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ON A DISCRETE WEIGHTED MIXED ARITHMETIC–GEOMETRIC MEAN INEQUALITY

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Abstract. Let $n \geqslant 2$. For $1 \leqslant i \leqslant n$, let x_i, $w_i \geqslant 0$ with $w_1 > 0$. Further let $W_i = \sum_{k=1}^{i} w_k$, $M_{i,1} = \sum_{k=1}^{i} w_k x_k / W_k$, $M_{i,0} = \prod_{k=1}^{i} x_k^{w_k/W_k}$, $M_{i,1}(\mathbf{M}_{i,0}) = \sum_{k=1}^{i} w_k M_{k,0}/W_k$, $M_{i,0}(\mathbf{M}_{i,1}) = \prod_{k=1}^{i} M_{k,1}^{w_k/W_k}$. A result of Holland states that when $W_{n-1}^2 \geqslant w_n \sum_{i=1}^{n-2} W_i$, then

$$W_{n-1}\Big(M_{n-1,0}(\mathbf{M}_{n-1,1}) - M_{n-1,1}(\mathbf{M}_{n-1,0})\Big) \leqslant W_n\Big(M_{n,0}(\mathbf{M}_{n,1}) - M_{n,1}(\mathbf{M}_{n,0})\Big).$$

The above result implies a discrete weighted mixed arithmetic-geometric mean inequality. In this paper, we extend the validity of the above inequality by considering the case $W_{n-1}^2 < w_n \sum_{i=1}^{n-2} W_i$.

1. Introduction

Let $M_{n,r}(\mathbf{w},\mathbf{x})$ be the generalized weighted power means: $M_{n,r}(\mathbf{w},\mathbf{x}) = (\sum_{i=1}^{n} w_i x_i^r)^{\frac{1}{r}}$, where $\mathbf{w} = (w_1, w_2, \cdots, w_n)$, $\mathbf{x} = (x_1, x_2, \cdots, x_n)$, $w_i > 0$, $1 \leqslant i \leqslant n$ with $\sum_{i=1}^{n} w_i = 1$. Here $M_{n,0}(\mathbf{w},\mathbf{x})$ denotes the limit of $M_{n,r}(\mathbf{w},\mathbf{x})$ as $r \to 0^+$. Unless specified, we always assume $x_i > 0$, $1 \leqslant i \leqslant n$. When there is no risk of confusion, we shall write $M_{n,r}$ for $M_{n,r}(\mathbf{w},\mathbf{x})$ and we also denote A_n, G_n for the arithmetic mean $M_{n,1}$, geometric mean $M_{n,0}$, respectively.

For fixed $\mathbf{x} = (x_1, \cdots, x_n)$, $\mathbf{w} = (w_1, \cdots, w_n)$ with $w_1 > 0$, $w_i \geqslant 0$, we define $\mathbf{x_i} = (x_1, \cdots, x_i)$, $\mathbf{w_i} = (w_1, \cdots, w_i)$, $W_i = \sum_{j=1}^{i} w_j$, $M_{i,r} = M_{i,r}(\mathbf{w}_i/W_i, \mathbf{x}_i)$, $\mathbf{M}_{i,r} = (M_{1,r}, \cdots, M_{i,r})$. The following result on mixed mean inequalities is due to Nanjundiah [5] (see also [1]):

THEOREM 1.1. *Let $r > s$ and $n \geqslant 2$. If for $2 \leqslant k \leqslant n-1$, $W_n w_k - W_k w_n > 0$. Then*

$$M_{n,s}(\mathbf{M}_{n,r}) \geqslant M_{n,r}(\mathbf{M}_{n,s}),$$

with equality holding if and only if $x_1 = \cdots = x_n$.

