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ON REVERSE ISOPERIMETRIC INEQUALITIES IN TWO–DIMENSIONAL SPACE FORMS AND RELATED RESULTS

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Abstract. In this paper, we investigate the curvature radius of strictly convex domains in two-dimensional space forms. We obtain a class of new reverse isoperimetric inequalities which provide a measure for the deviation of a strictly convex domain from a geodesic disc, via the maximum and the minimum of the curvature radius of its boundary.

1. Introduction

The geodesic circle is uniquely characterized by the following property: among all simple closed curves of given perimeter L in a surface of constant curvature c, the geodesic circle of perimeter L bounds maximum area. This property is succinctly expressed by the isoperimetric inequality

$$L^2 - 4\pi A + cA^2 \geqslant 0, \tag{1}$$

where A is the area bounded by a simple closed curve Γ of perimeter L, and where equality holds if and only if Γ is a geodesic circle [13, 14, 19].

Let $\mathbb{X}_c^2$ denote the two-dimensional space forms of constant curvature c, that is: the Euclidean plane $\mathbb{E}^2$ for $c = 0$; the sphere $\mathbb{S}_c^2$ of radius $\frac{1}{\sqrt{c}}$ for $c > 0$; the hyperbolic plane $\mathbb{H}_c^2$ for $c < 0$.

A more general isoperimetric inequality for the domain D of area A and perimeter L in $\mathbb{X}_c^2$ is

$$L^2 - 4\pi A + cA^2 \geqslant B_D,$$

where B_D is non-negative and vanishes only when D is a geodesic disc. This inequality is also called a Bonnesen-style isoperimetric inequality. Bonnesen-style isoperimetric inequalities are widely investigated by many mathematicians, see for instance [1, 2, 4, 10, 11, 12, 13, 14, 18, 21, 22].

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In contrast to the Bonnesen-style isoperimetric inequality, one may wish to consider the reverse Bonnesen-style isoperimetric inequality:

$$L^2 - 4\pi A + cA^2 \leqslant U_D,$$

where U_D is an invariant, and hopefully equality is attained if and only if D is a geodesic circle.

Only a few reverse Bonnesen-style isoperimetric inequalities for strictly convex domains in the Euclidean plane $\mathbb{E}^2$ are given by [3, 6, 7, 9, 15, 16, 17, 18]. In this paper, our purpose is to investigate the reverse isoperimetric inequalities for strictly convex domains in $\mathbb{X}_c^2$.

2. Preliminaries and main results

To deal simultaneously with the Euclidean plane, the sphere, and the hyperbolic plane, we consider the functions

$$\mathrm{sn}_c(t) := \begin{cases} \frac{1}{\sqrt{-c}}\sinh(\sqrt{-c}t), & c<0,\\ t, & c=0,\\ \frac{1}{\sqrt{c}}\sin(\sqrt{c}t), & c>0, \end{cases}$$

and

$$\mathrm{cn}_c(t) := \begin{cases} \cosh(\sqrt{-c}t), & c<0,\\ 1, & c=0,\\ \cos(\sqrt{c}t), & c>0. \end{cases}$$

It is natural to define the function

$$\mathrm{tn}_c(t) = \frac{\mathrm{sn}_c(t)}{\mathrm{cn}_c(t)}, \quad \mathrm{ctn}_c(t) = \frac{\mathrm{cn}_c(t)}{\mathrm{sn}_c(t)}.$$

Therefore we have the following identities:

$$\begin{aligned} &c\cdot \mathrm{sn}_c^2(t) + \mathrm{cn}_c^2(t) = 1, \quad \mathrm{sn}_c'(t) = \mathrm{cn}_c(t), \quad \mathrm{cn}_c'(t) = -c\cdot \mathrm{sn}_c(t),\\ &\mathrm{sn}_c(t) = \frac{2\mathrm{tn}_c(\frac{t}{2})}{1+c\cdot \mathrm{tn}_c^2(\frac{t}{2})}, \quad \mathrm{cn}_c(t) = \frac{1-c\cdot \mathrm{tn}_c^2(\frac{t}{2})}{1+c\cdot \mathrm{tn}_c^2(\frac{t}{2})}. \end{aligned} \tag{2}$$

A domain D in $\mathbb{X}_c^2$ is a (closed) simply connected subset of $\mathbb{X}_c^2$ such that its boundary is a regular, simple and closed curve. The curvature radius ρ at $x \in \partial D$ is defined by

$$\mathrm{ctn}_c(\rho(x)) = \kappa(x),$$

where $\kappa(x)$ is the curvature of ∂D at x.

