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A NOTE ON A RESULT OF I. GUSIĆ ON TWO INEQUALITIES IN LATTICE–ORDERED GROUPS

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Abstract. We complement some of results of Ivica Gusić from 1998 concerning Maligranda-Orlicz inequality in lattice-ordered groups.

A φ-function is a non-decreasing continuous function $f\colon [0,+\infty) \to [0,+\infty)$ such that $f(u) = 0$ if and only if $u = 0$ and $\lim_{t\to+\infty} f(t) = +\infty$ (we denote by $[0,+\infty)$ the closed halfline of nonnegative real numbers).

Lech Maligranda and Władysław Orlicz proved in [5] that if f is a convex φ-function, n is a positive integer, a_k are arbitrary non-negative numbers for $k = 1,\dots,n$ and $a_0 = 0$, then

$$\sum_{k=1}^{n} |f(a_k) - f(a_{k-1})| \leqslant f\left(\sum_{k=1}^{n} |a_k - a_{k-1}|\right). \tag{1}$$

Inequality (1) is called Maligranda-Orlicz inequality. The original proof of (1) goes by induction with respect to n. Moreover, the representation of φ-functions:

$$f(t) = \int_0^t p(s)ds, \quad t \geqslant 0,$$

where p is a non-negative and non-decreasing function has been utilized. The crucial role in their proof plays the following inequality (denoted by $(*)$ in [5]):

$$f(A) + |f(a_n) - f(a_{n-1})| \leqslant f(A + |a_n - a_{n-1}|), \tag{$*$}$$

where $A = \sum_{k=1}^{n} |a_k - a_{k-1}|$.

Josip Pečarić and Ivica Gusić in [6] extended Maligranda-Orlicz inequality for functions defined on a set of the form $[0,\beta_1] \times \cdots \times [0,\beta_r]$, having increasing increments and attaining values in $[0,+\infty)$. Then, Ivica Gusić in [3] proved this inequality for mappings f acting between positive cones of lattice-ordered groups, having increasing increments and vanishing at zero. Moreover, he studied the converse inequality for

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increasing functions with decreasing increments. He considered inequalities (2) and (3) below with three independent variables, which are analogous to $(*)$ above, and then he proved the general case inductively.

The notion of lattice-ordered groups (ℓ-groups for short) goes back to Garrett Birkhoff [2]. An *ℓ-group* is a group which is ordered by a lattice order which is compatible with the group operation. In particular, the notion of *absolute value* $|\cdot|$ is well defined for elements of an ℓ-group by $|a| = a \vee -a$.

If G is an ℓ-group, then G^+ denotes the set of all elements of G which are greater or equal to 0 (we will use the additive notation in groups, even if no commutativity is assumed). A group G is termed *uniquely divisible by* 2 if the function $G \ni a \mapsto 2a = a + a \in G$ is bijective. Then, the value $\frac{a}{2}$ is well-defined for every $a \in G$. A partially ordered group is called *Archimedean*, if condition $na \leqslant b$ satisfied for all $n \in \mathbb{N}$ implies $a \leqslant 0$ for every $a, b \in G$.

If G and H are partially ordered groups, then a map $f\colon G \to H$ is called *increasing*, if $f(a) \leqslant f(b)$ in H whenever $a \leqslant b$ in G. A map $f\colon G \to H$ is *decreasing* if $-f$ is increasing. If $f\colon G \to H$, then the *difference operator* Δ_h is defined by

$$\Delta_h f(x) = f(x+h) - f(x), \quad x, h \in G.$$

Next, $f\colon G \to H$ is said to have *increasing increments* [*decreasing increments*, respectively] if for every $h \in G^+$ the map $G \ni x \mapsto \Delta_h f(x)$ is increasing [decreasing, respectively]. Function $f\colon G \to H$ is *subadditive*, if

$$f(x+y) \leqslant f(x) + f(y), \quad x, y \in G.$$

If G and H are uniquely divisible by 2 and $D \subset G$ is a *midpoint-convex set* (i.e. $\frac{1}{2}(x+y) \in D$ whenever $x, y \in D$), then $f\colon G \to H$ is called *midpoint-convex*, if

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2}, \quad x, y \in D.$$

A function $f\colon G \to H$ is called *midpoint-concave*, if $-f$ is midpoint-convex.

Every function with increasing increments is midpoint-convex and every function with decreasing increments is midpoint-concave (see Marek Kuczma [4, page 430]). However, the converse is not true. If $a\colon \mathbb{R} \to \mathbb{R}$ is a discontinuous additive mapping, then its absolute value $|a|$ is a midpoint convex map, but does not have all increments increasing. Moreover, a itself provides an example which shows that a function with increasing increments does not need to be continuous. The Bernstein-Doetsch theorem says that a real midpoint-convex function defined on an open and convex subset of a linear-topological space which is bounded from above on some set with non-empty interior is continuous and convex, see [4, Chapter 6.4].

Throughout the paper we will keep assuming that G and H are ℓ-groups. We are interested in the following two functional inequalities:

$$f(a) + |f(b) - f(c)| \leqslant f(a + |b - c|), \tag{2}$$

$$f(a) + |f(b) - f(c)| \geqslant f(a + |b - c|), \tag{3}$$

which have been previously studied by Gusić in [3]. In our first result we show the necessity of assumptions of [3, Lemma 1].

