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ON WEIGHTED BERNSTEIN TYPE INEQUALITY IN GRAND VARIABLE EXPONENT LEBESGUE SPACES

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Abstract. In this paper a weighted Bernstein type inequality on derivatives of trigonometric polynomials is established in new function spaces unifying two nonstandard Banach function spaces, in particular, grand and variable exponent Lebegue spaces.

1. Introduction

The paper deals with the Bernstein type inequality in grand variable Lebesgue spaces $L^{p(\cdot),\theta}$ with weight (for $L^{p(\cdot),\theta}$ spaces see Definition 1). Namely, we show that for any $r \in \mathbb{N}$ and trigonometric polynomial T_n of degree less than or equal to n, the inequality

$$\|\sin^r t\, T_n'(t)\|_{L^{p(\cdot),\theta}(\mathbb{T})} \leqslant cn\|\sin^r t\, T_n(t)\|_{L^{p(\cdot),\theta}(\mathbb{T})} \tag{1}$$

holds with a constant c independent of T_n.

The latter inequality in the classical Lebesgue spaces L^p $(0 < p < \infty)$, was proved by Z. Ditzian and V. Totik [4], Section 8.4 (we refer also to [2], Theorem 7.5).

An improved Bernstein inequality in variable exponent Lebesgue spaces with Muckenhoupt type weights, suited to this spaces, was proved in [1]. It should be emphasized that even for constant p, the special weights inside of norms in inequality (1) are more general than the Muckenhoupt weights of the same type.

Let $s(\cdot)$ be continuous, 2π-periodic function defined on $\mathbb{R}$. It is said that s satisfies the log-Hölder continuity condition if there exists a positive constant A such that for all $x, y \in \mathbb{R}$, $|x-y| < 1/2$, the inequality

$$|s(x)-s(y)| \leqslant \frac{A}{-\log|x-y|} \tag{2}$$

holds. In the sequel we denote the class of 2π-periodic functions satisfying the log-Hölder continuity condition by $\mathscr{P}^{\log}$. Further, we say that $s \in \mathscr{P}$ if

$$1 < s_- \leqslant s_+ < \infty,$$

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where

$$s_- := \inf_{\mathbb{T}} s(x); \quad s_+ := \sup_{\mathbb{T}} s(x), \quad \mathbb{T} := [-\pi, \pi].$$

Recently, the authors of this paper in [10] (see also [11]) introduced a new functions space being the mixture of two well-known spaces: variable exponent Lebesgue space and grand Lebesgue space. Let us recall the definition of that space.

DEFINITION 1. Let $p \in \mathscr{P}$ and let $\theta > 0$. Denote by $L^{p(\cdot),\theta}(\mathbb{T})$ the class of those measurable functions for which

$$\|f\|_{L^{p(\cdot),\theta}(\mathbb{T})} := \sup_{0<\varepsilon<p_- -1} \varepsilon^{\frac{\theta}{p_- -\varepsilon}} \|f\|_{L^{p(\cdot)-\varepsilon}(\mathbb{T})} < \infty,$$

where

$$\|f\|_{L^{s(\cdot)}(\mathbb{T})} = \inf_{\lambda>0} \Big\{ \lambda : \int_{\mathbb{T}} \Big| \frac{f(x)}{\lambda} \Big|^{s(x)} dx \leqslant 1 \Big\}.$$

In the same paper [10] (see also [11]) the authors showed that the space $L^{p(\cdot),\theta}(\mathbb{T})$ is a Banach space. It is worth mentioning that the space $L^{p(\cdot),\theta}$ is non-reflective and non-separable. Different from the constant exponent case this space is also non-rearrangement invariant.

For constant p and $\theta = 1$, the space $L^{p(\cdot),\theta}$, is the Iwaniec-Sbordone space $L^{p)}$ introduced in [9] in their studies related with the integrability properties of the Jacobian in a bounded open set Ω. The generalized version of that space, $L^{p),\theta}$ appeared in L. Greco, T. Iwaniec and C. Sbordone [8]. For structural properties of $L^{p),\theta}$ spaces we refer to [6]. The boundedness of the Hardy–Littlewood maximal operator in $L^{p)}_w(\Omega)$ spaces, $1 < p < \infty$, for bounded open Ω, under the Muckenhoupt A_p condition was proved in [7].