If the curvature is further restricted, then we have the following definition from [5].

DEFINITION 1. A domain $D \subset \mathbb{X}_c^2$ with at least C^2 smooth boundary ∂D is convex if the curvature at every point of ∂D is nonnegative; if the curvature at every point of ∂D is positive, D is strictly convex.

If $c > 0$, a convex domain on the sphere $\mathbb{S}_c^2$ is always assumed to lie inside an open hemisphere. If $c < 0$, we need a stronger convexity notion.

DEFINITION 2. A domain $D \subset \mathbb{H}_c^2$ with at least C^2 smooth boundary ∂D is h-convex if the curvature at every point of ∂D is greater than or equal to $\sqrt{-c}$; if the curvature at every point of ∂D is greater than $\sqrt{-c}$, D is strictly h-convex.

For example, discs of radius r in $\mathbb{H}_c^2$ have curvature equal to $\sqrt{-c}\coth(\sqrt{-c}r)$, thus they are strictly h-convex.

We also need a lemma from [5, Theorem 4.1].

LEMMA 1. *Let D be a strictly convex domain of area A and perimeter L in $\mathbb{X}_c^2$, if $c \geqslant 0$, or strictly h-convex if $c < 0$. Then*

$$\int_{\partial D} \mathrm{tn}_c \rho(s) ds \geqslant A \frac{4\pi - cA}{2\pi - cA}, \tag{3}$$

where s is the arc length parameter.

It follows from (3) that

THEOREM 1. *Let D be a strictly h-convex domain of area A and perimeter L in $\mathbb{H}_c^2$, let ρ be the curvature radius at ∂D. Then*

$$A < \int_{\partial D} \mathrm{tn}_c \rho(s) ds < \frac{L}{\sqrt{-c}}. \tag{4}$$

Proof. It is obvious that

$$1 < \frac{4\pi - cA}{2\pi - cA} < 2 \quad \text{for } c < 0.$$

We have from (3)

$$\int_{\partial D} \mathrm{tn}_c \rho(s) ds > A. \tag{5}$$

Since

$$1 + \tanh^2\left(\frac{\sqrt{-c}\rho}{2}\right) > 2\tanh\left(\frac{\sqrt{-c}\rho}{2}\right)$$

and, for $c < 0$,

$$\mathrm{tn}_c \rho = \frac{2\mathrm{tn}_c(\frac{\rho}{2})}{1 - c \cdot \mathrm{tn}_c^2(\frac{\rho}{2})} = \frac{2\tanh(\frac{\sqrt{-c}\rho}{2})}{1 + \tanh^2(\frac{\sqrt{-c}\rho}{2})} \frac{1}{\sqrt{-c}} < \frac{1}{\sqrt{-c}},$$

we have

$$\int_{\partial D} \mathrm{tn}_c \rho(s)ds < \frac{L}{\sqrt{-c}}. \tag{6}$$

Combining (5), (6) gives the inequality (4). □

Let D be a bounded convex domain with perimeter L and area A in $\mathbb{X}_c^2$. Let $N(x) \in T_x\mathbb{X}_c^2$ denote the outward unit normal vector at the point x belonging to ∂D. For each $x \in \partial D$ let $\gamma(t) = \exp_x(tN(x))$ be the unit speed geodesic starting at x with $\gamma(0) = x$, $\gamma'(0) = N(x)$. For any $t \geqslant 0$, each point $x \in \partial D$ moves the same distance t along the geodesic through x. Then one has a convex domain D_t parallel to D at distance $t(\geqslant 0)$. The area $A(t)$ of D_t in $\mathbb{X}_c^2 (c \neq 0)$ is

$$A(t) = \frac{2\pi}{c} + L\mathrm{sn}_c(t) - \left(\frac{2\pi}{c} - A\right)\mathrm{cn}_c(t).$$

