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JESSEN'S FUNCTIONAL AND MAJORIZATION

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Abstract. In this note, we prove a Sherman type inequality for the Jessen's functional by using a majorization method. In consequence, we obtain a Hardy-Littlewood-Pólya-Karamata type inequality, which says that some n-sums generated by the Jessen's functional are Schur-concave with respect to its n-tuple of weight vectors.

1. Introduction

Let E be a nonempty set and $\mathscr{L}(E,\mathbb{R})$ be a linear class of real functions $f : E \to \mathbb{R}$ such that

(i): $f,g \in \mathscr{L}(E,\mathbb{R})$ implies $\alpha f + \beta g \in \mathscr{L}(E,\mathbb{R})$ for $\alpha,\beta \in \mathbb{R}$,

(ii): $1 \in \mathscr{L}(E,\mathbb{R})$, i.e., the constant function $f(t) = 1$ for $t \in E$ is in $\mathscr{L}(E,\mathbb{R})$.

A real functional $A : \mathscr{L}(E,\mathbb{R}) \to \mathbb{R}$ is said to be a *positive linear functional* if

(iii): $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for $f,g \in \mathscr{L}(E,\mathbb{R})$ and $\alpha,\beta \in \mathbb{R}$,

(iv): $f \in \mathscr{L}(E,\mathbb{R})$ and $f(t) \geqslant 0$ for $t \in E$ imply $A(f) \geqslant 0$.

If in addition

(v): A (1) = 1

then A is called *normalized*.

We denote

$$\mathscr{P}^0_{E,A} = \{p \in \mathscr{L}(E,\mathbb{R}) : p \geqslant 0,\ A(p) > 0\}.$$

Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on interval $I \subset \mathbb{R}$. If A is a normalized positive linear functional on $\mathscr{L}(E,\mathbb{R})$, then *Jessen's inequality* [12, pp. 47–48] holds as follows:

$$\Phi(A(f)) \leqslant A(\Phi(f)) \quad \text{for } f \in \mathscr{L}(E,\mathbb{R}) \text{ such that } \Phi(f) \in \mathscr{L}(E,\mathbb{R}). \tag{1}$$

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Consider a function $\Phi : I \to \mathbb{R}$ defined on an interval $I \subset \mathbb{R}$. Assume A is a positive linear functional on $\mathscr{L}(E,\mathbb{R})$. The *Jessen's functional* is defined by

$$J(\Phi, f, p, A) = A(p\Phi(f)) - A(p)\Phi\left(\frac{A(pf)}{A(p)}\right) \tag{2}$$

(see [6]). Here and hereafter it is assumed that the Jessen functional is well-defined, i.e., $f, pf, p\Phi(f) \in \mathscr{L}(E,\mathbb{R})$ and $f \in I^E$.

Jessen's inequality (1) ensures that

$$J(\Phi, f, p, A) \geqslant 0 \quad \text{for a continuous convex function } \Phi.$$

THEOREM A. [7, Theorem 1] *Let $\Phi : I = [a,b] \to \mathbb{R}$ be a continuous convex function, A be a positive linear functional on $\mathscr{L}(E,\mathbb{R})$, $f \in I^E \cap \mathscr{L}(E,\mathbb{R})$, and the map $p \to J(\Phi, f, p, A)$, $p \in \mathscr{P}^0_{E,A}$, be well-defined. Then the map is superadditive, i.e.,*

$$J(\Phi, f, p+q, A) \geqslant J(\Phi, f, p, A) + J(\Phi, f, q, A) \quad \textit{for } p,q \in \mathscr{P}^0_{E,A}. \tag{3}$$

In consequence, the map $p \to J(\Phi, f, p, A)$, $p \in \mathscr{P}^0_{E,A}$, is monotone, i.e.,

$$J(\Phi, f, p+q, A) \geqslant J(\Phi, f, p, A) \quad \textit{for } p,q \in \mathscr{P}^0_{E,A}.$$

A more attentive analysis shows that inequality (3) is equivalent to the subadditivity of the function

$$p \to A(p)\Phi\left(\frac{A(pf)}{A(p)}\right) \quad \text{for } p \in \mathscr{P}^0_{E,A},$$

i.e.,

$$A(p+q)\Phi\left(\frac{A((p+q)f)}{A(p+q)}\right) \leqslant A(p)\Phi\left(\frac{A(pf)}{A(p)}\right) + A(q)\Phi\left(\frac{A(qf)}{A(q)}\right) \quad \text{for } p,q \in \mathscr{P}^0_{E,A}.$$