Let J be a bounded interval in $\mathbb{R}$. It follows from the celebrated paper by L. Diening [3] that the Hardy–Littlewood maximal operator defined on J is bounded in $L^{r(\cdot)}(J)$ provided that r belongs to the classes $\mathscr{P}(J)$ and $\mathscr{P}^{log}(J)$ defined on J. The key point of the proof of the latter result is the following lemma:

LEMMA 1. [3] *Let J be a bounded interval. A positive function s belongs to $\mathscr{P}^{log}(J)$ if and only if there is a positive constant C such that for all subintervals $I \subseteq J$ the inequality*

$$|I|^{s_-(I)-s_+(I)} \leqslant C \tag{3}$$

holds. Moreover, if $s \in \mathscr{P}^{log}(J)$ with the constant A (see (2)), then the constant C in (3) can be taken equal to $\max\{e^A, 2^{(s_+-s_-)}\}$. Further, inequality (3) holds for $I \subseteq J$ satisfying $|I| \leqslant 1/2$ with the constant $C = e^A$.

Since for us it is interesting the constant C, let us see that $C = \max\{e^A, 2^{(s_+-s_-)}\}$ in (3). Let I be a subinterval of J. Suppose that $x, y \in I$. Let $|I| \leqslant 1/2$. Then

$$|I|^{s_-(I)-s_+(I)} \leqslant |I|^{\frac{A}{\ln I}} = e^A.$$

Let $|I| \geqslant \frac{1}{2}$. Then

$$|I|^{s_-(I)-s_+(I)} \leqslant 2^{s_+-s_-}.$$

Finally we mention that in the sequel constants (often different constants in one and the same lines of inequalities) will be denoted by c or C. The symbol r' stands for the conjugate number of r; under the symbol $s'(\cdot)$ we mean the function $s(\cdot)/(s(\cdot)-1)$; the set of natural numbers will be denoted by $\mathbb{N}$; by the symbol $\mathbb{T}$ we denote the interval $[-\pi,\pi]$; the symbol $C(\mathbb{T})$ means the space of all continuous functions on $\mathbb{T}$; f' is a derivative of a function f; $f_I := \frac{1}{|I|}\int_I |f(t)|dt$; $s_- := \inf_{\mathbb{T}} s$, $s_+ := \sup_{\mathbb{T}} s$; $s_-(J) := \inf_J s$; $s_+(J) := \sup_J s$, where J is an interval.

2. Preliminaries

Let T_n be any polynomial with degree less than or equal to n. Suppose that $M := \|T_n\|_{L^\infty(\mathbb{T})}$ and let t_0 be a point at which $|T_n(t_0)| = M$. Let us introduce the notation:

$$I := \left[t_0 - \frac{1}{2n}, t_0 + \frac{1}{2n}\right],\ B := \left\{t \in \mathbb{T} : |\sin t| \leqslant \sin\frac{1}{8n}\right\},\ B_1 := \mathbb{T}\setminus B,\ B_2 := B_1 \cap I. \quad (4)$$

Observe that

$$|B| = \frac{1}{2n}, \quad \frac{1}{n} \geqslant |B_2| \geqslant \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n} = |B|. \quad (5)$$

In the sequel for an exponent $p \in \mathscr{P}$, we denote

$$p^{(\varepsilon)}(\cdot) := \frac{1}{p(\cdot)-\varepsilon}.$$

REMARK 1. It is easy to see that if $p \in \mathscr{P}^{log}(\mathbb{T})$ with constant A, then $p^{(\varepsilon)} \in \mathscr{P}^{log}(\mathbb{T})$ with the constant $A^{(p)} := \frac{A}{p_-^2}$, i.e., for all $x, y \in \mathbb{T}$ with $|x-y| \leqslant 1/2$,

$$|p^{(\varepsilon)}(x) - p^{(\varepsilon)}(y)| \leqslant \frac{A^{(p)}}{-\log|x-y|}. \quad (6)$$

Now we prove some auxiliary statements.

LEMMA 2. *Let $p \in \mathscr{P} \cap \mathscr{P}^{log}$ and let p be π-periodic. Suppose that $0 < \varepsilon < p_- - 1$. Then there is a positive constant c independent of n and ε such that for all natural n,*

$$|B|^{p_-^{(\varepsilon)}(B)-p_+^{(\varepsilon)}(B)} \leqslant c. \quad (7)$$

Proof. Let us observe that $B = I_1 \cup I_2 \cup I_3$, where

$$I_1 := [-\pi, -\pi + 1/(8n)],\ I_2 := [-1/(8n), 1/(8n)],\ I_3 := [\pi - 1/(8n), \pi].$$

Suppose that x' and x'' be points in B such that

$$p_-^{(\varepsilon)}(B) = p^{(\varepsilon)}(x'), \quad p_+^{(\varepsilon)}(B) = p^{(\varepsilon)}(x''),$$

It is obvious that $x', x'' \in B$ since the set B is compact. Consider the following cases

(i) $x' \in I_1$ and $x'' \in I_3$, or $x'' \in I_1$ and $x' \in I_3$;

(ii) $x' \in I_1$ and $x'' \in I_2$, or $x'' \in I_1$ and $x' \in I_2$;

(iii) $x' \in I_2$ and $x'' \in I_3$, or $x'' \in I_2$ and $x' \in I_3$;

(iv) $x', x'' \in I_1$, or $x', x'' \in I_2$, or $x', x'' \in I_3$.

