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CHARACTERIZATIONS OF INNER PRODUCT SPACES BY INEQUALITIES INVOLVING SEMI–INNER PRODUCT

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Abstract. Using the notion of semi-inner product in normed spaces, in this paper we provide some new characterizations of inner product spaces. We answer a question posed by Dragomir, whether the property (N) is characteristic for inner product spaces. We show that the space X is of (N)-type if and only if the norm in X comes from an inner product.

1. Introduction

Let $(X,\|\cdot\|)$ be a normed space over $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$. G. Lumer [6] and J. R. Giles [4] proved that in a normed space X there always exists a mapping $[\cdot|\cdot]:X\times X\to\mathbb{K}$ satisfying the following properties:

(sip1) $\forall_{x,y,z\in X}\ \forall_{\alpha,\beta\in\mathbb{K}}: \quad [\alpha x+\beta y|z]=\alpha[x|z]+\beta[y|z]$;
(sip2) $\forall_{x,y\in X}\ \forall_{\alpha\in\mathbb{K}}: \quad [x|\alpha y]=\overline{\alpha}[x|y]$;
(sip3) $\forall_{x,y\in X}: \quad |[x|y]|\leqslant\|x\|\cdot\|y\|$;
(sip4) $\forall_{x\in X}: \quad [x|x]=\|x\|^2$.

Such a mapping is called a *semi–inner–product* (s.i.p.) in X (generating the norm $\|\cdot\|$). There may exist infinitely many different semi-inner-products in X. There is a unique one if and only if X is *smooth* (i.e., there is a unique supporting hyperplane at each point of the unit sphere S). If X is an inner product space, the only s.i.p. on X is the inner-product itself.

The *semi–orthogonality* of vectors x and y in X (with respect to a given semi-inner product), is defined as follows:

$$x\perp_s y \quad :\Leftrightarrow \quad [y|x]=0.$$

Of course, in an inner product space we have $\perp_s=\perp$.

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2. Open problem

By [3, Proposition 4, p.21] "a normed space X is smooth if and only if there exists a unique Lumer-Giles semi-inner product which generates the norm".

DEFINITION 1. [3, Definition 28, p. 165] Assume that $(X, \|\cdot\|)$ is a smooth normed linear space. It is said to be of *(N)-type* if the s.i.p. $[\cdot|\cdot]$ that generates the norm satisfies the condition:

$$\forall_{x,y,z\in X}: \quad |[x|y+z]| \leqslant |[x|y]| + |[x|z]|. \tag{1}$$

It is obvious that any inner product space is a smooth normed space of (N)-type.

PROBLEM 1. [3, Remark 22, p. 165] Is it true that a normed space of (N)-type has to be inner product space?

We will solve this open problem. Namely, we will give a characterization of inner product spaces. In fact, we will prove that Problem 1 can be strengthen. Indeed, it is not necessary to assume that X is smooth.

3. Main results

In this section, we treat the Problem 1. Thus, we derive a new condition characterizing inner product spaces. The following theorem will be useful to derive the main result.

THEOREM 1. [5], [2] *Let X be a normed space such that* $\dim X \geqslant 3$. *The normed linear space X is an inner product space if and only if for each two-dimensional subspace X_1 of X there exists a linear operator $P\colon X \to X_1$ with the following properties:*
(i) $P = P\circ P$;
(ii) $\|P\| = 1$.

For real spaces it has been proved by Kakutani [5]. Bohnenblust [2] extended it to complex spaces.

Next we prove the main result of this section.