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It is easy to see that the case $r=1$, $s=0$ of Theorem 1.1 follows from the following Popoviciu-type inequalities established in [4] (see also [1, Theorem 9]):

THEOREM 1.2. *Let $n \geqslant 2$. If for $2 \leqslant k \leqslant n-1$, $W_n w_k - W_k w_n > 0$, then*

$$W_{n-1}\Big(\ln M_{n-1,0}(\mathbf{M}_{n-1,1}) - \ln M_{n-1,1}(\mathbf{M}_{n-1,0})\Big) \leqslant W_n\Big(\ln M_{n,0}(\mathbf{M}_{n,1}) - \ln M_{n,1}(\mathbf{M}_{n,0})\Big)$$

with equality holding if and only if $x_n = M_{n-1,0} = M_{n-1,1}(\mathbf{M}_{n-1,0})$.

In [6], the following Rado-type inequalities were established:

THEOREM 1.3. *Let $s<1$ and $n \geqslant 2$. If for $2 \leqslant k \leqslant n-1$, $W_n w_k - W_k w_n > 0$, then*

$$W_{n-1}\Big(M_{n-1,s}(\mathbf{M}_{n-1,1}) - M_{n-1,1}(\mathbf{M}_{n-1,s})\Big) \leqslant W_n\Big(M_{n,s}(\mathbf{M}_{n,1}) - M_{n,1}(\mathbf{M}_{n,s})\Big)$$

with equality holding if and only if $x_1 = \cdots = x_n$ and the above inequality reverses when $s>1$.

The above theorem is readily seen to imply Theorem 1.1. As pointed out in [4] (see also [3]), it is not possible to establish Theorem 1.1 without any constraint on the weights. It follows that both Theorem 1.3 and Theorem 1.4 can not be valid for general weights. It is therefore interesting to establish Theorem 1.4 under less stringent conditions on the weights. In [3], Holland further improved the condition in Theorem 1.3 for the case $s=0$ by proving the following(although his theorem was given in terms of mixed-means inequality):

THEOREM 1.4. *Let $n \geqslant 2$. If $W_{n-1}^2 \geqslant w_n \sum_{i=1}^{n-2} W_i$ with the empty sum being 0, then*

$$W_{n-1}\Big(M_{n-1,0}(\mathbf{M}_{n-1,1}) - M_{n-1,1}(\mathbf{M}_{n-1,0})\Big) \leqslant W_n\Big(M_{n,0}(\mathbf{M}_{n,1}) - M_{n,1}(\mathbf{M}_{n,0})\Big) \tag{1.1}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

It is our goal in this paper to extend the above result of Holland by considering the validity of inequality (1.1) for the case $W_{n-1}^2 < w_n \sum_{i=1}^{n-2} W_i$. Note that this only happens when $n \geqslant 3$. In the next section, we apply the approach in [2] to prove the following

THEOREM 1.5. *Let $n \geqslant 3$. Inequality (1.1) holds when the following conditions are satisfied:*

$$\frac{w_n \sum_{i=1}^{n-2} W_i}{W_{n-1}^2} - 1 \leqslant \frac{w_1}{w_n}, \qquad \frac{W_{n-1}}{W_n}\prod_{i=1}^{n-2}\left(\frac{W_{i+1}}{W_i}\right)^{\frac{W_i w_n}{W_{n-1}^2}} \leqslant 1, \tag{1.2}$$
$$\left(\frac{W_{n-1}w_n}{W_n w_1}\left(\sum_{i=1}^{n-2}\frac{W_i w_n}{W_{n-1}^2} - 1\right) + \frac{w_n}{W_n}\right)\prod_{i=1}^{n-2}\left(\frac{W_{i+1}}{w_{i+1}}\right)^{\frac{w_{i+1}}{W_{n-1}}} \leqslant 1.$$

It is natural to ask whether there exists weights that satisfy the conditions given in (1.2). The answer is affirmative, as we show in Section 3 that there does exist sequences $\{w_i\}_{i=1}^n$ that satisfy the conditions of Theorem 1.5.