Let the case (i) hold. Then by the fact that $p^{(\varepsilon)}$ is 2π-periodic, it is clear that $p^{(\varepsilon)}(x') = p^{(\varepsilon)}(\bar{x}')$, where $\bar{x}' \in \bar{I}_1$, $\bar{x}' = x' + 2\pi$ and $\bar{I}_1 = I_1 + 2\pi$. Then $I_3 \cup \bar{I}_1$ is an interval of length $\frac{1}{4n}$ containing $\bar{x}'$ and x''. Then by Remark 1 we have that

$$\begin{aligned} |B|^{p_-^{(\varepsilon)}(B) - p_+^{(\varepsilon)}(B)} &\leqslant 2^{p_+} n^{|p^{(\varepsilon)}(x') - p^{(\varepsilon)}(x'')|} = 2^{p_+} n^{|p^{(\varepsilon)}(\bar{x}') - p^{(\varepsilon)}(x'')|} \\ &\leqslant 2^{p_+} n^{\frac{A^{(p)}}{-\log|\bar{x}' - x''|}} \leqslant 2^{p_+} n^{\frac{A^{(p)}}{\log 4n}} \leqslant c_p, \end{aligned}$$

where $c_p = 2^{p_+} e^{A^{(p)}}$ and $A^{(p)}$ is the constant arisen in (6).

The case when $x'' \in I_1$ and $x' \in I_3$ is proved analogously.

(ii), (iii): These cases are analogous to (i) but here it is used π-periodicity of $p^{(\varepsilon)}$ instead of 2π-periodicity. For example, in the case (ii) when $x' \in I_1$ and $x'' \in I_2$, denoting $\bar{I}_1 := \pi + I_1$ and $\bar{x}' = \pi + x'$, then we have that $\bar{I}_1 \subset I_2$ and $\bar{x}', x'' \in I_2$.

(iv) This case is easier to prove because in this case we deal with an interval. For example, if $x', x'' \in I_1$, then by Lemma 1 we have that (7) holds with constant $c = \max\{e^A, 2^{1-1/p_+}\}$, where A is from (2). □

LEMMA 3. *Let $p \in \mathscr{P} \cap \mathscr{P}^{log}$ and let p be π-periodic. Then there is a positive constant c depending only on p such that*

$$|B|^{p_-^{(\varepsilon)}(B)} \leqslant c\, |B_2|^{p_+^{(\varepsilon)}(B_2)} \tag{8}$$

provided that $I \cap B \neq \emptyset$.

Proof. Since $B_2 \subset I$ and $|I| = \frac{1}{n}$, therefore by the condition $p^{(\varepsilon)} \in \mathscr{P}^{log}$, Lemma 1, Remark 1, Lemma 2 and (5) we have that

$$\begin{aligned} |B_2|^{p_+^{(\varepsilon)}(B_2)} &\geqslant |B_2|^{p_+^{(\varepsilon)}(I)} \geqslant 2^{-p_+^{(\varepsilon)}(I)} |I|^{p_+^{(\varepsilon)}(I)} \\ &\geqslant 2^{-1}|I|^{p_+^{(\varepsilon)}(I)} \geqslant 2^{-1}e^{-A}|I|^{p_-^{(\varepsilon)}(I)} \geqslant 2^{-1}e^{-A}|I|^{p_-^{(\varepsilon)}(I\cap B)} \\ &\geqslant 2^{-1}e^{-A}|I|^{p_+^{(\varepsilon)}(I\cap B)} \geqslant 2^{-1}e^{-A}|I|^{p_+^{(\varepsilon)}(B)} \geqslant 4^{-1}e^{-A}|B|^{p_-^{(\varepsilon)}(B)}, \end{aligned}$$

where A is the constant of the condition $p \in \mathscr{P}^{log}$. We also have that $c = 4^{-1}e^A$. □

LEMMA 4. *Let $p \in \mathscr{P} \cap \mathscr{P}^{log}$ and let p be π-periodic. Suppose also that $p_- = p(0)$. Then there is constant $c > 0$ such that for all $n \in \mathbb{N}$ and $\varepsilon \in (0, p_- - 1)$ inequality (8) holds provided that $I \cap B = \emptyset$.*

Proof. Since 0 belongs to B we have

$$p_+^{(\varepsilon)}(B) = p^{(\varepsilon)}(0) = p_+^{(\varepsilon)} \geqslant p_+^{(\varepsilon)}(B_2).$$

Then by using Lemma 2 we find that

$$|B|^{p_-^{(\varepsilon)}(B)} \leqslant c|B|^{p_+^{(\varepsilon)}(B)} = c|B|^{p_+^{(\varepsilon)}} \leqslant c|B_2|^{p_+^{(\varepsilon)}(B_2)},$$

where the constant c depends only on p. □

Summarizing Lemmas 2–4 we see that the following lemma is true.