THEOREM 2. *Let X be a normed space such that* $\dim X = 3$ *and let $[\cdot|\cdot]$ be a given semi-inner product in X. The space X satisfies (1) if and only if the norm in X comes from an inner product.*

Proof. Assume that X satisfies (1). Fix arbitrarily a subspace $M \subset X$ and suppose that $\dim M = 2$. Fix arbitrarily two linearly independent vectors $a, b \in M$. Using (sip1) and (sip4) we have for $c := -\frac{[b|a]}{\|a\|^2}a + b$

$$[c|a] \overset{(\text{sip1})}{=} -\frac{[b|a]}{\|a\|^2}[a|a] + [b|a] \overset{(\text{sip4})}{=} -\frac{[b|a]}{\|a\|^2}\|a\|^2 + [b|a] = 0, \quad \text{i.e.,} \quad a \perp_s c. \tag{2}$$

Fix $p \notin M$. Define $d := -\frac{[p|a]}{\|a\|^2}a + p$. Applying again (sip1) and (sip4) we get

$$[d|a] \overset{\text{(sip1)}}{=} -\frac{[p|a]}{\|a\|^2}[a|a] + [p|a] \overset{\text{(sip4)}}{=} -\frac{[p|a]}{\|a\|^2}\|a\|^2 + [p|a] = 0, \quad \text{i.e.,} \quad a \perp_s d. \tag{3}$$

It is easy to see that $\operatorname{span}\{a,c\} = M$ and $d \notin M$. Therefore $\operatorname{span}\{a,c,d\} = X$. Define $u := -\frac{[d|c]}{\|c\|^2}c + d$. In a similar way we obtain

$$[u|c] = 0, \quad \text{i.e.,} \quad c \perp_s u. \tag{4}$$

Moreover,

$$[u|a] = \left[-\frac{[d|c]}{\|c\|^2}c + d\middle|a\right] = -\frac{[d|c]}{\|c\|^2}[c|a] + [d|a] \overset{(2),(3)}{=} 0, \quad \text{i.e.,} \qquad a \perp_s u. \tag{5}$$

We will show that for any $\gamma \in \mathbb{K}$ we have $M \perp_s \gamma u$. Fix $m \in M$ and $\gamma \in \mathbb{K}$. It follows that, for some $\alpha, \beta \in \mathbb{K}$, one has $m = \alpha a + \beta c$. Using (1), we get

$$\begin{aligned}
|\,[\gamma u|m]\,| &= |\,[\gamma u|\alpha a + \beta c]\,| \overset{(1)}{\leqslant} |\,[\gamma u|\alpha a]\,| + |\,[\gamma u|\beta c]\,| \overset{\text{(sip1)}}{=} \\
&= |\gamma| \cdot (|\,[u|\alpha a]\,| + |\,[u|\beta c]\,|) \overset{\text{(sip2)}}{=} |\gamma| \cdot \left(|\overline{\alpha} \cdot [u|a]\,| + |\overline{\beta} \cdot [u|c]\,|\right) \overset{(4),(5)}{=} \\
&= |\gamma| \cdot \left(|\overline{\alpha} \cdot 0| + |\overline{\beta} \cdot 0|\right) = 0,
\end{aligned}$$

hence $M \perp_s \gamma u$.

It is easy to see that $u \notin M$. Therefore $\operatorname{span}\{a,c,u\} = X$. Define a linear mapping $P\colon X \to X$ by $P(\alpha_1 a + \alpha_2 c + \alpha_3 u) := \alpha_1 a + \alpha_2 c$. It is easy to see that $P(X) = M$ and $P = P \circ P$. Thus P is a projection from X onto M. Applying properties $M \perp_s \gamma u$ and $P(X) = M$ we obtain

$$\forall_{\gamma \in \mathbb{K}} : P(X) \perp_s \gamma u. \tag{6}$$

Since $\|P\| = \|P \circ P\| \leqslant \|P\| \cdot \|P\|$, we get $1 \leqslant \|P\|$. Now we prove the converse, i.e., $\|P\| \leqslant 1$. Fix $x \in X$. It follows that, for some $\alpha, \beta, \gamma \in \mathbb{K}$, one has $x = \alpha a + \beta c + \gamma u$. Since $[Px|Px] \overset{\text{(sip4)}}{=} \|Px\|^2 \geqslant 0$, we get $[Px|Px] = |\,[Px|Px]\,|$. Thus we have