To motivate our further investigations, consider the following problem. Let $m,n \in \mathbb{N}$. For a given vector $c \in \mathscr{P}^0_{E,A}$, take two sum decompositions

$$c = p_1 + p_2 + \ldots + p_m \quad \text{and} \quad c = q_1 + q_2 + \ldots + q_n,$$

where $p_1, p_2, \ldots, p_m \in \mathscr{P}^0_{E,A}$ and $q_1, q_2, \ldots, q_n \in \mathscr{P}^0_{E,A}$. By virtue of Theorem A we have

$$J(\Phi, f, c, A) \geqslant J(\Phi, f, p_1, A) + J(\Phi, f, p_2, A) + \ldots + J(\Phi, f, p_m, A),$$
$$J(\Phi, f, c, A) \geqslant J(\Phi, f, q_1, A) + J(\Phi, f, q_2, A) + \ldots + J(\Phi, f, q_n, A).$$

In this situation, we are interested in comparing the right-hand sides of the last two inequalities.

In this note our goal is to give conditions on the vector tuples $(p_1, p_2, \ldots, p_m)$ and $(q_1, q_2, \ldots, q_n)$ for the following inequality (4) to hold:

$$\begin{aligned} &J(\Phi, f, p_1, A) + J(\Phi, f, p_2, A) + \ldots + J(\Phi, f, p_m, A) \\ &\leqslant J(\Phi, f, q_1, A) + J(\Phi, f, q_2, A) + \ldots + J(\Phi, f, q_n, A). \end{aligned} \tag{4}$$

Thus for the Jessen's functional, an inequality of the Hardy-Littlewood-Pólya-Karamata type arises. This is a complement of the superadditivity property described in Theorem A.

In fact, a more general class of inequalities can be obtained. Namely, we will provide conditions for real scalars $a_1, a_2, \ldots, a_m$, $b_1, b_2, \ldots, b_n$ and for vectors $p_1, p_2, \ldots,$ p_m, $q_1, q_2, \ldots, q_n$ in $\mathscr{P}^0_{E,A}$ so that the following Sherman type inequality (5) is satisfied:

$$\begin{aligned} &a_1 J(\Phi, f, p_1, A) + a_2 J(\Phi, f, p_2, A) + \ldots + a_m J(\Phi, f, p_m, A) \\ &\leqslant b_1 J(\Phi, f, q_1, A) + b_2 J(\Phi, f, q_2, A) + \ldots + b_n J(\Phi, f, q_n, A). \end{aligned} \tag{5}$$

As will be seen in Section 2, a key for solving the above-mentioned problem is the notion of *vector majorization* for comparing the vector tuples $(p_1, p_2, \ldots, p_m)$ and $(q_1, q_2, \ldots, q_n)$ [4, 11].

2. Sherman inequality for Jessen's functional

Throughout we denote $\mathscr{L} = \mathscr{L}(E, \mathbb{R})$ and $\mathscr{P} = \mathscr{P}^0_{E,A}$ for abbreviation.

We begin with a lemma, which, in fact, is based on the proof of [7, Theorem 1].

LEMMA 2.1. *Let $\Phi : I \to \mathbb{R}$ be a convex function on an interval $I = [a,b] \subset \mathbb{R}$. Let $\mathbf{P} = (p_1, p_2, \ldots, p_l) \in \mathscr{P}^l$. Let A be a positive linear functional on $\mathscr{L}(E, \mathbb{R})$, and $f \in I^E \cap \mathscr{L}(E, \mathbb{R})$.*

If the below expressions are well-defined then

$$A\left(\sum_{i=1}^{l} p_i\right) \Phi\left(\frac{A\left(\sum\limits_{i=1}^{l} p_i f\right)}{A\left(\sum\limits_{i=1}^{l} p_i\right)}\right) \leqslant \sum_{i=1}^{l} A(p_i) \Phi\left(\frac{A(p_i f)}{A(p_i)}\right). \tag{6}$$

If Φ is concave, then the inequality (6) is reversed.