This formula is called the Steiner formula of D in $\mathbb{X}_c^2$ ([20]; see also [18, p. 322]). By (2), $A(t)$ can be written as

$$A(t) = \frac{(4\pi - cA)\mathrm{tn}_c^2(\frac{t}{2}) + 2L\mathrm{tn}_c\frac{t}{2} + A}{1 + c\cdot \mathrm{tn}_c^2(\frac{t}{2})}.$$

It is worth noting that the behavior of the Steiner formula of D for negative values of t is quite interesting. Setting

$$A(t) = 0,$$

we have

$$\mathrm{tn}_c\left(\frac{t_1}{2}\right) = \frac{-L + \sqrt{\Delta_c(D)}}{4\pi - cA}, \quad \mathrm{tn}_c\left(\frac{t_2}{2}\right) = \frac{-L - \sqrt{\Delta_c(D)}}{4\pi - cA},$$

where $\Delta_c(D) = L^2 - 4\pi A + cA^2$. We notice that when $c > 0$ we have $4\pi - cA > 0$ because $D \subset \mathbb{S}_c^2$ is always assumed to lie in an open hemisphere. The fact that the roots are real is equivalent to the isoperimetric inequality (1).

Let $D \subset \mathbb{X}_c^2$ be a convex domain (make it h-convex if $c < 0$). The *focal set* $F(\partial D)$ of ∂D is the set

$$F(\partial D) = \{\exp_x(-\rho(x)N(x)) : x \in \partial D\} \subset \mathbb{X}_c^2,$$

where $N(x)$ is the outward unit normal vector at $x \in \partial D$ and $\rho(x)$ is the curvature radius at $x \in \partial D$. The definition of the *focal set* also can be found in [5].

LEMMA 2. *Let D be a strictly convex domain of area A and perimeter L in $\mathbb{X}_c^2$, if $c \geqslant 0$, or strictly h-convex if $c < 0$. Let ρ_M and ρ_m be the maximum and the minimum of the curvature radius ρ at ∂D. Then we have*

$$\mathrm{tn}_c\left(\frac{\rho_m}{2}\right) \leqslant \frac{L - \sqrt{\Delta_c(D)}}{4\pi - cA} \leqslant \frac{L + \sqrt{\Delta_c(D)}}{4\pi - cA} \leqslant \mathrm{tn}_c\left(\frac{\rho_M}{2}\right). \tag{7}$$

Each inequality holds as equality if and only if D is a geodesic disc.

In particular,

$$A(-\rho_m) \geqslant 0, \quad A(-\rho_M) \geqslant 0.$$

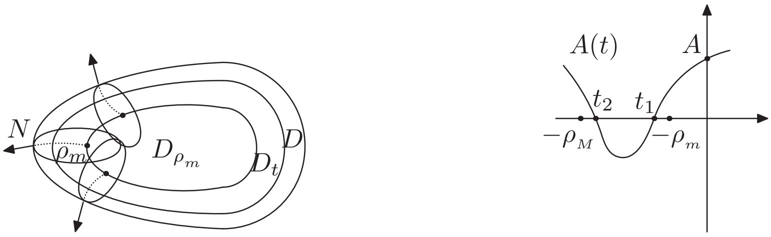

Figure 1: *The convex domain D_t for $-\rho_m \leqslant t \leqslant 0$ and its area $A(t)$ for $t \leqslant 0$.*

Proof. Let ρ_m be the minimum of the curvature radius ρ at $x_0 \in \partial D$, that is, $\rho_m = \rho(x_0)$ at $x_0 \in \partial D$. Then the curvature

$$\kappa(x) \leqslant \kappa(x_0) = \mathrm{ctn}_c(\rho_m) \quad \text{for } x \in \partial D.$$