PROPOSITION 1. *Assume that* $f\colon G^+ \to H$ *is an arbitrary function. Then* f *satisfies inequality* (2) *for all* $a,b,c \in G^+$ *such that* $a \geqslant c$ *if and only if the following conditions are satisfied:*

(a1) $f(a) - f(0) \geqslant 0$ *for all* $a \in G^+$,

(b1) f *has increasing increments.*

Proof. The "if" part has been proved by Gusić in [3, Lemma 1] under an additional assumption that $f(0) = 0$. Note that f satisfies (2) if and only if $f - f(0)$ satisfies (2), and also the condition (b1) remains untouched, therefore this assumption can be omitted in Gusić's result.

We will prove the "only if" part. Apply (2) for $a = c = 0$. We get

$$f(0) + |f(b) - f(0)| \leqslant f(b),$$

which means that $f(b) - f(0) \geqslant 0$ for all $b \in G^+$.

Next, we will show that f is increasing. Fix some $a_1, a_2 \in G^+$ such that $a_1 \leqslant a_2$. Put $a = a_1$, $b = a_2 - a_1$ and let $c = 0$. Note that $a,b,c \in G^+$ and $a \geqslant c$. From (2) we derive that

$$f(a_1) \leqslant f(a_1) + |f(a_2 - a_1) - f(0)| \leqslant f(a_1 + a_2 - a_1) = f(a_2),$$

as desired.

To prove (b1) fix arbitrary $a_1, a_2 \in G^+$ such that $a_1 \leqslant a_2$ and some $h \in G^+$. Put $a = a_2$, $b = a_1 + h$ and $c = a_1$. Note that $a,b,c \in G^+$ and $a \geqslant c$. Therefore, one can apply inequality (2) for these points. Using the facts that $b - c = h \geqslant 0$ and f is increasing we obtain

$$\begin{aligned}\Delta_h f(a_1) &= f(a_1 + h) - f(a_1) = f(b) - f(c) = |f(b) - f(c)| \\ &\leqslant f(a + |b - c|) - f(a) = f(a_2 + h) - f(a_2) = \Delta_h f(a_2).\end{aligned}$$

Therefore, f has increasing increments, which completes the proof. □

Concerning functional inequality (3), the situation is much less comfortable. Gusić proved in [3, Lemma 2] that if G is linearly ordered, then inequality (3) is satisfied for all $a,b,c \in G^+$ such that $a \geqslant c$ by every function $f\colon G^+ \to H^+$ which fulfils the following assumptions:

(a2) $f(a) \in H^+$ for all $a \in G$,

(b2) $f(0) = 0$,

(c2) f is increasing,

(d2) f has decreasing increments.

One can easily observe that f satisfies (3) if and only if $f - f(0)$ does so. Therefore, the condition (b2) can be weakened to:

(b2') $f(a) \geqslant f(0)$.

In the next examples we show that there exist solutions of inequality (3) which fail to satisfy one or more of the remaining conditions.

EXAMPLE 1. Every decreasing function f, defined on the whole group G or on its arbitrary subset, satisfies (3). Indeed, for arbitrary $a,b,c, \in G^+$ one has

$$f(a+|b-c|) \leqslant f(a) \leqslant f(a) + |f(b) - f(c)|.$$

Therefore, (c2) does not follow from inequality (3), even if the remaining conditions are fulfilled.

EXAMPLE 2. Assume that G is an arbitrary ℓ-group and $H = \mathbb{R}$ with the standard order and addition. Then every function $f\colon G^+ \to [1,2]$ is a solution of inequality (3). Indeed, it is easy to see that for arbitrary $a,b,c, \in G^+$ we have $f(b) - f(a) \in [-1,1]$ and therefore

$$f(a+|b-c|) \leqslant 2 = 1+1 \leqslant f(a) + |f(b) - f(c)|.$$

This example shows that there exist solutions of (3) which satisfy (a2), do not satisfy neither (b2) nor (b2') and may but do not need to satisfy (c2) and (d2).

PROPOSITION 2. *Assume that* $f\colon G^+ \to H^+$ *satisfies condition* (c2) *and inequality* (3) *for all* $a,b,c \in G^+$ *such that* $a \geqslant c$. *Then* f *is subadditive.*

Proof. Put $c = 0$ in (3) and use the fact that $f \geqslant 0$ to get

$$f(a) + f(b) = f(a) + |f(b)| \geqslant f(a+|b|) = f(a+b)$$

for all $a,b \in G^+$. □

EXAMPLE 3. The converse of Proposition 2 is not true in general. There exist subadditive functions $f\colon G^+ \to H^+$ which does not satisfy inequality (3), even if conditions (b2) and (b2') are fulfilled. To see this one can take $G = H = \mathbb{R}$ with standard order and addition and $f\colon G^+ \to H^+$ given by

$$f(x) = \lceil x \rceil = \min\{k \in \mathbb{Z} : x \leqslant k\}.$$

Clearly, f is subadditive and $f(0) = 0$, whereas (3) is not satisfied (to visualize this one can for example take points $a = 2$, $b = 1$ and $c = \frac{1}{2}$).