Proof. We introduce

$$\alpha_i = \frac{A(p_i)}{\sum\limits_{j=1}^{l} A(p_j)} \quad \text{for } i = 1, 2, \ldots, l. \tag{7}$$

Then the following identity holds:

$$\frac{A\left(\sum\limits_{i=1}^{l} p_i f\right)}{A\left(\sum\limits_{i=1}^{l} p_i\right)} = \sum_{i=1}^{l} \alpha_i \frac{A(p_i f)}{A(p_i)}. \tag{8}$$

Since $\Phi : I \to \mathbb{R}$ is convex on I, and $\sum_{i=1}^{l} \alpha_i = 1$ with $\alpha_i \geqslant 0$ for $i = 1,2,\ldots,l$, from (7)–(8) we find that

$$\Phi\left(\frac{A\left(\sum_{i=1}^{l} p_i f\right)}{A\left(\sum_{i=1}^{l} p_i\right)}\right) = \Phi\left(\sum_{i=1}^{l} \alpha_i \frac{A(p_i f)}{A(p_i)}\right) \leqslant \sum_{i=1}^{l} \alpha_i \Phi\left(\frac{A(p_i f)}{A(p_i)}\right)$$

$$= \sum_{i=1}^{l} \frac{A(p_i)}{\sum_{j=1}^{l} A(p_j)} \Phi\left(\frac{A(p_i f)}{A(p_i)}\right) = \frac{1}{\sum_{j=1}^{l} A(p_j)} \sum_{i=1}^{l} A(p_i) \Phi\left(\frac{A(p_i f)}{A(p_i)}\right). \tag{9}$$

It now follows that (9) implies (6). □

Before giving the next result some notation is needed. Throughout $m,n \in \mathbb{N}$.

An $m \times n$ real matrix $\mathbf{S} = (s_{ij})$ is said to be *column stochastic* if $s_{ij} \geqslant 0$ for $i = 1,2,\ldots,m,\ j = 1,2,\ldots,n$, and all column sums of $\mathbf{S}$ are equal to 1, i.e., $\sum_{i=1}^{m} s_{ij} = 1$ for $j = 1,2,\ldots,n$.

An $n \times n$ real matrix $\mathbf{S} = (s_{ij})$ is said to be *doubly stochastic* if $s_{ij} \geqslant 0$ for $i = 1,2,\ldots,n,\ j = 1,2,\ldots,n$, and all row and column sums of $\mathbf{S}$ are equal to 1, i.e., $\sum_{j=1}^{n} s_{ij} = 1$ for $i = 1,2,\ldots,n$, and $\sum_{i=1}^{n} s_{ij} = 1$ for $j = 1,2,\ldots,n$.

Given an $m \times n$ real matrix $\mathbf{S} = (s_{ij})$ and $(p_1,p_2,\ldots,p_m) \in \mathscr{L}^m$ and $(q_1,q_2,\ldots,q_n) \in \mathscr{L}^n$, we write

$$(q_1,q_2,\ldots,q_n) = (p_1,p_2,\ldots,p_m)\mathbf{S} \tag{10}$$

in the sense that

$$q_j = s_{1j}p_1 + s_{2j}p_2 + \ldots + s_{mj}p_m \quad \text{for } j = 1,2,\ldots,n. \tag{11}$$

An n-tuple $\mathbf{Q} = (q_1,q_2,\ldots,q_n) \in \mathscr{L}^n$ is said to be *majorized* by n-tuple $\mathbf{P} = (p_1,p_2,\ldots,p_n) \in \mathscr{L}^n$, written as $\mathbf{Q} \prec \mathbf{P}$, if there exists an $n \times n$ doubly stochastic matrix $\mathbf{S} = (s_{ij})$ such that

$$(q_1,q_2,\ldots,q_n) = (p_1,p_2,\ldots,p_n)\mathbf{S} \tag{12}$$

(cf. [8, p. 33]).

Below we show a Sherman like inequality (14) for the functional $p \to A(p)\Phi\left(\frac{A(pf)}{A(p)}\right)$, $p \in \mathscr{P}$, (cf. [13], see also [1, 3, 10]). The basic hypothesis is condition (13) which defines the notion of *weighted majorization* for pairs $(\mathbf{P},\mathbf{a})$ and $(\mathbf{Q},\mathbf{b})$ (see [1, 3, 11] for details).

THEOREM 2.2. *Let $\Phi : I \to \mathbb{R}$ be a convex function on an interval $I = [a,b] \subset \mathbb{R}$. Let $\mathbf{P} = (p_1,p_2,\ldots,p_m) \in \mathscr{P}^m$, $\mathbf{Q} = (q_1,q_2,\ldots,q_n) \in \mathscr{P}^n$, $\mathbf{a} = (a_1,a_2,\ldots,a_m) \in \mathbb{R}_+^m$*

and $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}_+^n$, *where* $\mathscr{P} = \mathscr{P}_{E,A}^0$ *and* $m, n \in \mathbb{N}$. *Let* A *be a positive linear functional on* $\mathscr{L}(E,\mathbb{R})$, *and* $f \in I^E \cap \mathscr{L}(E,\mathbb{R})$.