LEMMA 5. *Let $p \in \mathscr{P} \cap \mathscr{P}^{log}$ and let p be π-periodic. Suppose also that $p_- = p(0)$. Then there is a constant c depending only on p such that for all $n \in \mathbb{N}$ and $\varepsilon \in (0, p_- - 1)$ inequality (8) holds.*

The next lemma will be also useful for us.

LEMMA 6. *Let $p \in \mathscr{P} \cap \mathscr{P}^{log}$ and let $\theta > 0$. Suppose that p is π-periodic and that $p_- = p(0)$. Then there is a positive constant c depending only on p such that for any polynomial T_n with degree less than or equal to n, the inequality*

$$\|T_n\|_{L^{p(\cdot),\theta}(\mathbb{T})} \leqslant cn\|\sin t\; T_n\|_{L^{p(\cdot),\theta}(\mathbb{T})}$$

holds.

Proof. Using notation (4) we have

$$\|T_n\|_{L^{p(\cdot),\theta}(\mathbb{T})} \leqslant c\|T_n\|_{L^{p(\cdot),\theta)}(B_1)} + c\|T_n\|_{L^{p(\cdot),\theta}(B)} =: A_1 + A_2, \tag{9}$$

where the constant c depends only on p. We estimate A_1 and A_2 separately. To estimate A_1, observe that

$$B_1 \subset \left\{t : n|\sin t| \geqslant \frac{1}{4\pi}\right\}. \tag{10}$$

Hence,

$$\|T_n\|_{L^{p(\cdot),\theta}(B_1)} \leqslant 4\pi n\|\sin t T_n(t)\|_{L^{p(\cdot),\theta}(B_1)}. \tag{11}$$

Now we estimate A_2. Recall that t_0 is a point at which $|T_n(t_0)| = M := \|T_n\|_{L^\infty(\mathbb{T})}$ and $I := [t_0 - 1/(2n), t_0 + 1/(2n)]$. Applying the Bernstein inequality for the space $C(\mathbb{T})$ we have that for $t \in I$,

$$|T_n(t) - T_n(t_0)| \leqslant \frac{1}{2n}|T'(\xi)| \leqslant \frac{1}{2n}\|T'(t)\|_{L^\infty(I)} \leqslant \frac{1}{2}|T_n(t)|_{L^\infty(\mathbb{T})} \leqslant \frac{1}{2}M.$$

Consequently,

$$|T_n(t)| > \frac{M}{2}, \quad t \in I. \tag{12}$$

Further,

$$\begin{aligned}\|T_n\|_{L^{p(\cdot),\theta}(B)} &\leqslant M\|\chi_B\|_{L^{p(\cdot),\theta}} = M \sup_{\varepsilon\in(0,p_- -1)} \varepsilon^{\frac{\theta}{p_- -\varepsilon}} \|\chi_B\|_{L^{p(\cdot)-\varepsilon}} \\ &\leqslant M \sup_{\varepsilon\in(0,p_- -1)} \varepsilon^{\frac{\theta}{p_- -\varepsilon}} \max\left\{|B|^{\frac{1}{(p(\cdot)-\varepsilon)_+(B)}}, |B|^{\frac{1}{(p(\cdot)-\varepsilon)_-(B)}}\right\} \\ &\leqslant cM \sup_{\varepsilon\in(0,p_- -1)} \varepsilon^{\frac{\theta}{p_- -\varepsilon}} |B|^{\frac{1}{(p(\cdot)-\varepsilon)_+(B)}}.\end{aligned}$$

In the latter inequality we used the estimate

$$|B|^{\frac{1}{(p(\cdot)-\varepsilon)_-(B)}} \leqslant (2\pi)^{1-1/p_+}|B|^{\frac{1}{(p(\cdot)-\varepsilon)_+(B)}}.$$