2. Proof of Theorem 1.5

We may assume that $x_i > 0$, $w_i > 0$, $1 \leqslant i \leqslant n$ and the case $x_i = 0$ or $w_i = 0$ for some i follows by continuity. We recast (1.1) as

$$G_n(\mathbf{A}_n) - \frac{W_{n-1}}{W_n} G_{n-1}(\mathbf{A}_{n-1}) - \frac{w_n}{W_n} G_n \geqslant 0. \tag{2.1}$$

Note that

$$G_n(\mathbf{A}_n) = \Big(G_{n-1}(\mathbf{A}_{n-1})\Big)^{W_{n-1}/W_n} A_n^{w_n/W_n}, \quad G_{n-1}(\mathbf{A}_{n-1}) = A_n \prod_{i=1}^{n-1} \left(\frac{A_i}{A_{i+1}}\right)^{W_i/W_{n-1}}. \tag{2.2}$$

Dividing $G_n(\mathbf{A}_n)$ on both sides of (2.1) and using (2.2), we can recast (2.1) as:

$$\frac{W_{n-1}}{W_n} \prod_{i=1}^{n-1} \left(\frac{A_i}{A_{i+1}}\right)^{W_i w_n/(W_{n-1}W_n)} + \frac{w_n}{W_n} \prod_{i=1}^{n} \left(\frac{x_i}{A_i}\right)^{w_i/W_n} \leqslant 1. \tag{2.3}$$

We express $x_i = (W_i A_i - W_{i-1}A_{i-1})/w_i$, $1 \leqslant i \leqslant n$ with $W_0 = A_0 = 0$ to recast (2.3) as

$$\frac{W_{n-1}}{W_n} \prod_{i=1}^{n-1} \left(\frac{A_i}{A_{i+1}}\right)^{W_i w_n/(W_{n-1}W_n)} + \frac{w_n}{W_n} \prod_{i=1}^{n} \left(\frac{W_i A_i - W_{i-1}A_{i-1}}{w_i A_i}\right)^{w_i/W_n} \leqslant 1.$$

We set $y_i = A_i/A_{i+1}$, $1 \leqslant i \leqslant 2$ to further recast the above inequality as

$$\frac{W_{n-1}}{W_n} \prod_{i=1}^{n-1} y_i^{W_i w_n/(W_{n-1}W_n)} + \frac{w_n}{W_n} \prod_{i=1}^{n-1} \left(\frac{W_{i+1}}{w_{i+1}} - \frac{W_i}{w_{i+1}} y_i\right)^{w_{i+1}/W_n} \leqslant 1. \tag{2.4}$$

We now regard the right-hand side expression above as a function of y_{n-1} only and define

$$f(y_{n-1}) = \frac{W_{n-1}}{W_n} c \cdot y_{n-1}^{w_n/W_n} + \frac{w_n}{W_n} c' \cdot \left(\frac{W_n}{w_n} - \frac{W_{n-1}}{w_n} y_{n-1}\right)^{w_n/W_n},$$

where

$$c = \prod_{i=1}^{n-2} y_i^{W_i w_n/(W_{n-1}W_n)}, \quad c' = \prod_{i=1}^{n-2} \left(\frac{W_{i+1}}{w_{i+1}} - \frac{W_i}{w_{i+1}} y_i\right)^{w_{i+1}/W_n}.$$

On setting $f'(y_{n-1}) = 0$, we find that

$$y_{n-1} = \left(\frac{W_{n-1}}{W_n} + \frac{w_n}{W_n} \left(\frac{c'}{c}\right)^{W_n/W_{n-1}}\right)^{-1}.$$

It is easy to see that $f(y_{n-1})$ is maximized at the above value with its maximal value being

$$\left(\frac{W_{n-1}}{W_n} c^{W_n/W_{n-1}} + \frac{w_n}{W_n} (c')^{W_n/W_{n-1}}\right)^{W_{n-1}/W_n}.$$