If

$$\mathbf{Q} = \mathbf{PS} \quad \textit{and} \quad \mathbf{a} = \mathbf{b}\mathbf{S}^T \tag{13}$$

for some $m \times n$ *column stochastic matrix* $\mathbf{S} = (s_{ij})$, *and the below expressions are well-defined, then*

$$\sum_{j=1}^{n} b_j A(q_j)\Phi\left(\frac{A(q_j f)}{A(q_j)}\right) \leqslant \sum_{i=1}^{m} a_i A(p_i)\Phi\left(\frac{A(p_i f)}{A(p_i)}\right). \tag{14}$$

If Φ *is concave, then the inequality* (14) *is reversed.*

Proof. Note that $\frac{A(pf)}{A(p)} \in I = [a,b]$ for $f \in I^E \cap \mathscr{L}(E,\mathbb{R})$. In fact, we have $f(t) \in I$, i.e., $a \leqslant f(t) \leqslant b$ for $t \in E$. Therefore $a1 \leqslant f \leqslant b1$ with $\leqslant$ denoting the componentwise ordering on $\mathscr{L}(E,\mathbb{R})$. Now, we can see that

$$\frac{A(pa1)}{A(p)} \leqslant \frac{A(pf)}{A(p)} \leqslant \frac{A(pb1)}{A(p)}.$$

Hence

$$a \leqslant \frac{A(pf)}{A(p)} \leqslant b.$$

Finally we obtain $\frac{A(pf)}{A(p)} \in I$, as claimed.

Therefore the function

$$p \to \Psi(p) = A(p)\Phi\left(\frac{A(pf)}{A(p)}\right), \quad p \in \mathscr{P}_{E,A}^0, \tag{15}$$

is well-defined. So, both sides of the inequality (14) are well-defined.

We shall prove that the function (15) is convex on $\mathscr{P}$. To do so, we use Lemma 2.1 for $l = 2$. Namely, for any $p, q \in \mathscr{P}$ and $\alpha \in (0,1)$, we have $\alpha p, (1-\alpha)q \in \mathscr{P}$. Thanks to (6) applied to the vectors αp and $(1-\alpha)q$ in place of p_1 and p_2, we derive

$$\begin{aligned}
\Psi(\alpha p + (1-\alpha)q) &= A(\alpha p + (1-\alpha)q)\Phi\left(\frac{A(\alpha p + (1-\alpha)qf)}{A(\alpha p + (1-\alpha)q)}\right) \\
&\leqslant A(\alpha p)\Phi\left(\frac{A(\alpha pf)}{A(\alpha p)}\right) + A((1-\alpha)q)\Phi\left(\frac{A((1-\alpha)qf)}{A((1-\alpha)q)}\right) \\
&= \alpha A(p)\Phi\left(\frac{A(pf)}{A(p)}\right) + (1-\alpha)A(q)\Phi\left(\frac{A(qf)}{A(q)}\right) \\
&= \alpha\Psi(p) + (1-\alpha)\Psi(q).
\end{aligned} \tag{16}$$

Observe that for $\alpha = 0$ or $\alpha = 1$, inequality (16) is satisfied trivially. Thus we have shown that Ψ is convex on $\mathscr{P}$.

By (13), $(q_1,q_2,\dots,q_n)=(p_1,p_2,\dots,p_m)\mathbf{S}$. According to (11) we deduce that $q_j=\sum_{i=1}^{m}s_{ij}p_i$ with $\sum_{i=1}^{m}s_{ij}=1$, $j=1,2,\dots,n$, and $s_{ij}\geqslant 0$. Therefore by convexity of Ψ we get

$$\Psi(q_j)=\Psi\left(\sum_{i=1}^{m}s_{ij}p_i\right)\leqslant\sum_{i=1}^{m}s_{ij}\Psi(p_i)\quad\text{for } j=1,2,\dots,n.$$

Consequently,

$$\sum_{j=1}^{n}b_j\Psi(q_j)\leqslant\sum_{j=1}^{n}b_j\sum_{i=1}^{m}s_{ij}\Psi(p_i)=\sum_{i=1}^{m}\sum_{j=1}^{n}b_js_{ij}\Psi(p_i).$$

But the second equation of (13) gives $a_i=\sum_{j=1}^{n}b_js_{ij}$, $i=1,2,\dots,m$. So, we see that

$$\sum_{j=1}^{n}b_j\Psi(q_j)\leqslant\sum_{i=1}^{m}a_i\Psi(p_i).$$

Combining this with (15) completes the proof of inequality (14). □

The following Hardy-Littlewood-Pólya-Karamata type result is a special case of Theorem 2.2.