Further, notice that inequality (12) holds for all $t \in B_2$, because $B_2 \subset I$. Hence, applying (10), (12), the inclusion $B_2 \subset B_1$ and obvious inequality

$$n\|\sin t\, T_n(t)\|_{L^{p(\cdot),\theta}(\mathbb{T})} \geqslant n\|\sin t\, T_n(t)\|_{L^{p(\cdot),\theta}(B_1)}$$

we find that

$$\begin{aligned}n\|\sin t\, T_n(t)\|_{L^{p(\cdot),\theta}(\mathbb{T})} &\geqslant \frac{1}{4\pi}\|T_n(t)\|_{L^{p(\cdot),\theta}(B_1)} \geqslant \frac{1}{4\pi}\|T_n(t)\|_{L^{p(\cdot),\theta}(B_2)} \geqslant \frac{M}{8\pi}\|\chi_{B_2}\|_{L^{p(\cdot),\theta}} \\ &\geqslant \frac{M}{8\pi} \sup_{\varepsilon\in(0,p_- -1)} \varepsilon^{\frac{\theta}{p_- -\varepsilon}} \min\left\{|B_2|^{\frac{1}{(p(\cdot)-\varepsilon)_+(B_2)}}, |B_2|^{\frac{1}{(p(\cdot)-\varepsilon)_-(B_2)}}\right\} \\ &\geqslant \frac{M}{16\pi}(2\pi)^{p_- - p_+} \sup_{\varepsilon\in(0,p_- -1)} \varepsilon^{\frac{\theta}{p_- -\varepsilon}} |B_2|^{\frac{1}{(p(\cdot)-\varepsilon)_-(B_2)}}.\end{aligned}$$

Taking into account these estimates and Lemma 5 we conclude that

$$\|T_n\|_{L^{p(\cdot),\theta}(B)} \leqslant cn\,\|\sin t\, T_n(t)\|_{L^{p(\cdot),\theta}(\mathbb{T})} \tag{13}$$

with the constant c depending only on p.

Finally, inequalities (9), (13) and (11) yield that

$$\|T_n\|_{L^{p(\cdot),\theta)}(\mathbb{T})} \leqslant cn\,\|\sin t\, T_n(t)\|_{L^{p(\cdot),\theta}(\mathbb{T})}$$

with the positive constant c is independent of T_n. □

For 2π-periodic f, it is defined its Hardy–Littlewood maximal operator as

$$\mathscr{M}f(x) = \sup_{|t|\leqslant\pi} \frac{1}{2t}\int_{-t}^{t} |f(x+u)|du.$$

It is easy to see that $\mathscr{M}f$ is also 2π-periodic.

To prove the main result of this paper we to prove the boundedness of $\mathscr{M}$ in $L^{p(\cdot),\theta}(\mathbb{T})$.

PROPOSITION 1. *Let $p \in \mathscr{P} \cap \mathscr{P}^{log}$. Then for 2π-periodic $f \in L^{p(\cdot)}(\mathbb{T})$,*

$$\|\mathscr{M} f\|_{L^{p(\cdot),\theta}(\mathbb{T})} \leqslant c\|f\|_{L^{p(\cdot),\theta}(\mathbb{T})}. \tag{14}$$

To prove Proposition 1 we need some auxiliary statements.

Let J be a bounded interval in $\mathbb{R}$ and let s be a positive function on J. We denote by $\mathscr{M}_J$ the Hardy–Littlewood maximal function defined on J:

$$\mathscr{M}_J f(x) = \sup_{\substack{I \ni x \\ I \subseteq J}} \frac{1}{|I|} \int_I |f(t)| dt,$$

where the supremum is taken over all subintervals $I \subset J$ containing x.

LEMMA 7. *Let s be an exponent defined on a bounded interval J such that $1 \leqslant s_-(J) \leqslant s_+(J) < \infty$. Suppose that $s \in \mathscr{P}^{\log}(J)$. Then*

(a) *there is a positive constant $\bar{c}_p$ such that for all f, $\|f\|_{L^{s(\cdot)}(J)} \leqslant 1$, $f \geqslant 0$, $I \subset J$, $x \in J$,*

$$(f_I)^{s(x)} \leqslant \bar{c}_p \Big[\big(f^{s(\cdot)}(\cdot)\big)_I + 1 \Big],$$

where

$$\bar{c}_s = \max\{4^{s_+-1}, 2^{s_+-1}C\},$$

and C is defined in Lemma 1 (see (3)).