$$\begin{aligned}
\|Px\|^2 &= [Px|Px] = |\,[Px|Px]\,| = |\,[Px|Px] + 0| \overset{(6)}{=} \\
&= |\,[Px|Px] + [\gamma u|Px]\,| \overset{\text{(sip1)}}{=} |\,[P(x) + \gamma u|Px]\,| \overset{\text{(sip3)}}{\leqslant} \\
&\leqslant \|P(x) + \gamma u\| \cdot \|P(x)\| = \|\alpha a + \beta c + \gamma u\| \cdot \|Px\| = \\
&= \|x\| \cdot \|Px\|,
\end{aligned}$$

whence $\|Px\| \leqslant \|x\|$. Thus $\|P\| \leqslant 1$ and finally $\|P\| = 1$.

From Theorem 1, the norm in X comes from an inner product. The converse implication is obvious. $\square$

Now, let us prove the general case.

THEOREM 3. *Let X be a normed space such that $\dim X \geqslant 3$ and let $[\cdot|\cdot]$ be a given semi-inner product in X. Then, the following conditions are equivalent:*

(a) $\forall_{x,y,z\in X}: \quad |[x|y+z]| \leqslant |[x|y]| + |[x|z]|$;

(b) $\forall_{x,y,z\in X}: \quad [x|y+z] = [x|y] + [x|z]$;

(c) *The norm in X comes from an inner product.*

Proof. Implications (c)$\Rightarrow$(b)$\Rightarrow$(a) are obvious. We show (a)$\Rightarrow$(c). Assume that X satisfies (a). We show that the parallelogram law holds. Fix arbitrarily two vectors $x,y\in X$. Fix a subspace $Y\subset X$ such that $x,y\in Y$ and $\dim Y = 3$. We define the norm $\|\cdot\|_Y : Y \to \mathbb{R}$ by $\|\cdot\|_Y := \|\cdot\|$. According to Theorem 2, the norm $\|\cdot\|_Y$ satisfies the parallelogram law. Thus

$$\|x+y\|^2 + \|x-y\|^2 = \|x+y\|_Y^2 + \|x-y\|_Y^2 = 2\|x\|_Y^2 + 2\|y\|_Y^2 = 2\|x\|^2 + 2\|y\|^2.$$

We have proved that the parallelogram law holds. Therefore, the norm has to come from an inner product. $\square$

4. Norm derivatives

From now on we assume that the considered normed spaces are real. Let $(X,\|\cdot\|)$ be a real normed space. We define two mappings $\rho'_+, \rho'_- : X\times X \to \mathbb{R}$:

$$\rho'_\pm(x,y) := \lim_{t\to 0^\pm} \frac{\|x+ty\|^2 - \|x\|^2}{2t} = \|x\| \cdot \lim_{t\to 0^\pm} \frac{\|x+ty\| - \|x\|}{t}.$$

This mappings are called *norm derivatives*. Now, we recall their useful properties (the proofs can be found in [1] and [3]):

(nd1) $\forall_{x,y\in X}\ \forall_{\alpha\in\mathbb{R}}: \quad \rho'_\pm(x,\alpha x+y) = \alpha\|x\|^2 + \rho'_\pm(x,y)$;

(nd2) $\forall_{x,y\in X}\ \forall_{\alpha\geqslant 0}: \quad \rho'_\pm(\alpha x,y) = \alpha\rho'_\pm(x,y) = \rho'_\pm(x,\alpha y)$;

(nd2') $\forall_{x,y\in X}\ \forall_{\alpha< 0}: \quad \rho'_\pm(\alpha x,y) = \alpha\rho'_\mp(x,y) = \rho'_\pm(x,\alpha y)$;

(nd3) $\forall_{x,y\in X}: \quad |\rho'_\pm(x,y)| \leqslant \|x\|\cdot\|y\|$;

(nd4) $\forall_{x\in X}: \quad \rho'_\pm(x,x) = \|x\|^2$;

Moreover, the mappings ρ'_+, ρ'_- are continuous with respect to the second variable, but not necessarily with respect to the first one.