THEOREM 2.3. *Let $\Phi:I\to\mathbb{R}$ be a convex function on a closed interval $I\subset\mathbb{R}$. Let $\mathbf{P}=(p_1,p_2,\dots,p_n)\in\mathscr{P}^n$ and $\mathbf{Q}=(q_1,q_2,\dots,q_n)\in\mathscr{P}^n$, where $\mathscr{P}=\mathscr{P}^0_{E,A}$ and $n\in\mathbb{N}$. Let A be a positive linear functional on $\mathscr{L}(E,\mathbb{R})$, and $f\in I^E\cap\mathscr{L}(E,\mathbb{R})$.*

If the below expressions are well-defined, then

$$\mathbf{Q}\prec\mathbf{P}\quad\textit{implies}\quad\sum_{i=1}^{n}A(q_i)\Phi\left(\frac{A(q_if)}{A(q_i)}\right)\leqslant\sum_{i=1}^{n}A(p_i)\Phi\left(\frac{A(p_if)}{A(p_i)}\right).\tag{17}$$

If f is concave, then the inequality (17) is reversed.

Proof. We take $m=n$ and $\mathbf{b}=\mathbf{e}=(1,\dots,1)\in\mathbb{R}^n$. It follows from the vector majorization $(q_1,q_2,\dots,q_n)\prec(p_1,p_2,\dots,p_n)$ that $(q_1,q_2,\dots,q_n)=(p_1,p_2,\dots,p_n)\mathbf{S}$ for some doubly stochastic matrix $\mathbf{S}$ (see (12)).

We also define $\mathbf{a}=\mathbf{b}\mathbf{S}^T$. Then $\mathbf{a}=\mathbf{e}\mathbf{S}^T=\mathbf{e}$. So, by making use of Theorem 2.2, Eq. (14), we obtain the required inequality (17). □

A function $F:\mathscr{P}^n\to\mathbb{R}$ with a convex set $\mathscr{P}\subset\mathscr{L}$ is said to be *Schur-convex* on $\mathscr{P}^n$ if for $\mathbf{P},\mathbf{Q}\in\mathscr{P}^n$,

$$\mathbf{Q}\prec\mathbf{P}\ \text{ implies }\ F(\mathbf{Q})\leqslant F(\mathbf{P}).$$

A function $F:\mathscr{P}^n\to\mathbb{R}$ with a convex set $\mathscr{P}\subset\mathscr{L}$ is said to be *Schur-concave* on $\mathscr{P}^n$, if $-F$ is *Schur-convex* on $\mathscr{P}^n$.

The statement (17) states that the function

$$F(p_1,p_2,\dots,p_n)=\sum_{i=1}^{n}A(p_i)\Phi\left(\frac{A(p_if)}{A(p_i)}\right),\quad p_i\in\mathscr{P},$$

is Schur-convex on $\mathscr{P}^n$.

For a real function $\Phi: I\to\mathbb{R}$, $p\in\mathscr{P}$ and $f\in I^E\cap\mathscr{L}(E,\mathbb{R})$ with $\Phi(f)\in\mathscr{L}(E,\mathbb{R})$, the *Jessen functional* is defined by

$$J(\Phi,f,p,A)=A(p\Phi(f))-A(p)\Phi\left(\frac{A(pf)}{A(p)}\right), \tag{18}$$

where $\Phi(f)=\Phi\circ f$, i.e., $\Phi(f)(t)=\Phi(f(t))$ for $t\in I$ (cf. (2)).

We now illustrate the previous results in terms of the Jessen's functional.

