\mathbf{a} = 0$, the proof above shows that one still obtained inequality by replacing the strict inequality $<$ in (15), (16), (25) and (26) all with $\leqslant$. The constants of these inequalities are best possible.

4. Some examples and applications

4.1. First example

A typical example of a homogeneous kernel with the negative degree of homogeneity is the function $K_1 : \mathbb{R}^n_+ \to \mathbb{R}$, defined by

$$K_1(\mathbf{x}) = \frac{1}{\left(\sum_{i=1}^n x_i\right)^\lambda}, \quad \lambda > 0.$$

Clearly, K_1 is a homogeneous function of degree $-\lambda$, and the constant factor $k_1(\mathbf{s}-\mathbf{1})$, can be expressed in terms of the usual Gamma function Γ. Namely,

$$k_1(\mathbf{s}-\mathbf{1}) = \int_{\mathbb{R}^{n-1}_+} \frac{\prod_{i=2}^n u_i^{s_i-1}}{\left(1 + \sum_{i=2}^n u_i\right)^\lambda} \hat{d}^1\mathbf{u} = \frac{\prod_{i=1}^n \Gamma(s_i)}{\Gamma(\lambda)},$$

which holds for $s_i > 0$, $i = 1, 2, \ldots n$.

4.2. Second example

Another example of a homogeneous kernel with degree $-\lambda$, is the function

$$K_2(\mathbf{x}) = \frac{1}{\max\{x_1^\lambda, \ldots, x_n^\lambda\}}, \quad \lambda > 0.$$

Hence,

$$k_1(\mathbf{s}-\mathbf{1}) = \int_{\mathbb{R}^{n-1}_+} \frac{\prod_{i=2}^n u_i^{s_i-1}}{\max\{1, u_2^\lambda, \ldots, u_n^\lambda\}} \hat{d}^1\mathbf{u} = \frac{\lambda}{s_1 \cdots s_n}.$$

4.3. Application

Let $n \geqslant 2$. Suppose that $F_1 : \mathbb{R}^n \to \mathbb{R}_+$ is a radial function and that $F_1^* : \mathbb{R}^n \to \mathbb{R}_+$ satisfies

$$\Delta F_1^*(x) = F_1(x).$$

Let us write $f_1^*(r) = F_1^*(r, 0, \ldots, 0)$ and $f_1(r) = F_1(r, 0, \ldots, 0)$. Then we have

$$(f_1^*)''(r) + \frac{n-1}{r}(f_1^*)'(r) = f_1(r).$$

Notice that

$$f_1(r) = \frac{1}{r^{n-1}} \frac{d}{dr}\left(r^{n-1} \frac{df_1^*}{dr}\right).$$

This implies

$$f_1^*(r) = \int_0^r \frac{1}{s^{n-1}}\left(\int_0^s t^{n-1} f_1(t)dt\right)ds$$

$$= \frac{1}{n-2}\int_0^r \frac{t(r^{n-2}-t^{n-2})}{r^{n-2}} f_1(t)dt$$

$$= \frac{r^2}{n-2}(\mathscr{H}_{n,n-2}f_1)(r).$$

If we define $F_i, i = 2, \ldots, n$ and so on likewise, then we obtain

$$\int_{\mathbb{R}^n_+} K(\mathbf{x}) \prod_{i=1}^{n} x_i^{s_i - 2\mu_i - 1} [f_i^*(x_i)]^{\mu_i} d\mathbf{x} \leqslant k_1(\mathbf{s} - \mathbf{1})(2n)^{-\Sigma_{i=1}^n \mu_i} \prod_{i=1}^{n} \|x_i^{-1/p_i} f_i^{\mu_i}\|_{p_i},$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{q_n(s_n - 1)} \left(\int_{\mathbb{R}^{n-1}_+} K(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{s_i - 2\mu_i - 1} [f_i^*(x_i)]^{\mu_i} \hat{d}^n \mathbf{x} \right)^{q_n} dx_n \right]^{1/q_n}$$

$$\leqslant k_1(\mathbf{s} - \mathbf{1})(2n)^{-\Sigma_{i=1}^{n-1} \mu_i} \prod_{i=1}^{n-1} \|x_i^{-1/p_i} f_i^{\mu_i}\|_{p_i}.$$