Thus, in order for inequality (2.4) to hold, it suffices to have

$$\frac{W_{n-1}}{W_n} c^{W_n/W_{n-1}} + \frac{w_n}{W_n} (c')^{W_n/W_{n-1}} \leqslant 1.$$

Explicitly, the above inequality is

$$g(y_1, y_2, \ldots, y_{n-2}) := \frac{W_{n-1}}{W_n} \prod_{i=1}^{n-2} y_i^{W_i w_n / W_{n-1}^2} + \frac{w_n}{W_n} \prod_{i=1}^{n-2} \left(\frac{W_{i+1}}{w_{i+1}} - \frac{W_i}{w_{i+1}} y_i \right)^{w_{i+1}/W_{n-1}} \leqslant 1.$$

Let $(a_1, a_2, \ldots, a_{n-2}) \in [0, W_2/W_1] \times [0, W_3/W_2] \times \ldots \times [0, W_{n-1}/W_{n-2}]$ be the point in which the absolute maximum of g is reached. If one of the a_i equals 0 or W_{i+1}/W_i, then it is easy to see that we have

$$g(a_1, a_2, \ldots, a_{n-2}) \leqslant \max \left(\frac{W_{n-1}}{W_n} \prod_{i=1}^{n-2} \left(\frac{W_{i+1}}{W_i} \right)^{W_i w_n / W_{n-1}^2}, \quad \frac{w_n}{W_n} \prod_{i=1}^{n-2} \left(\frac{W_{i+1}}{w_{i+1}} \right)^{w_{i+1}/W_{n-1}} \right). \tag{2.5}$$

If the point $(a_1, a_2, \ldots, a_{n-2})$ is an interior point, then we have

$$\nabla g(a_1, a_2, \ldots, a_{n-2}) = 0.$$

It follows that for every $1 \leqslant i \leqslant n-2$, we have

$$\frac{\prod_{i=1}^{n-2} a_i^{W_i w_n / W_{n-1}^2}}{\prod_{i=1}^{n-2} \left(\frac{W_{i+1}}{w_{i+1}} - \frac{W_i}{w_{i+1}} a_i \right)^{w_{i+1}/W_{n-1}}} = \frac{a_i}{\frac{W_{i+1}}{w_{i+1}} - \frac{W_i}{w_{i+1}} a_i} := \frac{1}{d}, \tag{2.6}$$

where $d > 0$ is a constant (depending on the w_i). In terms of d, we have

$$a_i = \frac{W_{i+1}}{d w_{i+1} + W_i}.$$

We use this to recast the first equation in (2.6) as

$$\prod_{i=1}^{n-2} \left(\frac{W_{i+1}}{d w_{i+1} + W_i} \right)^{W_i w_n / W_{n-1}^2} = \frac{1}{d} \prod_{i=1}^{n-2} \left(\frac{W_{i+1}}{w_{i+1}} \cdot \frac{d w_{i+1}}{d w_{i+1} + W_i} \right)^{w_{i+1}/W_{n-1}}.$$

We recast the above equality as

$$\begin{aligned} \ln \left(\prod_{i=1}^{n-2} (W_{i+1})^{W_i w_n / W_{n-1}^2 - w_{i+1}/W_{n-1}} \right) &= \sum_{i=1}^{n-2} \left(\frac{W_i w_n}{W_{n-1}^2} - \frac{w_{i+1}}{W_{n-1}} \right) \ln (d w_{i+1} + W_i) - \frac{w_1}{W_{n-1}} \ln d \\ &:= h(d). \end{aligned}$$

Note that the above equality holds when $d = 1$ and we have

$$h'(d) = \sum_{i=1}^{n-2} \left(\frac{W_i w_n}{W_{n-1}^2} - \frac{w_{i+1}}{W_{n-1}} \right) \frac{1}{d + W_i / w_{i+1}} - \frac{w_1}{W_{n-1}} \frac{1}{d}.$$

