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SOME GEOMETRIC PROPERTIES OF DIFFERENCE SEQUENCE SPACES OF ORDER m DERIVED BY GENERALIZED MEANS AND COMPACT OPERATORS

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Abstract. We have introduced a new sequence space $l(r,s,t,p;\Delta^{(m)})$ combining by using generalized means and difference operator of order m. Some topological properties as well as geometric properties namely Banach-Saks property of type p and uniform Opial property have been studied. Furthermore, the α-, β-, γ- duals of this space are computed and also obtained necessary and sufficient conditions for some matrix transformations from $l(r,s,t,p;\Delta^{(m)})$ to l_∞, l_1. Finally, we obtained some identities or estimates for operator norms and measure of noncompactness of some matrix operators on the BK space $l_p(r,s,t;\Delta^{(m)})$ by applying the Hausdorff measure of noncompactness.

1. Introduction

Let w be the space of all real or complex sequences $x = (x_n)$, $n \in \mathbb{N}_0$. For an infinite matrix A and a sequence space X, the matrix domain of A, denoted by X_A, is defined as $X_A = \{x \in w : Ax \in X\}$ [37].

Let (p_k) be a bounded sequence of strictly positive real numbers such that $H = \sup_k p_k$ and $M = \max\{1,H\}$. The linear spaces $c(p), c_0(p)$, $l_\infty(p)$ and $l(p)$ are introduced and studied by Maddox [23], where

$$c(p) = \left\{x = (x_k) \in w : \lim_{k\to\infty} |x_k - l|^{p_k} = 0 \text{ for some scalar } l\right\},$$

$$c_0(p) = \left\{x = (x_k) \in w : \lim_{k\to\infty} |x_k|^{p_k} = 0\right\},$$

$$l_\infty(p) = \left\{x = (x_k) \in w : \sup_k |x_k|^{p_k} < \infty\right\} \text{ and}$$

$$l(p) = \left\{x = (x_k) \in w : \sum_{k=0}^{\infty} |x_k|^{p_k} < \infty\right\}.$$

The linear spaces $c(p), c_0(p)$, $l_\infty(p)$ are complete with the paranorm $g(x) = \sup_k |x_k|^{\frac{p_k}{M}}$ if and only if $\inf p_k > 0$ for all k while $l(p)$ is complete with the paranorm $\tilde{g}(x) =$

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$\left(\sum_k |x_k|^{p_k}\right)^{\frac{1}{M}}$. Recently, several authors introduced sequence spaces by using matrix domain; for example, Başar et al. [4] studied the space $bs(p) = [l_\infty(p)]_S$, where S is the summation matrix. Altay and Başar [1] studied the sequence spaces $r^t(p)$ and $r^t_\infty(p)$, which consist of all sequences whose Riesz transforms are in the spaces $l(p)$ and $l_\infty(p)$ respectively, i.e., $r^t(p) = [l(p)]_{R^t}$ and $r^t_\infty(p) = [l_\infty(p)]_{R^t}$.

Kızmaz [22] first introduced and studied the difference sequence space. Later on, several authors including Çolak and Et [10], Polat and Başar [33], Başar and Altay [5], Altay and Başar [2], Aydın and Başar [3] and others have studied sequence spaces defined by using difference operator.

On the other hand, sequence spaces are also defined by using generalized weighted mean (see [1], [27]). Mursaleen and Noman [30] also introduced a sequence space of generalized means, which includes most of the known sequence spaces. In 2011, Polat et al. [34] introduced and studied a sequence space which is generated by combining both the weighted mean and the difference operator. Later on Başarır and Kara (see [7], [8]) generalized the sequence spaces of Polat et al. [34] to an mth-order difference sequence.

Recently, mathematicians are interested to study the geometric properties of Banach sequence spaces as these have various applications in the several areas of mathematics. In literature, a lot of papers about the geometric properties can be found; for example one can see the papers of Cui and Hudzik ([11], [12]). Cui and Meng [13], Ng and Lee [31] investigated the geometric properties of Cesàro sequence spaces. Başar et al. [6] studied rotundity property of generalized space of the space bv_p of p-bounded variation sequences. Quite recently, Hudzik et al. [21], Şimşek and Karakaya [36], Et et al. [18] pursued the geometric properties of sequence spaces.