Note that the convex domain we consider is simple connected, and $\rho_m < \frac{\pi}{2\sqrt{c}} < \frac{\pi}{\sqrt{c}}$ when $c > 0$ since D is strictly convex. By applying Theorem 2.9 in [9] to $\mathbb{X}_c^2$ the focal point $y_0 = \exp_{x_0}(-\rho_m N(x_0))$ has a distance of at least ρ_m from ∂D, and D contains a geodesic disc of radius ρ_m. Then at every boundary point the tangent disc of radius ρ_m remains inside D, and the locus of centers of geodesic discs with radius ρ_m is a convex curve (see Figure 1). If each point $x \in \partial D$ moves inward along the geodesic $\gamma(t)$ for $0 \leqslant -t \leqslant \rho_m$, then D_t remains convex and of non-negative area. So $A(-t) \geqslant 0$ for $0 \leqslant -t \leqslant \rho_m$. Further,

$$\mathrm{tn}_c\left(\frac{\rho_m}{2}\right) \leqslant \frac{L - \sqrt{\Delta_c(D)}}{4\pi - cA}.$$

Equality holds if and only if $A(-\rho_m) = 0$, that is, D_{ρ_m} is a point since D is strictly convex. Thus equality holds if and only if D is a geodesic disc. For $-t \geqslant \rho_M$, D_t remains again convex and of non-negative area. So $A(-t) \geqslant 0$ for $-t \geqslant \rho_M$. It follows that

$$\mathrm{tn}_c\left(\frac{\rho_M}{2}\right) \geqslant \frac{L + \sqrt{\Delta_c(D)}}{4\pi - cA}.$$

Equality holds if and only if D is a geodesic disc. □

When $c = 0$, (7) is written as

$$\rho_m \leqslant \frac{L - \sqrt{L^2 - 4\pi A}}{2\pi} \leqslant \frac{L + \sqrt{L^2 - 4\pi A}}{2\pi} \leqslant \rho_M,$$

which was proved by Green and Osher in [8, Theorem 1.10].

THEOREM 2. *Let D be a strictly convex domain of area A and perimeter L in $\mathbb{X}_c^2$, if $c \geqslant 0$, or strictly h-convex if $c < 0$. Let ρ_M and ρ_m be, respectively, the maximum and the minimum of the curvature radius ρ at ∂D. Then we have the following* *inequalities:*

$$L^2 - 4\pi A + cA^2 \leqslant \frac{A^2}{4}\left(\frac{1}{\mathrm{tn}_c(\frac{\rho_m}{2})} - \frac{1}{\mathrm{tn}_c(\frac{\rho_M}{2})}\right)^2,$$

$$L^2 - 4\pi A + cA^2 \leqslant L^2\left(\frac{\mathrm{tn}_c(\frac{\rho_m}{2}) - \mathrm{tn}_c(\frac{\rho_M}{2})}{\mathrm{tn}_c(\frac{\rho_m}{2}) + \mathrm{tn}_c(\frac{\rho_M}{2})}\right)^2.$$

Each inequality holds as equality if and only if D is a geodesic disc.

Proof. From (7) and since $4\pi - cA = (L^2 - \Delta_c(D))/A$, we have

$$\sqrt{\Delta_c(D)} \leqslant \frac{A}{\mathrm{tn}_c(\frac{\rho_m}{2})} - L \quad \text{and} \quad \sqrt{\Delta_c(D)} \leqslant L - \frac{A}{\mathrm{tn}_c(\frac{\rho_M}{2})}. \tag{8}$$

Adding the two inequalities gives

$$2\sqrt{\Delta_c(D)} \leqslant A\left(\frac{1}{\mathrm{tn}_c(\frac{\rho_m}{2})} - \frac{1}{\mathrm{tn}_c(\frac{\rho_M}{2})}\right),$$

that is,

$$L^2 - 4\pi A + cA^2 \leqslant \frac{A^2}{4}\left(\frac{1}{\mathrm{tn}_c(\frac{\rho_m}{2})} - \frac{1}{\mathrm{tn}_c(\frac{\rho_M}{2})}\right)^2. \tag{9}$$

Multiplying the first inequality in (8) by $\mathrm{tn}_c(\frac{\rho_m}{2})$, the second by $\mathrm{tn}_c(\frac{\rho_M}{2})$, we have

$$\mathrm{tn}_c\left(\frac{\rho_m}{2}\right)\sqrt{\Delta_c(D)} \leqslant A - L\mathrm{tn}_c\left(\frac{\rho_m}{2}\right),$$

$$\mathrm{tn}_c\left(\frac{\rho_M}{2}\right)\sqrt{\Delta_c(D)} \leqslant L\mathrm{tn}_c\left(\frac{\rho_M}{2}\right) - A.$$