Note that

$$\frac{W_i w_n}{W_{n-1}^2} - \frac{w_{i+1}}{W_{n-1}} \geqslant 0 \Leftrightarrow \frac{W_i}{w_{i+1}} \geqslant \frac{W_{n-1}}{w_n}.$$

In either case

$$\frac{W_i}{w_{i+1}} \geqslant \frac{W_{n-1}}{w_n} \quad \text{or} \quad \frac{W_i}{w_{i+1}} \leqslant \frac{W_{n-1}}{w_n},$$

it follows that

$$\begin{aligned} h'(d) &\leqslant \sum_{i=1}^{n-2} \left(\frac{W_i w_n}{W_{n-1}^2} - \frac{w_{i+1}}{W_{n-1}} \right) \frac{1}{d + W_{n-1}/w_n} - \frac{w_1}{W_{n-1}} \frac{1}{d} \\ &\leqslant \frac{\left(\sum_{i=1}^{n-2} \frac{W_i w_n}{W_{n-1}^2} - 1 \right) d - \frac{w_1}{w_n}}{d(d + W_{n-1}/w_n)}. \end{aligned}$$

Therefore, when

$$\sum_{i=1}^{n-2} \frac{W_i w_n}{W_{n-1}^2} - 1 \leqslant 0,$$

the function $h(d)$ is a decreasing function of d so that $d = 1$ is the only value that satisfies (2.6) and we have $a_i = 1$ correspondingly with $g(1, 1, \ldots, 1) = 1$ and this allows us to recover Theorem 1.4, by combining the observation that the right-hand side expression of (2.5) is an increasing function of w_n for fixed w_i, $1 \leqslant i \leqslant n-1$ with the discussion in the next section.

Suppose now

$$\sum_{i=1}^{n-2} \frac{W_i w_n}{W_{n-1}^2} - 1 > 0, \tag{2.7}$$

then the function $h(d)$ is a decreasing function of d for

$$d \leqslant \frac{\frac{w_1}{w_n}}{\sum_{i=1}^{n-2} \frac{W_i w_n}{W_{n-1}^2} - 1} := d_0.$$

If follows that if $d_0 \geqslant 1$, then $d = 1$ is the only value $\leqslant d_0$ that satisfies (2.6) and we have $a_i = 1$ correspondingly with $g(1, 1, \ldots, 1) = 1$. We further note that for any $d \geqslant d_0$ satisfying (2.6), the value of g at the corresponding a_i satisfies

$$\begin{aligned} g(a_1, a_2, \ldots, a_{n-2}) &= \left(\frac{W_{n-1}}{dW_n} + \frac{w_n}{W_n} \right) \prod_{i=1}^{n-2} \left(\frac{W_{i+1}}{w_{i+1}} \cdot \frac{dw_{i+1}}{dw_{i+1} + W_i} \right)^{w_{i+1}/W_{n-1}} \\ &\leqslant \left(\frac{W_{n-1}}{dW_n} + \frac{w_n}{W_n} \right) \prod_{i=1}^{n-2} \left(\frac{W_{i+1}}{w_{i+1}} \right)^{w_{i+1}/W_{n-1}} \\ &\leqslant \left(\frac{W_{n-1}}{d_0 W_n} + \frac{w_n}{W_n} \right) \prod_{i=1}^{n-2} \left(\frac{W_{i+1}}{w_{i+1}} \right)^{w_{i+1}/W_{n-1}}. \end{aligned}$$

Combining this with (2.5), we see that inequality (2.4) holds when the conditions in (1.2) are satisfied and this completes the proof of Theorem 1.5.

3. A further discussion

We show in this section that there does exist sequences $\{w_i\}_{i=1}^n$ satisfying the conditions of Theorem 1.5 together with the condition given in (2.7), for otherwise we are back to the situation of Theorem 1.4.