The aim of this paper is to introduce a new sequence space $l(r,s,t,p;\Delta^{(m)})$ defined by using both the generalized means and the difference operator of order m. We investigate some topological properties as well as geometric properties namely Banach-Saks property of type p, uniform Opial property etc. We also compute α-, β-, γ-duals and characterize some matrix mappings on the new sequence space. Finally, we obtain some identities or estimates for operator norms and measure of noncompactness of some matrix operators on the BK space $l_p(r,s,t;\Delta^{(m)})$ by using the Hausdorff measure of noncompactness.

2. Preliminaries

Let l_∞, c and c_0 be the spaces of all bounded, convergent and null sequences $x = (x_n)$ respectively with norm $\|x\|_\infty = \sup_n |x_n|$. Let bs, cs and l_p be the sequence spaces of all bounded, convergent series and p-absolutely summable series respectively and bv be the space of bounded variation sequences. We denote by $e = (1,1,\cdots)$ and e_n for the sequence whose n-th term is 1 and others are zero and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of all natural numbers. For any subsets U and V of w, the multiplier space $M(U,V)$ of U and V is defined as

$$M(U,V) = \{a = (a_n) \in w : au = (a_n u_n) \in V \text{ for all } u = (u_n) \in U\}.$$

In particular,

$$U^{\alpha} = M(U, l_1), \; U^{\beta} = M(U, cs) \text{ and } U^{\gamma} = M(U, bs)$$

are called the α-, β- and γ- duals of U respectively [26].

Let $A = (a_{nk})_{n,k}$ be an infinite matrix with real or complex entries a_{nk}. We write A_n as the sequence of the n-th row of A, i.e., $A_n = (a_{nk})_k$ for every n. For $x = (x_n) \in w$, the A-transform of x is defined as the sequence $Ax = ((Ax)_n)$, where

$$A_n(x) = (Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k,$$

provided the series on the right side converges for each n. For any two sequence spaces U and V, we denote by (U, V), the class of all infinite matrices A that map from U into V. Therefore $A \in (U, V)$ if and only if $Ax \in V$ for all $x \in U$. In other words, $A \in (U, V)$ if and only if $A_n \in U^{\beta}$ for all n [37].

A sequence space X is called a BK space if it is a Banach space with continuous coordinates $p_n : X \to \mathbb{K}$, where $\mathbb{K}$ denotes the real or complex field and $p_n(x) = x_n$ for all $x = (x_n) \in X$ and each $n \in \mathbb{N}_0$. An infinite matrix $T = (t_{nk})_{n,k}$ is called a triangle if $t_{nn} \neq 0$ and $t_{nk} = 0$ for all $k > n$. Let T be a triangle and X be a BK space. Then X_T is also a BK space with the norm given by $\|x\|_{X_T} = \|Tx\|_X$ for all $x \in X_T$ [37].

A functional ρ on w is called a convex modular if the following conditions satisfy:

i) $\rho(x) = 0$ if and only if $x = \theta$

ii) $\rho(\alpha x) = \rho(x)$ for all scalar α with $|\alpha| = 1$ and for all $x \in w$

iii) $\rho(\alpha x + \beta y) \leqslant \alpha\rho(x) + \beta\rho(y)$ for all scalar $\alpha, \beta \geqslant 0$ with $\alpha + \beta = 1$ and for all $x, y \in w$.

If the functional ρ satisfies *i*), *ii*) and

$\rho(\alpha x + \beta y) \leqslant \rho(x) + \rho(y)$ for all scalar $\alpha, \beta \geqslant 0$ with $\alpha + \beta = 1$ and for all $x, y \in w$,

then ρ is called a modular. It is obvious that every convex modular is a modular but the converse is not so.

For any modular ρ on a vector space X, the modular space is defined as

$$X_{\rho} = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0^{+}\}.$$

If ρ is a convex modular on X, then the functionals

$$\|x\|_L = \inf\left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leqslant 1\right\}$$

and

$$\|x\|_A = \inf \frac{1}{k}\left(1 + \rho(kx)\right)$$

become norms on X_{ρ} which are called Luxemburg norm and Amemiya norm respectively. In fact, these norms are equivalent and $\|x\|_L \leqslant \|x\|_A \leqslant 2\|x\|_L$ for all $x \in X_{\rho}$.

A modular ρ is said to satisfy the Δ_2 condition ($\rho \in \Delta_2$ in short) if for any $\varepsilon > 0$ there exist constants $K \geqslant 2$ and $a > 0$ such that

$$\rho(2x) \leqslant K\rho(x) + \varepsilon$$

for all $x \in X_\rho$ with $\rho(x) \leqslant a$. If ρ satisfies Δ_2-condition for all $a > 0$ with $K \geqslant 2$ dependent on a, then ρ is said to satisfy strong Δ_2-condition ($\rho \in \Delta_2^s$ in short).