Adding the two inequalities above yields

$$\left(\mathrm{tn}_c(\frac{\rho_m}{2}) + \mathrm{tn}_c\left(\frac{\rho_M}{2}\right)\right)\sqrt{\Delta_c(D)} \leqslant L\left(\mathrm{tn}_c(\frac{\rho_M}{2}) - \mathrm{tn}_c\left(\frac{\rho_m}{2}\right)\right).$$

Therefore,

$$L^2 - 4\pi A + cA^2 \leqslant L^2\left(\frac{\mathrm{tn}_c(\frac{\rho_m}{2}) - \mathrm{tn}_c(\frac{\rho_M}{2})}{\mathrm{tn}_c(\frac{\rho_m}{2}) + \mathrm{tn}_c(\frac{\rho_M}{2})}\right)^2. \tag{10}$$

The fact that equality holds in (9) or (10) implies that equalities hold in (8), that is,

$$\mathrm{tn}_c\left(\frac{\rho_m}{2}\right) = \frac{L - \sqrt{\Delta_c(D)}}{4\pi - cA}, \quad \mathrm{tn}_c\left(\frac{\rho_M}{2}\right) = \frac{L + \sqrt{\Delta_c(D)}}{4\pi - cA}.$$

Therefore, equality in (9) or (10) holds if and only if D is a geodesic disc, by Lemma 2. The proof of Theorem 1 is complete. □

From Theorem 1 we get the following corollary.

COROLLARY 1. *Let D be a strictly convex domain of area A and perimeter L in $\mathbb{E}^2$. Let ρ_M and ρ_m be, respectively, the maximum and the minimum of the curvature radius ρ at ∂D. Then we have the following inequalities:*

$$L^2 - 4\pi A \leqslant A^2 \left(\tfrac{1}{\rho_m} - \tfrac{1}{\rho_M}\right)^2,$$
$$L^2 - 4\pi A \leqslant L^2 \left(\tfrac{\rho_m - \rho_M}{\rho_m + \rho_M}\right)^2.$$

Each inequality holds as equality if and only if D is a disc.

THEOREM 3. *Let D be a strictly convex domain of area A and perimeter L in $\mathbb{X}_c^2$, if $c \geqslant 0$, or strictly h-convex if $c < 0$. Let ρ_M and ρ_m be, respectively, the maximum and the minimum of the curvature radius ρ at ∂D. Then we have*

$$\Delta_c(D) \leqslant \left(2\pi - \frac{c}{2}A\right)^2 \left(\mathrm{tn_c}\left(\frac{\rho_{\mathrm{M}}}{2}\right) - \mathrm{tn_c}\left(\frac{\rho_{\mathrm{m}}}{2}\right)\right)^2. \tag{11}$$

The inequality holds as equality if and only if D is a disc.

Proof. It follows from (7) that

$$\frac{L + \sqrt{\Delta_c(D)}}{4\pi - cA} - \frac{L - \sqrt{\Delta_c(D)}}{4\pi - cA} \leqslant \mathrm{tn}_c\left(\frac{\rho_M}{2}\right) - \mathrm{tn}_c\left(\frac{\rho_m}{2}\right),$$

that is,

$$\Delta_c(D) \leqslant \left(2\pi - \frac{c}{2}A\right)^2 \left(\mathrm{tn_c}\left(\frac{\rho_{\mathrm{M}}}{2}\right) - \mathrm{tn_c}\left(\frac{\rho_{\mathrm{m}}}{2}\right)\right)^2.$$

The inequality holds as equality if and only if D is a disc. □

Inequality (11) was already obtained by Li and Zhou in [12]. Here we prove it using a different method. When $c = 0$, this inequality becomes the classical inequality

$$L^2 - 4\pi A \leqslant \pi^2(\rho_M - \rho_m)^2,$$

attributed to Bottema [3].

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