(b)

$$\big(\mathscr{M}_J f(x)\big)^{s(x)} \leqslant \bar{c}_s \Big[\mathscr{M}_J(f^{s(\cdot)})(x) + 1 \Big], \quad x \in J, \ f \geqslant 0.$$

Proof. We prove (a). Part (b) is a direct consequence of (a). Let $|I| \geqslant 1/2$. Let us denote

$$\rho_s(f) := \int_J f^{s(x)}(x) dx.$$

Observe that since $|I| \geqslant 1/2$ and $\rho_s(f) \leqslant 1$ we have that $\left(\frac{f^{s(\cdot)}}{2}\right)_I \leqslant 1$. Then

$$\begin{aligned}
(f_I)^{s(x)} &\leqslant \left[\frac{1}{|I|} \left(\int_{I \cap \{f \geqslant 1\}} f(y) dy + \int_{I \cap \{f < 1\}} f(y) dy \right) \right]^{s(x)} \\
&\leqslant \left[\frac{1}{|I|} \int_I \left(f^{s(y)}(y) + 1 \right) dy \right]^{s(x)} \leqslant 2^{s(x)-1} \left[\left(f^{s(\cdot)} \right)_I^{s(x)} + 1 \right] \\
&= 2^{2s(x)-1} \left(\frac{f^{s(\cdot)}}{2} \right)_I^{s(x)} + 2^{s(x)-1} \leqslant 4^{s(x)-1} \left(f^{s(\cdot)}(\cdot) \right)_I + 2^{s(x)-1} \\
&\leqslant 4^{s_+-1} \left[\left(f^{s(\cdot)}(\cdot) \right)_I + 1 \right].
\end{aligned}$$

Let now $|I| < 1/2$. By Hölder's inequality and Lemma 1 we find that

$$\begin{aligned}
\big(f_I\big)^{s(x)} &\leqslant \left(\frac{1}{|I|}\int\limits_I f^{s_-(I)}(y)dy\right)^{s(x)/s_-(I)} \\
&\leqslant 2^{s(x)/s_-(I)}|I|^{-s(x)/s_-(I)}\left[\frac{1}{2}\int\limits_I \big(f(y)\big)^{s(y)}dy + \frac{1}{2}|I|\right]^{s(x)/s_-(I)} \\
&\leqslant 2^{s(x)/s_-(I)}|I|^{-s(x)/s_-(I)}\left[\frac{1}{2}\int\limits_I \big(f(y)\big)^{s(y)}dy + \frac{|I|}{2}\right] \\
&\leqslant 2^{s(x)/s_-(I)-1}|I|^{1-s(x)/s_-(I)}\left[\big(f^{s(\cdot)}(\cdot)\big)_I + 1\right] \\
&\leqslant 2^{s(x)/s_-(I)-1}|I|^{(s_-(I)-s(x))/s_-(I)}\left[\big(f^{s(\cdot)}(\cdot)\big)_I + 1\right] \\
&\leqslant 2^{s(x)/s_-(I)-1}C^{1/s_-(I)}\left[\big(f^{s(\cdot)}(\cdot)\big)_I + 1\right],
\end{aligned}$$

where C is defined in (3). Finally we conclude that

$$\bar{c}_s = \max\{4^{4s_+-1}, 2^{s_+-1}C\}. \quad \square$$

REMARK 2. It is well-known that $\mathscr{M}_J$ is bounded in $L^{p_0}(J)$ for $1 < p_0 < \infty$. Moreover, by Marcinkiewicz interpolation theorem the norm of $\mathscr{M}_J$ can be estimated by $C_0(p_0)'^{1/p_0}$, where the constant C_0 does not depend on p_0. We refer to e.g. [5], pp. 29–30.

PROPOSITION 2. *Let J be a bounded interval and let $p \in \mathscr{P}(J) \cap \mathscr{P}^{log}(J)$. Then*

$$\|\mathscr{M}_J\|_{L^{p(\cdot)}(J)\to L^{p(\cdot)}(J)} \leqslant \left[2^{p_--1}\widetilde{c}_p^{\,p_-}\left[C_0(p_-)' + |J|\right]\right]^{1/p_-},$$

where

$$\widetilde{c}_p = \max\{4^{p_+/p_--1}, 2^{p_+/p_--1}C\},$$

C is the constant (3) replacing s by p/p_- and the constant C_0 does not depend on p.

Proof. First we show that

$$\big(\mathscr{M}_J f(x)\big)^{p(x)/p_-} \leqslant c(p)\Big(\mathscr{M}_J\big(f^{p(\cdot)/p_-}\big)(x) + 1\Big), \;\; x \in J, \;\; f \geqslant 0,$$

where

$$c(p) = \left[2^{p_--1}(\widetilde{c}_p)^{p_-}[(p_-)' + |J|]\right]^{1/p_-}.$$

Indeed, by Lemma 7 (taking $s(x)=\frac{p(x)}{p_-}$) we have

$$\big(\mathscr{M}_J f(x)\big)^{p(x)/p_-} \leqslant \max\{4^{p_+/p_- -1}, 2^{p_+/p_- -1}C\}\Big[\big(\mathscr{M}_J f^{p(\cdot)/p_-}\big)(x)+1\Big]$$
$$=\widetilde{c}_p\Big[\big(\mathscr{M}_J f^{p(\cdot)/p_-}\big)(x)+1\Big].$$