LEMMA 1. [32] *Convergence in norm and in modular are equivalent in X_ρ if $\rho \in \Delta_2$.*

LEMMA 2. [32] *If $\rho \in \Delta_2^s$, then for any $L > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|\rho(u+v) - \rho(u)| < \varepsilon$$

whenever $u, v \in X_\rho$ with $\rho(u) \leqslant L$ and $\rho(v) \leqslant \delta$.

LEMMA 3. [32] *If $\rho \in \Delta_2^s$, then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x\| \geqslant 1 + \delta$ whenever $\rho(x) \geqslant 1 + \varepsilon$.*

Given any $p \in (1, \infty)$, we say that a Banach space X has the Banach-Saks property of type p (see [21]) if there exists a constant $C > 0$ such that every weakly null sequence (x_k) there exists a subsequence (x_{k_r}) such that

$$\Big\| \sum_{r=1}^{n} x_{k_r} \Big\| \leqslant C n^{\frac{1}{p}} \quad \text{for all } n \in \mathbb{N}.$$

A Banach space X is said to have Opial property if for every weakly null sequence (x_n) and for every $x \neq \theta$ in X, we have

$$\liminf_{n \to \infty} \|x_n\| < \liminf_{n \to \infty} \|x_n + x\|.$$

A Banach space X is said to have the uniform Opial property if for every $\varepsilon > 0$ there exists $\mu > 0$ such that

$$1 + \mu \leqslant \liminf_{n \to \infty} \|x_n + x\|$$

for any weakly null sequence $(x_n) \in S(X)$ and $x \in X$ with $\|x\| \geqslant \varepsilon$ (see [35]).

3. Difference sequence space $l(r, s, t, p; \Delta^{(m)})$

We first begin with the notion of generalized means given by Mursaleen et al. [30]. Consider the sets $\mathscr{U}$ and $\mathscr{U}_0$ as

$$\mathscr{U} = \Big\{ u = (u_n) \in w : u_n \neq 0 \text{ for all } n \Big\} \text{ and } \mathscr{U}_0 = \Big\{ u = (u_n) \in w : u_0 \neq 0 \Big\}.$$

Let $r=(r_n), t=(t_n)\in\mathscr{U}$ and $s=(s_n)\in\mathscr{U}_0$. The sequence $y=(y_n)$ of generalized means of a sequence $x=(x_n)$ is defined by

$$y_n=\frac{1}{r_n}\sum_{k=0}^{n}s_{n-k}t_kx_k \qquad (n\in\mathbb{N}_0).$$

The infinite matrix $A(r,s,t)$ of generalized means is defined by

$$(A(r,s,t))_{nk}=\begin{cases}\frac{s_{n-k}t_k}{r_n} & 0\leqslant k\leqslant n,\\ 0 & k>n.\end{cases}$$

Since $A(r,s,t)$ is a triangle, it has a unique inverse and the inverse is also a triangle [20]. Take $D_0^{(s)}=\frac{1}{s_0}$ and

$$D_n^{(s)}=\frac{1}{s_0^{n+1}}\begin{vmatrix} s_1 & s_0 & 0 & 0\cdots & 0\\ s_2 & s_1 & s_0 & 0\cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \\ s_{n-1} & s_{n-2} & s_{n-3} & s_{n-4}\cdots & s_0\\ s_n & s_{n-1} & s_{n-2} & s_{n-3}\cdots & s_1\end{vmatrix} \qquad \text{for } n=1,2,3,\cdots$$

Then the inverse of $A(r,s,t)$ is the triangle $B=(b_{nk})_{n,k}$, which is defined as

$$b_{nk}=\begin{cases}(-1)^{n-k}\,\frac{D_{n-k}^{(s)}}{t_n}r_k & 0\leqslant k\leqslant n,\\ 0 & k>n.\end{cases}$$

Combining both the generalized means and the difference operator of order m, we now introduce a sequence space $l(r,s,t,p;\Delta^{(m)})$ defined as

$$l(r,s,t,p;\Delta^{(m)})=\Big\{x=(x_n)\in w:[A(r,s,t).\Delta^{(m)}]x\in l(p)\Big\},$$

where $p=(p_k)$ is a bounded sequence of strictly positive real numbers. The sequence $y=(y_n)$ is $A(r,s,t).\Delta^{(m)}$-transform of a sequence $x=(x_n)$, i.e.,

$$y_n=\frac{1}{r_n}\sum_{j=0}^{n}\left[\sum_{i=j}^{n}(-1)^{i-j}\binom{m}{i-j}s_{n-i}t_i\right]x_j. \tag{1}$$