COROLLARY 2.4. *Let $\Phi: I\to\mathbb{R}$ be a convex function on a closed interval $I\subset\mathbb{R}$. Let $\mathbf{P}=(p_1,p_2,\dots,p_m)\in\mathscr{P}^m$, $\mathbf{Q}=(q_1,q_2,\dots,q_n)\in\mathscr{P}^n$, $\mathbf{a}=(a_1,a_2,\dots,a_m)\in\mathbb{R}_+^m$ and $\mathbf{b}=(b_1,b_2,\dots,b_n)\in\mathbb{R}_+^n$, where $\mathscr{P}=\mathscr{P}_{E,A}^0$ and $m,n\in\mathbb{N}$. Let A be a positive linear functional on $\mathscr{L}(E,\mathbb{R})$, and $f\in I^E\cap\mathscr{L}(E,\mathbb{R})$.*

If

$$\mathbf{Q}=\mathbf{PS}\quad\textit{and}\quad\mathbf{a}=\mathbf{bS}^T \tag{19}$$

for some $m\times n$ column stochastic matrix $\mathbf{S}=(s_{ij})$, and the below expressions are well-defined, then

$$\sum_{j=1}^{n}b_jJ(\Phi,f,q_j,A)\geqslant\sum_{i=1}^{m}a_iJ(\Phi,f,p_i,A). \tag{20}$$

If f is concave, then the inequality (20) is reversed.

Proof. It follows from (18) that

$$\begin{aligned}\sum_{j=1}^{n}b_jJ(\Phi,f,q_j,A)&=\sum_{j=1}^{n}b_j\left(A(q_j\Phi(f))-A(q_j)\Phi\left(\frac{A(q_jf)}{A(q_j)}\right)\right)\\&=A\left(\sum_{j=1}^{n}b_jq_j\Phi(f)\right)-\sum_{j=1}^{n}b_jA(q_j)\Phi\left(\frac{A(q_jf)}{A(q_j)}\right).\end{aligned} \tag{21}$$

Analogously,

$$\begin{aligned}\sum_{i=1}^{m}a_iJ(\Phi,f,p_i,A)&=\sum_{i=1}^{m}a_i\left(A(p_i\Phi(f))-A(p_i)\Phi\left(\frac{A(p_if)}{A(p_i)}\right)\right)\\&=A\left(\sum_{i=1}^{m}a_ip_i\Phi(f)\right)-\sum_{i=1}^{m}a_iA(p_i)\Phi\left(\frac{A(p_if)}{A(p_i)}\right).\end{aligned} \tag{22}$$

It is not hard to verify by (19) that

$$\sum_{j=1}^{n}b_jq_j=\sum_{i=1}^{m}a_ip_i.$$

Now, thanks to (21), (22) and Theorem 2.2, we infer that inequality (20) holds, as wanted. □

By setting $\mathbf{a}$ and $\mathbf{b}$ to be the all ones vector $\mathbf{e}$, with the aid of a doubly stochastic matrix $\mathbf{S}$, one obtains the following result from Corollary 2.4.

COROLLARY 2.5. *Let $\Phi : I \to \mathbb{R}$ be a convex function on a closed interval $I \subset \mathbb{R}$. Let $\mathbf{P} = (p_1, p_2, \dots, p_n) \in \mathscr{P}^n$ and $\mathbf{Q} = (q_1, q_2, \dots, q_n) \in \mathscr{P}^n$, where $\mathscr{P} = \mathscr{P}^0_{E,A}$ and $n \in \mathbb{N}$. Let A be a positive linear functional on $\mathscr{L}(E,\mathbb{R})$, and $f \in I^E \cap \mathscr{L}(E,\mathbb{R})$.*

If the below expressions are well-defined, then

$$\mathbf{Q} \prec \mathbf{P} \quad \textit{implies} \quad \sum_{i=1}^{n} J(\Phi, f, q_i, A) \geqslant \sum_{i=1}^{n} J(\Phi, f, p_i, A). \tag{23}$$

If Φ is concave, then the inequality (23) is reversed.

Proof. From $\mathbf{Q} \prec \mathbf{P}$ we have $\mathbf{Q} = \mathbf{PS}$ for some doubly stochastic matrix $\mathbf{S}$ (cf. (12)). By putting $m = n$, $\mathbf{b} = \mathbf{e} = (1, \dots, 1) \in \mathbb{R}^n$, and $\mathbf{a} = \mathbf{b}\mathbf{S}^T$ (see 19), we obtain $\mathbf{a} = \mathbf{e}\mathbf{S}^T = \mathbf{e}$. By using Corollary 2.4, (20), we can see that the desired inequality (23) holds. □

Notice that the statement (23) says that the function

$$J(p_1, p_2, \dots, p_n) = \sum_{i=1}^{n} J(\Phi, f, p_i, A), \quad p_i \in I,$$

is Schur-concave on $\mathscr{P}^n$.

REMARK 2.6. The above corollaries extend analogous results for the Jensen's functional and Jensen-Mercer's functional (see [5, 6, 9, 11] for details).

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