Let $\|f\|_{L^{p(\cdot)}(X)} \leqslant 1$. Let us estimate $\rho_p(\mathscr{M}_J f)=\int\limits_J (\mathscr{M}_J f(x))^{p(x)}dx$. By using Remark 2 we find that

$$\begin{aligned}\rho_p(\mathscr{M}_J f) &= \|\big(\mathscr{M}_J f\big)^{p(\cdot)/p_-}\|^{p_-}_{L^{p_-}(J)}\\ &\leqslant 2^{p_- -1}(\widetilde{c}_p)^{p_-}\big(\|\big(\mathscr{M}_J f^{p(\cdot)/p_-}\big)\|^{p_-}_{L^{p_-}(J)}+\|1\|^{p_-}_{L^{p_-}(J)}\big)\\ &\leqslant 2^{p_- -1}(\widetilde{c}_p)^{p_-}\Big(C_0(p_-)'\|f^{p(\cdot)/p_-}\|^{p_-}_{L^{p_-}(J)}+|J|\Big)\\ &= 2^{p_- -1}(\widetilde{c}_p)^{p_-}\Big(C_0(p_-)'\int\limits_J (f(x))^{p(x)}dx+|J|\Big)\\ &\leqslant 2^{p_- -1}(\widetilde{c}_p)^{p_-}\Big[C_0(p_-)'+|J|\Big].\end{aligned}$$

Thus,

$$\rho_p(\mathscr{M}_J f)\leqslant 2^{p_- -1}(\widetilde{c}_p)^{p_-}\Big[C_0(p_-)'+|J|\Big]=:\bar{C}_p.$$

Hence

$$\rho_p\big((\mathscr{M}_J f)/\bar{C}_p^{1/(p(\cdot))}\big)\leqslant 1.$$

Consequently,

$$\|(\mathscr{M}_J f)/\bar{C}_p^{1/(p(\cdot))}\|^{p_+}_{L^{p(\cdot)}(J)}\leqslant 1.$$

Finally,

$$\|\mathscr{M}_J\|_{L^{p(\cdot)}(J)\to L^{p(\cdot)}(J)}\leqslant (\bar{C}_p)^{1/p_-}. \quad \square$$

COROLLARY 1. *Let J be a bounded interval and let $p\in\mathscr{P}(J)\cap\mathscr{P}^{log}(J)$. Then for $0<\varepsilon<p_- -1$,*

$$\|\mathscr{M}_J f\|_{L^{p(\cdot)-\varepsilon}(J)\to L^{p(\cdot)-\varepsilon}(J)}\leqslant C_{p,\varepsilon},$$

where $C_{p,\varepsilon}$ is the constant depending only on p and ε such that

$$\sup_{0<\varepsilon\leqslant\delta} C_{p,\varepsilon}<\infty,$$

for some small positive number δ.

PROPOSITION 3. *Let J be a bounded interval and let $p\in\mathscr{P}(J)\cap\mathscr{P}^{log}(J)$. Then for $f\in L^{p(\cdot),\theta}(J)$,*

$$\|\mathscr{M}_J f\|_{L^{p(\cdot),\theta}(J)}\leqslant c\|f\|_{L^{p(\cdot),\theta}(J)}. \tag{15}$$

Proof. First recall that (see e.g. [12])

$$L^{q(\cdot)}(J) \hookrightarrow L^{p(\cdot)}(J) \tag{16}$$

holds if and only if $q(x) \geqslant p(x)$.

By using (16) and Corollary 1 we see that

$$\begin{aligned}
&\sup_{0<\varepsilon<p_- -1} \varepsilon^{\frac{\theta}{p_- -\varepsilon}} \|\mathscr{M}_J f\|_{L^{p(\cdot)-\varepsilon}(J)} \\
&= \max\{\sup_{0<\varepsilon<\sigma}(\cdots), \sup_{\sigma\leqslant\varepsilon<p_- -1}(\cdots)\} \\
&\leqslant \max\{\sup_{0<\varepsilon<\sigma}(\cdots), (1+|J|)\sup_{\sigma\leqslant\varepsilon<p_- -1} \varepsilon^{\frac{\theta}{p_- -\varepsilon}} \|\mathscr{M}_J f\|_{L^{p(\cdot)-\sigma}(J)}\} \\
&\leqslant C(\sigma,|J|,\theta,p)\sup_{0<\varepsilon<\sigma} \varepsilon^{\frac{\theta}{p_- -\varepsilon}} \|\mathscr{M}_J f\|_{L^{p(\cdot)-\varepsilon}(J)} \\
&\leqslant C(\sigma,|J|,\theta,p)\sup_{0<\varepsilon<\sigma} \varepsilon^{\frac{\theta}{p_- -\varepsilon}} \|f\|_{L^{p(\cdot)-\varepsilon}(J)} \\
&\leqslant C(\sigma,|J|,\theta,p)\|f\|_{L^{p(\cdot),\theta}(J)}. \quad \square
\end{aligned}$$