Since $A(r,s,t).\Delta^{(m)}$ is a triangle, it has a unique inverse and the inverse transform is

$$x_n=\sum_{j=0}^{n}\sum_{k=j}^{n}(-1)^{k-j}\binom{m+n-k-1}{n-k}\frac{D_{k-j}^{(s)}}{t_k}r_jy_j, \qquad n\in\mathbb{N}_0. \tag{2}$$

In particular if $p_k=p\ (1\leqslant p<\infty)$ then we write the space as $l_p(r,s,t;\Delta^{(m)})$ instead of $l(r,s,t,p;\Delta^{(m)})$. This sequence space includes many earlier known sequence spaces studied by several authors with the particular choices of the parameters. One can see some recent developments in this direction by the authors (see [24], [28]).

Using modular, we can define the sequence space $l(r,s,t,p;\Delta^{(m)})$ as

$$l(r,s,t,p;\Delta^{(m)}) = \{x \in w : \rho_{\Delta^{(m)}}(\sigma x) < \infty \text{ for some } \sigma > 0\},$$

where

$$\rho_{\Delta^{(m)}}(x) = \sum_{n=0}^{\infty} \left| \frac{1}{r_n} \sum_{j=0}^{n} \left[\sum_{i=j}^{n} (-1)^{i-j} \binom{m}{i-j} s_{n-i} t_i \right] x_j \right|^{p_n}.$$

Clearly $\rho_{\Delta^{(m)}}$ is a convex modular and $\rho_{\Delta^{(m)}} \in \Delta_2^s$ if $\sup_{k \geqslant 1} p_k < \infty$. The space $l(r,s,t,p;\Delta^{(m)})$ is complete equipped with the Luxemburg norm

$$\|x\| = \inf\left\{\sigma > 0 : \rho_{\Delta^{(m)}}\left(\frac{x}{\sigma}\right) \leqslant 1\right\}.$$

The rest of this paper we assume that $p = (p_k)$ is bounded with $p_k > 1$ for all k and $M = \sup_{k \geqslant 1} p_k$. We also denote $x|_i = (x_1, x_2, \ldots, x_i, 0, 0, \ldots)$ and $x|_{\mathbb{N}-i} = (0, 0, \ldots, x_{i+1}, x_{i+2}, \ldots)$ for $i \in \mathbb{N}$ and $x \in w$.

4. Main results

In this section, we begin with some topological results of the newly defined sequence space.

THEOREM 1. *(a) The sequence space $l(r,s,t,p;\Delta^{(m)})$ is a complete linear metric space paranormed by $\tilde{h}$ defined as*

$$\tilde{h}(x) = \left(\sum_{n=0}^{\infty} \left| \frac{1}{r_n} \sum_{j=0}^{n} \left[\sum_{i=j}^{n} (-1)^{i-j} \binom{m}{i-j} s_{n-i} t_i \right] x_j \right|^{p_n} \right)^{\frac{1}{M}}.$$

(b) The sequence space $l_p(r,s,t;\Delta^{(m)})$, $1 \leqslant p < \infty$ is a BK space with the norm given by

$$\|x\|_{l_p(r,s,t;\Delta^{(m)})} = \|y\|_{l_p},$$

where $y = (y_n)$ is defined in (1).

Proof. We leave the proof as it is a routine verification. □

THEOREM 2. *The sequence space $l(r,s,t,p;\Delta^{(m)})$ is linearly isomorphic to the space $l(p)$, i.e., $l(r,s,t,p;\Delta^{(m)}) \cong l(p)$.*

Proof. We omit the proof as it is trivial. □

PROPOSITION 1. *Let $p = (p_k)$ be a bounded sequence with $p_k > 1$ for all k and $M = \sup_{k \geqslant 1} p_k$. Then*

1). if $0 < \lambda < 1$ then $\lambda^M \rho_{\Delta^{(m)}}(\frac{x}{\lambda}) \leqslant \rho_{\Delta^{(m)}}(x)$ and $\rho_{\Delta^{(m)}}(\lambda x) \leqslant \lambda \rho_{\Delta^{(m)}}(x)$.