In particular, by letting $\mu_i = 1$, $i = 1 \ldots, n$, we obtain

$$\int_{\mathbb{R}^n_+} K(\mathbf{x}) \prod_{i=1}^{n} x_i^{s_i - 3} f_i^*(x_i) d\mathbf{x} \leqslant k_1(\mathbf{s} - \mathbf{1})(2n)^{-n} \prod_{i=1}^{n} \|x_i^{-1/p_i} f_i^{\mu_i}\|_{p_i}$$

$$\leqslant \frac{k_1(\mathbf{s} - \mathbf{1})(2n)^{-n}}{|S^n|} \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} |\mathbf{x_i}|^{-n} F^{p_i}(\mathbf{x_i}) d\mathbf{x_i} \right)^{\frac{1}{p_i}}$$

similarly

$$\left[\int_{\mathbb{R}_+} x_n^{q_n(s_n - 1)} \left(\int_{\mathbb{R}^{n-1}_+} K(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{s_i - 3} f_i^*(x_i) \hat{d}^n \mathbf{x} \right)^{q_n} dx_n \right]^{1/q_n}$$

$$\leqslant \frac{k_1(\mathbf{s} - \mathbf{1})(2n)^{-n+1}}{|S^n|^{1/q_n}} \prod_{i=1}^{n-1} \left(\int_{\mathbb{R}^n} |\mathbf{x_i}|^{-n} F^{p_i}(\mathbf{x_i}) d\mathbf{x_i} \right)^{\frac{1}{p_i}}.$$

REFERENCES

[1] V. Adiyasuren, Ts. Batbold, *Some new inequalities of Hardy-Hilbert type*, Mongolian Math. J. **15**, 12–19 (2011)

[2] V. Adiyasuren, Ts. Batbold, *Some new inequalities similar to Hilbert-type integral inequality with a homogeneous kernel*, J. Math. Inequal. **6** (2012), 183–193.

[3] V. Adiyasuren, Ts. Batbold, *New multiple inequality similar to Hardy-Hilbert's integral inequality*, Tamsui Oxf. J. Inf. Math. Sci. **28** (2012), 281–292.

[4] V. Adiyasuren, Ts. Batbold, M. Krnić, *Hilbert-type inequalities including some operators, the best possible constants and applications: a survey*, in "Topics in Mathematical Analysis and Applications", L. Toth and Th. M. Rassias, Eds., Springer, 2014, pp. 17–52.

[5] I. Brnetić, M. Krnić, J. Pečarić, *Multiple Hilbert and Hardy-Hilbert inequalities with non-conjugate parameters*, Bull. Austral. Math. Soc. **71** (2005), 447–457.

[6] N. Das, S. Sahoo, *New inequalities similar to Hardy-Hilbert's inequality*, Turk. J. Math. **34** (2010), 153–165.

[7] G. D. Handley, *Hilbert and Hardy type inequalities*, thesis, The University of Melbourne, 2006.

[8] G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press Cambridge, 1952.

[9] M. Krnić, *On the multidimensional Hilbert-type inequalities involving the Hardy operator*, Filomat 26:4 (2012), 845–857.

[10] M. KRNIĆ, J. PEČARIĆ, I. PERIĆ, P. VUKOVIĆ, *Recent Advances in Hilbert-type Inequalities*, Element, Zagreb, 2012.

[11] A. KUFNER, L. MALIGRANDA, L. E. PERSSON, *The Hardy inequality – About its history and some related results*, Vydavatelský servis, Pilsen, 2007.

[12] X. LIU, B. YANG, *On a new Hilbert-Hardy-type integral operator and applications*, J. Inequal. Appl. **2010** (Article ID 812636), 10 pp, 2010.

[13] D. S. MITRINOVIC, J. E. PECARIC, A. M. FINK, *Classical and new inequalities in analysis*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.

[14] A. R. MOAZZEN, R. LASHKARIPOUR, *Some new extensions of Hardy's inequality*, Int. J. Nonlinear Anal. Appl. **5** (1) (2014), 98–109.

[15] I. PERIĆ, P. VUKOVIĆ, *Multiple Hilbert's type inequalities with a homogeneous kernel*, Banach J. Math. Anal. **5** (2011), 33–43.

[16] P. VUKOVIĆ, *Note on Hilbert-type inequalities*, Turk. J. Math. **36** (2012), 253–262.

[17] B. YANG, I. BRNETIĆ, M. KRNIĆ, J. PEČARIĆ, *Generalization of Hilbert and Hardy-Hilbert integral inequalities*, Math. Inequal. Appl. **8** (2005), 259–272.

[18] B. YANG, T. M. RASSIAS, *On a Hilbert-type integral inequality in the subinterval and its operator expression*, Banach J. Math. Anal. **4** (2010), 100–110.

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