In the next result we will show that (c2) implies (d2).

PROPOSITION 3. *Assume that* $f\colon G^+ \to H^+$ *satisfies condition* (c2) *and inequality* (3) *for all* $a,b,c \in G^+$ *such that* $a \geqslant c$. *Then* f *satisfies condition* (d2).

Proof. Fix some $x,y,h \in G^+$ and assume that $x \leqslant y$. We will apply (3) with the substitutions $a = x+h$, $b = y$ and $c = x$. We get

$$\begin{aligned}\Delta_h f(x) &= f(x+h) - f(x) = f(a) - f(c)\\ &\geqslant f(a+|b-c|) - |f(b)-f(c)| - f(c)\\ &= f(y+h) - f(y) + f(x) - f(x) = \Delta_h f(y).\end{aligned}$$

Above we applied the monotonicity of f. Observe also that condition $a \geqslant c$ was satisfied, since $x \leqslant y$. □

On the half-line of nonnegative reals we can apply the Bernstein-Doetsch theorem to get that f is continuous on $(0,+\infty)$ and concave on $[0,+\infty)$.

COROLLARY 1. *If $f\colon [0,+\infty) \to [0,+\infty)$ satisfies condition* (c2) *and inequality* (3) *for all $a,b,c \in [0,+\infty)$ such that $a \geqslant c$, then f is concave.*

Proof. From Proposition 3 we get that f is midpoint-concave on $[0,+\infty)$. Clearly, f is bounded from below by assumption. The Bernstein-Doetsch theorem guarantees that f is concave and continuous on the open half-line $(0,+\infty)$. Note that f needs not to be continuous at 0, but clearly, by (c2) f is concave on the closed half-line $[0,+\infty)$. □

Next, we will prove that solutions of (3) defined on a linearly ordered Archimedean group are either strictly increasing or are constant on elements large enough.

PROPOSITION 4. *Assume that G^+ is linearly ordered and Archimedean and $f\colon G^+ \to H^+$ satisfies inequality* (3) *for all $a,b,c \in G^+$ such that $a \geqslant c$ jointly with condition* (c2)*. Then, either f is strictly increasing, or there exists some $x \in G^+$ such that f is constant on the set $\{y \in G : y \geqslant x\}$.*

Proof. Assume that $f(b) = f(c)$ for some $b,c \in G^+$. Then, from (3) we obtain

$$f(a) = f(a) + |f(b) - f(c)| \geqslant f(a+|b-c|)$$

for all $a \geqslant c$. On the other hand, since f is increasing, then $f(a) = f(a+|b-c|)$. By induction we obtain $f(a) = f(a+n|b-c|)$ for all $a \geqslant c$ and all $n \in \mathbb{N}$. Now, fix arbitrary element $u \geqslant a$. Since G is Archimedean and linearly ordered, then we can find some $n \in \mathbb{N}$ such that $u \leqslant n|b-c|$. Now, from the fact that f is increasing we get $f(a) \leqslant f(u) \leqslant f(a+n|b-c|)$. Therefore, f is constant on the set $\{y \in G^+ : y \geqslant x\}$ for some $x \in G^+$. □

The following example was suggested by the Referee, and shows that in the preceding statement one cannot take $x = 0$.

EXAMPLE 4. Function $f\colon [0,+\infty) \to [0,+\infty)$ given by

$$f(x) = \begin{cases} x, & 0 \leqslant x \leqslant 1,\\ 1, & 1 < x,\end{cases}$$

satisfies all assumptions of Proposition 4, but is neither constant nor strictly increasing.

We will terminate the paper by a result on a functional equation corresponding to inequalities (2) and (3) for mappings acting between positive cones of vector lattices. For the definition of a vector lattice and related notions the reader is referred to the monograph of Abramovich and Aliprantis [1].

PROPOSITION 5. *Assume that* G *and* H *are vector lattices, the cone* G^+ *is generating in* G *and* H *is Archimedean. If* $f\colon G^+ \to H$ *satisfies functional equation*

$$f(a)+|f(b)-f(c)| = f(a+|b-c|) \tag{4}$$

for all $a,b,c \in G^+$ *such that* $a \geqslant c$*, then there exists a unique positive linear operator* $T\colon G \to H$ *such that*

$$T(a) = f(a)-f(0), \quad a \in G^+.$$

Proof. Define $g = f - f(0)$ and observe that g satisfies equation (4). We will show that g is additive. To do this put $a = 0$ and $c = 0$ in (4). We obtain $|g(b)| = g(b)$ for all $b \in G^+$. Next, apply (4) for $c = 0$ to derive

$$f(a)+g(b) = f(a+b)$$

for $a,b \in G^+$. This easily implies that $g\colon G^+ \to H^+$ is an additive mapping.

The assertion follows from the Kantorovich theorem (see [1, Theorem 1.15]), which says that every additive mapping $g\colon G^+ \to H^+$ extends uniquely to a positive operator. □

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