2). if $\lambda \geqslant 1$, then $\rho_{\Delta^{(m)}}(x) \geqslant \lambda \rho_{\Delta^{(m)}}(\frac{x}{\lambda})$ and $\rho_{\Delta^{(m)}}(x) \leqslant \lambda^M \rho_{\Delta^{(m)}}(\frac{x}{\lambda})$.

Proof. We leave the proof as it follows from the definition of $\rho_{\Delta^{(m)}}$. □

PROPOSITION 2. *Let $p=(p_k)$ be a bounded sequence with $p_k>1$ for all k and $M=\sup_{k\geqslant 1} p_k$. Then*
1). if $\|x\|<1$ then $\rho_{\Delta^{(m)}}(x)\leqslant\|x\|$.
2). if $\|x\|>1$ then $\rho_{\Delta^{(m)}}(x)\geqslant\|x\|$.
3). if $\|x\|=1$ then $\rho_{\Delta^{(m)}}(x)=\|x\|$.
4). if $0<\lambda<1$ and $\|x\|>\lambda$ then $\rho_{\Delta^{(m)}}(x)>\lambda^M$.
5). if $\lambda\geqslant 1$ and $\|x\|\leqslant\lambda$ then $\rho_{\Delta^{(m)}}(x)<\lambda^M$.

Proof. The proof follows from Proposition 1. □

THEOREM 3. *Let $p=(p_k)$ be a bounded sequence with $p_k>1$ for all k. Then the sequence space $l(r,s,t,p;\Delta^{(m)})$ equipped with the Luxemburg norm has the uniform Opial property.*

Proof. Let $X=l(r,s,t,p;\Delta^{(m)})$ and $S(X)$ be the unit sphere of X. Let (x^k) be any weakly null sequence in $S(X)$. We will show that for any $\varepsilon>0$ there exists $\mu>0$ such that

$$\liminf_{l\to\infty}\|x^k+x\|\geqslant 1+\mu,$$

where $x\in X$ satisfying $\|x\|\geqslant\varepsilon$. Since $\sup_{k\geqslant 1}p_k<\infty$, so $\rho_{\Delta^{(m)}}\in\Delta_2^s$ and hence the convergence in norm and the convergence in modular are equivalent by Lemma 1. Therefore there exists $\delta\in(0,1)$ such that $\rho_{\Delta^{(m)}}(x)\geqslant\delta$ as $\|x\|\geqslant\varepsilon$. Again from Lemma 2, we can find $\delta_1\in(0,\delta)$ such that $\rho_{\Delta^{(m)}}(y)\leqslant 1$, $\rho_{\Delta^{(m)}}(z)\leqslant\delta_1$ implies

$$\left|\rho_{\Delta^{(m)}}(y+z)-\rho_{\Delta^{(m)}}(y)\right|<\frac{\delta}{6}. \tag{3}$$

Since $x\in X$, there exists $n_0\in\mathbb{N}$ such that

$$\sum_{n=n_0+1}^{\infty}\left|\frac{1}{r_n}\sum_{j=0}^{n}\left[\sum_{i=j}^{n}(-1)^{i-j}\binom{m}{i-j}s_{n-i}t_i\right]x_j\right|^{p_n}<\frac{\delta_1}{6}.$$

Now

$$\begin{aligned}\delta\leqslant\rho_{\Delta^{(m)}}(x)&=\sum_{n=0}^{n_0}\left|\frac{1}{r_n}\sum_{j=0}^{n}\left[\sum_{i=j}^{n}(-1)^{i-j}\binom{m}{i-j}s_{n-i}t_i\right]x_j\right|^{p_n}\\&\quad+\sum_{n=n_0+1}^{\infty}\left|\frac{1}{r_n}\sum_{j=0}^{n}\left[\sum_{i=j}^{n}(-1)^{i-j}\binom{m}{i-j}s_{n-i}t_i\right]x_j\right|^{p_n}\\&\leqslant\sum_{n=0}^{n_0}\left|\frac{1}{r_n}\sum_{j=0}^{n}\left[\sum_{i=j}^{n}(-1)^{i-j}\binom{m}{i-j}s_{n-i}t_i\right]x_j\right|^{p_n}+\frac{\delta_1}{6},\end{aligned}$$

which implies

$$\sum_{n=0}^{n_0}\left|\frac{1}{r_n}\sum_{j=0}^{n}\left[\sum_{i=j}^{n}(-1)^{i-j}\binom{m}{i-j}s_{n-i}t_i\right]x_j\right|^{p_n} \geqslant \delta-\frac{\delta}{6}=\frac{5\delta}{6}.$$