Now, fix the semi-inner product $[\cdot|\cdot]$. Then,

$$\forall_{x,y\in X} \quad \rho'_\pm(x,y) = \lim_{t\to 0^\pm} [y|x+ty]. \tag{7}$$

It is known that, X is smooth if and only if $\rho'_+(x,y) = \rho'_-(x,y) = [y|x]$ for all $x,y\in X$.

The following mapping $\langle\cdot|\cdot\rangle_g : X\times X \to \mathbb{R}$ was introduced by Miličić [7]:

$$\langle y|x\rangle_g := \tfrac{1}{2}\left(\rho'_+(x,y) + \rho'_-(x,y)\right)$$

and is called a *M-semi-inner product* (briefly *M-s.i.p.*). From the above properties of the mappings ρ'_+, ρ'_- we get:

(Msip1) $\forall_{x,y\in X}\ \forall_{\alpha\in\mathbb{R}}: \quad \langle\alpha x+y|x\rangle_g = \alpha\|x\|^2 + \langle y|x\rangle_g$;

(Msip2) $\forall_{x,y\in X}\ \forall_{\alpha\in\mathbb{R}}: \quad \langle \alpha x|y\rangle_g = \alpha\langle x|y\rangle_g = \langle x|\alpha y\rangle_g$;
(Msip3) $\forall_{x,y\in X}: \quad |\langle x|y\rangle_g| \leqslant \|x\|\cdot\|y\|$;
(Msip4) $\forall_{x\in X}: \quad \langle x|x\rangle_g = \|x\|^2$;
In a similar way as earlier, we introduce ρ*-orthogonality*:

$$x\perp_\rho y \ :\Leftrightarrow\ \langle y|x\rangle_g = 0.$$

If $(X,\langle\cdot|\cdot\rangle)$ is an inner product space, then $\langle y|x\rangle = [y|x] = \langle y|x\rangle_g$ for arbitrary $x,y\in X$. Hence we have $\perp = \perp_s = \perp_\rho$.

5. Semi-smooth spaces

A normed space X is called *semi-smooth* if the M-semi-inner-product is additive with respect to the first variable, i.e.,

$$\forall_{x,y\in X} \quad \langle x+y|z\rangle_g = \langle x|z\rangle_g + \langle y|z\rangle_g.$$

Each smooth space is semi-smooth in the above sense but not conversely (l^1 is a suitable example). If X is a real semi-smooth space, then the M-semi-inner-product is a real semi-inner-product in sense of Lumer-Giles.

THEOREM 4. *Assume that* $(X,\|\cdot\|)$ *is a semi-smooth normed linear space such that* $\dim X \geqslant 3$. *Then, the following conditions are equivalent:*
(a) $\forall_{x,y,z\in X}: \ |\langle x|y+z\rangle_g| \leqslant |\langle x|y\rangle_g| + |\langle x|z\rangle_g|$;
(b) $\forall_{x,y,z\in X}: \ \langle x|y+z\rangle_g = \langle x|y\rangle_g + \langle x|z\rangle_g$;
(c) *The norm in* X *comes from an inner product.*

Proof. The space X is semi-smooth. Therefore, the M-semi-inner-product is a real semi-inner-product. Now by applying Theorem 3 we arrive at the desired assertion. □

6. Concluding remarks and problems

Let us now consider the conditions:

(A) $\forall_{x,y,z\in X}: \ |\rho'_+(x+y,z)| \leqslant |\rho'_+(x,z)| + |\rho'_+(y,z)|$;

(B) $\forall_{x,y,z\in X}: \ |\rho'_-(x+y,z)| \leqslant |\rho'_-(x,z)| + |\rho'_-(y,z)|$;