Proof of Proposition 1. Following, for example, the arguments from [13], p. 244, let g be a function equal to f in $(-2\pi, 2\pi)$ and to 0 otherwise. Then

$$(\mathscr{M}f)(x) \leqslant (\mathscr{M}_{(-2\pi,2\pi)}g)(x), \quad x \in \mathbb{T}.$$

Now taking into account the fact that p and f are periodic, and Proposition 3, we are done. $\square$

3. The main result

The main result of this paper reads as follows:

THEOREM 1. *Suppose that p is π-periodic and that $p_- = p(0)$. Let $p \in \mathscr{P} \cap \mathscr{P}^{log}$ and let $\theta > 0$. Then for any $r \in \mathbb{N}$ and arbitrary trigonometric polynomials T_n of degree less than or equal to n, inequality* (1) *holds with a constant c independent of T_n.*

To prove this theorem we need the next statement.

PROPOSITION 4. *Let p be 2π-periodic, $p \in \mathscr{P} \cap \mathscr{P}^{log}$ and let $\theta > 0$. Then for arbitrary trigonometric polynomials T_n of degree less than or equal to n, the inequality*

$$\|T_n'(t)\|_{L^{p(\cdot),\theta}(\mathbb{T})} \leqslant cn\, \|T_n(t)\|_{L^{p(\cdot),\theta}(\mathbb{T})} \tag{17}$$

holds, where the constant c does not depend on T_n.

Proof. It is well-known (see e.g., [13], Ch.12) that for the Cesáro means of f,

$$\sup_n |\sigma_n(f,x)| \leqslant c(\mathscr{M} f)(x). \tag{18}$$

Further, by using the representation

$$T_n(x) = \frac{1}{\pi} \int_{\mathbb{T}} T_n(t) D_n(t-x)\,dt,$$

where

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt,$$

we have that

$$\begin{aligned}
T_n'(x) &= -\frac{1}{\pi} \int_{\mathbb{T}} T_n(t) D_n'(t-x)\,dt = -\frac{1}{\pi} \int_{\mathbb{T}} T_n(x+t) D_n'(t)\,dt \\
&= \frac{1}{\pi} \int_{\mathbb{T}} T_n(x+t) \Big\{ \sum_{k=1}^{n} k \sin kt \Big\} dt \\
&= \frac{1}{\pi} \int_{\mathbb{T}} T_n(x+t) 2n \sin nt \Big\{ \frac{1}{2} + \sum_{k=1}^{n-1} \frac{n-k}{n} \cos kt \Big\} dt \\
&= 2n \frac{1}{\pi} \int_{\mathbb{T}} T_n(x+t) \sin nt K_{n-1}(t)\,dt = 2n \frac{1}{\pi} \int_{\mathbb{T}} T_n(t) \sin n(t-x) K_{n-1}(t-x)\,dt.
\end{aligned}$$

Hence, taking estimate (18) and Proposition 1 into account we find that

$$\|T_n'(t)\|_{L^{p(\cdot),\theta}(\mathbb{T})} \leqslant 2n \|\sigma_{n-1}(|T_n|,\cdot)\|_{L^{p(\cdot),\theta}(\mathbb{T})} \leqslant cn \|T_n(t)\|_{L^{p(\cdot),\theta}(\mathbb{T})}. \qquad \square$$

Proof of Theorem 1. For $n \in \mathbb{N}$, we have

$$\sin^r T_n'(t) = (\sin^r t T_n(t))' - r \sin^{r-1} t \cos t T_n(t).$$

Therefore,

$$\|\sin^r t\, T_n'(t)\|_{L^{p(\cdot),\theta}(\mathbb{T})} \leqslant \|(\sin^r t\, T_n(t))'\|_{L^{p(\cdot),\theta}(\mathbb{T})} + r \|\sin^{r-1} t\, T_n(t)\|_{L^{p(\cdot),\theta}(\mathbb{T})}.$$

Now applying Proposition 4 and Lemma 6 we obtain

$$\|\sin^r t\, T_n'(t)\|_{L^{p(\cdot),\theta}(\mathbb{T})} \leqslant c_1 n \|\sin^r t\, T_n(t)\|_{L^{p(\cdot),\theta}(\mathbb{T})} + c_2 n \|\sin^r t\, T_n(t)\|_{L^{p(\cdot),\theta}(\mathbb{T})}.$$

Finally we have (1) with a constant c independent of T_n. $\square$

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