Since weak convergence implies coordinate wise convergence, so we have

$$\sum_{n=0}^{n_0}\left|\frac{1}{r_n}\sum_{j=0}^{n}\left[\sum_{i=j}^{n}(-1)^{i-j}\binom{m}{i-j}s_{n-i}t_i\right](x_j^k+x_j)\right|^{p_n} \geqslant \frac{5\delta}{6}$$

for all $k \geqslant k_0$. Since $x^k \to 0$ weakly as $k \to \infty$, so there exists $n_0 \in \mathbb{N}$ and $k_1 > k_0$ such that

$$\rho_{\Delta^{(m)}}(x^k|_{n_0}) \leqslant \delta_1 \text{ for all } k \geqslant k_1.$$

Since $(x^k) \subset S(X)$, by Proposition 2, we have $\rho_{\Delta^{(m)}}(x^k)=1$. Therefore for $n_0 \in \mathbb{N}$, we have $\rho_{\Delta^{(m)}}(x^k|_{\mathbb{N}-n_0}) \leqslant 1$. Let us set $u=x^k|_{\mathbb{N}-n_0}$ and $v=x^k|_{n_0}$. Then $u,v \in X$ and $\rho_{\Delta^{(m)}}(u) \leqslant 1$, $\rho_{\Delta^{(m)}}(v) \leqslant \delta_1$. Using (3), we have

$$\left|\rho_{\Delta^{(m)}}(x^k|_{\mathbb{N}-n_0}+x^k|_{n_0})-\rho_{\Delta^{(m)}}(x^k|_{\mathbb{N}-n_0})\right| < \tfrac{\delta}{6},$$

for all $k \geqslant k_1$. Thus $\rho_{\Delta^{(m)}}(x^k)-\frac{\delta}{6} < \rho_{\Delta^{(m)}}(x^k|_{\mathbb{N}-n_0})$ for all $k \geqslant k_1$, i.e.,

$$\sum_{n=n_0+1}^{\infty}\left|\frac{1}{r_n}\sum_{j=0}^{n}\left[\sum_{i=j}^{n}(-1)^{i-j}\binom{m}{i-j}s_{n-i}t_i\right]x_j^k\right|^{p_n} > 1-\frac{\delta}{6}$$

for all $k \geqslant k_1$. Again $\rho_{\Delta^{(m)}}(x^k|_{\mathbb{N}-n_0}) \leqslant 1$ and $\rho_{\Delta^{(m)}}(x|_{\mathbb{N}-n_0}) \leqslant \frac{\delta_1}{6} < \delta_1$, from the inequality (3), we have

$$\left|\rho_{\Delta^{(m)}}(x^k+x|_{\mathbb{N}-n_0})-\rho_{\Delta^{(m)}}(x^k|_{\mathbb{N}-n_0})\right| < \frac{\delta}{6}.$$

We have for $k \geqslant k_1$

$$\begin{aligned}
\rho_{\Delta^{(m)}}(x^k+x) &= \sum_{n=0}^{n_0}\left|\frac{1}{r_n}\sum_{j=0}^{n}\left[\sum_{i=j}^{n}(-1)^{i-j}\binom{m}{i-j}s_{n-i}t_i\right](x_j^k+x_j)\right|^{p_n} \\
&\quad + \sum_{n=n_0+1}^{\infty}\left|\frac{1}{r_n}\sum_{j=0}^{n}\left[\sum_{i=j}^{n}(-1)^{i-j}\binom{m}{i-j}s_{n-i}t_i\right](x_j^k+x_j)\right|^{p_n} \\
&> \sum_{n=0}^{n_0}\left|\frac{1}{r_n}\sum_{j=0}^{n}\left[\sum_{i=j}^{n}(-1)^{i-j}\binom{m}{i-j}s_{n-i}t_i\right](x_j^k+x_j)\right|^{p_n} \\
&\quad + \sum_{n=0}^{n_0}\left|\frac{1}{r_n}\sum_{j=0}^{n}\left[\sum_{i=j}^{n}(-1)^{i-j}\binom{m}{i-j}s_{n-i}t_i\right]x_j^k\right|^{p_n}-\frac{\delta}{6} \\
&> \frac{5\delta}{6}+\left(1-\frac{\delta}{6}\right)-\frac{\delta}{6}=1+\frac{\delta}{2}.
\end{aligned}$$