Now note that the left-hand side expression of (2.7) vanishes when

$$w_n = \frac{W_{n-1}^2}{\sum_{i=1}^{n-2} W_i}. \tag{3.1}$$

It follows by continuity that such sequences $\{w_i\}_{i=1}^n$ satisfying the conditions of Theorem 1.5 exist as long as the positive sequence $\{w_i\}_{i=1}^n$ with w_i, $1 \leqslant i \leqslant n-1$ being arbitrary and w_n defined by (3.1) satisfies the last two inequalities of (1.2) with strict inequalities there. It is readily checked that these inequalities become

$$\left(\frac{\sum_{i=1}^{n-2} W_i}{\sum_{i=1}^{n-1} W_i}\right)^{\sum_{i=1}^{n-2} W_i} W_{n-1}^{W_{n-2}} < \prod_{i=1}^{n-2} W_i^{w_i}, \tag{3.2}$$

$$\frac{W_{n-1}}{\sum_{i=1}^{n-1} W_i} \prod_{i=1}^{n-2} \left(\frac{W_{i+1}}{w_{i+1}}\right)^{w_{i+1}/W_{n-1}} < 1. \tag{3.3}$$

Now, it is easy to see that inequality (3.2) holds when $n = 3$. It follows that it holds for all $n \geqslant 3$ by induction as long as we have

$$\left(\frac{\sum_{i=1}^{n-2} W_i}{\sum_{i=1}^{n-1} W_i}\right)^{\sum_{i=1}^{n-2} W_i} W_{n-1}^{W_{n-1}} \geqslant \left(\frac{\sum_{i=1}^{n-1} W_i}{\sum_{i=1}^{n} W_i}\right)^{\sum_{i=1}^{n-1} W_i} W_n^{W_{n-1}}. \tag{3.4}$$

The right-hand side expression above when regarded as a function of W_n only is maximized at

$$W_n = \frac{W_{n-1} \sum_{i=1}^{n-1} W_i}{\sum_{i=1}^{n-2} W_i}.$$

As the inequality in (3.4) becomes an equality with this value of W_n, we see that inequality (3.2) does hold for all $n \geqslant 3$.

Note that it follows from the arithmetic-geometric mean inequality that

$$\frac{W_{n-1}}{\sum_{i=1}^{n-1} W_i} \prod_{i=1}^{n-2} \left(\frac{W_{i+1}}{w_{i+1}}\right)^{w_{i+1}/W_{n-1}} \leqslant \frac{W_{n-1}}{\sum_{i=1}^{n-1} W_i} \left(\sum_{i=0}^{n-2} \frac{W_{i+1}}{w_{i+1}} \cdot \frac{w_{i+1}}{W_{n-1}}\right) = 1.$$

As one checks easily that the above inequality is strict in our case, we see that inequality (3.3) also holds. We therefore conclude the existence of sequences $\{w_i\}_{i=1}^n$ satisfying the conditions of Theorem 1.5.

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REFERENCES

[1] P. S. BULLEN, *Inequalities due to T. S. Nanjundiah*, *Recent progress in inequalities*, Kluwer Acad. Publ., Dordrecht, 1998, 203–211.

[2] P. GAO, *A note on mixed-mean inequalities*, J. Inequal. Appl., **2010** (2010), Art. ID 509323, 8 pp.

[3] F. HOLLAND, *An inequality between compositions of weighted arithmetic and geometric means*, JIPAM. J. Inequal. Pure Appl. Math., **7** (2006), Article 159, 8 pp. (electronic).

[4] K. KEDLAYA, *A weighted mixed-mean inequality*, Amer. Math. Monthly, **106** (1999), 355–358.

[5] T. S. NANJUNDIAH, *Sharpening of some classical inequalities*, Math Student, **20** (1952), 24–25.

[6] C. TARNAVAS AND D. TARNAVAS, *An inequality for mixed power means*, Math. Inequal. Appl., **2** (1999), 175–181.

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