The latter two conditions are equivalent. Indeed, suppose that (A) holds. Fix arbitrarily three vectors x,y,z. Then we have

$$\begin{aligned}|\rho'_-(x+y,z)| &\overset{\text{(nd2')}}{=} |-\rho'_+((-x)+(-y),z)| = |\rho'_+((-x)+(-y),z)| \overset{\text{(A)}}{\leqslant}\\ &\leqslant |\rho'_+(-x,z)| + |\rho'_+(-y,z)| \overset{\text{(nd2')}}{=} |-\rho'_-(x,z)| + |-\rho'_-(y,z)| =\\ &= |\rho'_-(x,z)| + |\rho'_-(y,z)|.\end{aligned}$$

The reverse implication can be proved analogously.

In the case of a smooth space X we have $\rho'_+(\cdot,\diamond)=\rho'_-(\cdot,\diamond)=[\diamond|\cdot]$. Thus, if X is smooth, then (1)$\Leftrightarrow$(A)$\Leftrightarrow$(B). According to Theorem 3, both (A) and (B) characterizes X as an inner product space.

The case of non-smooth spaces remains an open problem. But on the other hand, the following assertion holds true.

THEOREM 5. *For an arbitrary real normed space X we have* (1)$\Rightarrow$(A) *and* (1)$\Rightarrow$(B).

Proof. Fix arbitrarily three vectors $x,y,z\in X$. Then we have

$$\begin{aligned}|\rho'_+(x+y,z)| &\overset{(7)}{=} \left|\lim_{t\to 0^+}[z|x+y+tz]\right| = \lim_{t\to 0^+}|[z|x+y+tz]| = \\ &= \lim_{t\to 0^+}\left|\left[z|x+\frac{1}{2}tz+y+\frac{1}{2}tz\right]\right| \overset{(1)}{\leqslant} \\ &\leqslant \lim_{t\to 0^+}\left(\left|\left[z|x+\frac{1}{2}tz\right]\right|+\left|\left[z|x+\frac{1}{2}tz\right]\right|\right) = \\ &= \lim_{t\to 0^+}\left|\left[z|x+\frac{1}{2}tz\right]\right| + \lim_{t\to 0^+}\left|\left[z|x+\frac{1}{2}tz\right]\right|.\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}\left|\rho'_+(x,z)\right|+\left|\rho'_+(y,z)\right| &\overset{\text{(nd2)}}{=} \left|2\rho'_+\left(x,\frac{1}{2}z\right)\right|+\left|2\rho'_+\left(y,\frac{1}{2}z\right)\right| \overset{(7)}{=} \\ &= \left|2\lim_{t\to 0^+}\left[\frac{1}{2}z|x+t\cdot\frac{1}{2}z\right]\right|+\left|2\lim_{t\to 0^+}\left[\frac{1}{2}z|y+t\cdot\frac{1}{2}z\right]\right| = \\ &= \left|\lim_{t\to 0^+}\left[z|x+\frac{1}{2}tz\right]\right|+\left|\lim_{t\to 0^+}\left[z|y+\frac{1}{2}tz\right]\right| = \\ &= \lim_{t\to 0^+}\left|\left[z|x+\frac{1}{2}tz\right]\right| + \lim_{t\to 0^+}\left|\left[z|x+\frac{1}{2}tz\right]\right|.\end{aligned}$$

Finally, we obtain $|\rho'_+(x+y,z)|\leqslant\left|\rho'_+(x,z)\right|+\left|\rho'_+(y,z)\right|$. In a similar way one can prove (1)$\Rightarrow$(B). $\square$

The problem arises whether the reverse is true.

PROBLEM 2. If the condition (A) holds, does $\|\cdot\|$ derive from an inner product?

There is the following natural problem connected with Theorems 2, 3.

PROBLEM 3. Is it necessary to assume that $\dim X\geqslant 3$? The two-dimensional case remains an open problem.

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