Using Lemma 3 there exists $\mu > 0$ depending only on δ such that $\|x^k+x\| > 1+\mu$ for $k \geqslant k_1$. Hence $\liminf\limits_{l\to\infty}\|x^k+x\| \geqslant 1+\mu$. This completes the proof. □

COROLLARY 1. *Let $p=(p_k)$ be a bounded sequence with $p_k>1$ for all k. Then the sequence space $l(r,s,t,p;\Delta^{(m)})$ equipped with the Amemiya norm has the uniform Opial property.*

THEOREM 4. *The sequence space $l_p(r,s,t;\Delta^{(m)})$, $1<p<\infty$ has the Banach-Saks property of type p.*

Proof. Let (ε_n) be a sequence of positive real numbers such that $\sum_{n=1}^{\infty}\varepsilon_n \leqslant \frac{1}{4}$. Let (x^k) be any weakly null sequence in the unit ball of $l_p(r,s,t;\Delta^{(m)})$, i.e., $\|x\|_p \leqslant 1$ (for the simplification of notation we here write $\|x\|_p$ instead of $\|x\|_{l_p(r,s,t;\Delta^{(m)})}$). Set $x^0=0$, $y^1=x^{n_1}$. Then there exists $\eta_1\in\mathbb{N}$ such that

$$\Big\|\sum_{i=\eta_1+1}^{\infty} y_i^1 e_i\Big\|_p<\varepsilon_1 .$$

Since weak convergence implies coordinate wise converges, so $x_n\to 0$ coordinate wise. Hence there exists $n_2\in\mathbb{N}$ such that

$$\Big\|\sum_{i=1}^{\eta_1} x_i^n e_i\Big\|_p<\varepsilon_1 \text{ for all } n\geqslant n_2 .$$

Set $y^2=x^{n_2}$. Then there exists $\eta_2\in\mathbb{N}$, $\eta_2>\eta_1$ such that

$$\Big\|\sum_{i=\eta_2+1}^{\infty} y_i^2 e_i\Big\|_p<\varepsilon_2 .$$

Using the coordinate wise converges of $x^n\to 0$, we can find $n_3\in\mathbb{N}$, $n_3>n_2$ such that

$$\Big\|\sum_{i=1}^{\eta_2} x_i^n e_i\Big\|_p<\varepsilon_2 \text{ for all } n\geqslant n_3 .$$

By repeating this process, we can find two increasing sequences (n_j) and (η_j) such that

$$\Big\|\sum_{i=1}^{\eta_{j-1}} x_i^n e_i\Big\|_p<\varepsilon_{j-1} \text{ for all } n\geqslant n_j, \text{ and}$$

$$\Big\|\sum_{i=\eta_{j+1}}^{\infty} y_i^j e_i\Big\|_p<\varepsilon_j, \text{ where } y^j=x^{n_j} .$$

Now

$$\Big\|\sum_{j=1}^{n} y^j\Big\|_p = \Big\|\sum_{j=1}^{n}\Big(\sum_{i=1}^{\eta_{j-1}} y_i^j e_i + \sum_{i=\eta_{j-1}+1}^{\eta_j} y_i^j e_i + \sum_{i=\eta_j+1}^{\infty} y_i^j e_i\Big)\Big\|_p$$
$$\leqslant \Big\|\sum_{j=1}^{n}\sum_{i=1}^{\eta_{j-1}} y_i^j e_i\Big\|_p + \Big\|\sum_{j=1}^{n}\sum_{i=\eta_{j-1}+1}^{\eta_j} y_i^j e_i\Big\|_p + \Big\|\sum_{j=1}^{n}\sum_{i=\eta_j+1}^{\infty} y_i^j e_i\Big)\Big\|_p$$
$$\leqslant \Big\|\sum_{j=1}^{n}\sum_{i=\eta_{j-1}+1}^{\eta_j} y_i^j e_i\Big\|_p + \sum_{j=1}^{n}(\varepsilon_{j-1}+\varepsilon_j). \tag{4}$$

Now $\|x^{n_j}\|_p = \Big(\sum_{n=0}^{\infty}\Big|\frac{1}{r_n}\sum_{k=0}^{n}\Big[\sum_{i=k}^{n}(-1)^{i-k}\binom{m}{i-k}s_{n-i}t_i\Big]x_k^{n_j}\Big|^p\Big)^{\frac{1}{p}}$ and $\|x^{n_j}\|_p \leqslant 1$. Therefore $\|x^{n_j}\|_p^p \leqslant 1$ and $\Big\|\sum_{j=1}^{n}\sum_{i=\eta_{j-1}+1}^{\eta_j} y_i^j e_i\Big\|_p^p \leqslant n$. Since $\sum_{n=1}^{\infty}\varepsilon_n \leqslant \frac{1}{4}$ and $\frac{1}{2} \leqslant n^{\frac{1}{p}}$ for all $n \geqslant 1$, we have from the relation (4)

$$\Big\|\sum_{j=1}^{n} y^j\Big\|_p \leqslant 2n^{\frac{1}{p}}.$$

Hence the space $l_p(r,s,t;\Delta^{(m)})$, $1 < p < \infty$ has the Banach-Saks property of type p. □

4.1. The α-, β-, γ-dual of $l(r,s,t,p;\Delta^{(m)})$

In 1999, K. G. Grosse-Erdmann [19] characterized the matrix transformations between the sequence spaces of Maddox, namely, $l(p)$, $c(p)$, $c_0(p)$ and $l_\infty(p)$. We list the following conditions:

Let L be any natural number, F denotes finite subset of $\mathbb{N}_0$ and α, α_k are complex numbers. Let $p = (p_k)$ be bounded sequence of strictly positive real numbers with $p_k > 1$ for all k, $A = (a_{nk})_{n,k}$ be an infinite matrix and $p'_k = \frac{p_k}{p_k - 1}$ for all k.

$$\exists L \ \sup_F \sum_k \Big|\sum_{n\in F} a_{nk}L^{-1}\Big|^{p'_k} < \infty \tag{5}$$

$$\lim_n a_{nk} = 0 \text{ for all } k \tag{6}$$

$$\forall L \ \sup_n \sum_k \big|a_{nk}L\big|^{p'_k} < \infty \tag{7}$$

$$\exists L \sup_n \sum_k \big|a_{nk}L^{-1}\big|^{p'_k} < \infty \tag{8}$$

$$\exists(\alpha_k) \ \lim_n |a_{nk} - \alpha_k| = 0 \text{ for all } k \tag{9}$$

$$\exists(\alpha_k) \ \forall L \ \sup_n \sum_k \big(|a_{nk} - \alpha_k|L\big)^{p'_k} < \infty \tag{10}$$

$$\exists L \sup_n \sum_k \left| a_{nk} L^{-1} \right|^{p'_k} < \infty \tag{11}$$

LEMMA 4. [19]
(i) $A \in (l(p), l_1)$ *if and only if* (5) *hold.*
(ii) $A \in (l(p), c_0)$ *if and only if* (6) *and* (7) *hold.*
(iii) $A \in (l(p), c)$ *if and only if* (8), (9) *and* (10) *hold.*
(iv) $A \in (l(p), l_\infty)$ *if and only if* (11) *hold.*

Now we compute the α-, γ-dual of $l(r,s,t,p;\Delta^{(m)})$. Consider the set

$$H(p) = \bigcup_{L \in \mathbb{N}} \left\{ a = (a_n) \in w : \sup_F \sum_{j=0}^{\infty} \left| \sum_{n \in F} \sum_{k=j}^{n} (-1)^{k-j} \binom{m+n-k-1}{n-k} \frac{D_{k-j}^{(s)}}{t_k} r_j a_n L^{-1} \right|^{p'_j} < \infty \right\}.$$

THEOREM 5. *(a) Let* $p = (p_k)$ *be a bounded sequence with* $p_k > 1$ *for all* k*. Then* $[l(r,s,t,p;\Delta^{(m)})]^\alpha = H(p)$.

Proof. (a) Let $p_k > 1$ for all k, $a = (a_n) \in w$, $x \in l(r,s,t,p;\Delta^{(m)})$ and $y \in l(p)$. Then for each n, we have

$$a_n x_n = \sum_{j=0}^{n} \sum_{k=j}^{n} (-1)^{k-j} \binom{m+n-k-1}{n-k} \frac{D_{k-j}^{(s)}}{t_k} r_j a_n y_j = (Cy)_n,$$

where the matrix $C = (c_{nj})_{n,j}$ is defined as

$$c_{nj} = \begin{cases} \sum_{k=j}^{n} (-1)^{k-j} \binom{m+n-k-1}{n-k} \frac{D_{k-j}^{(s)}}{t_k} r_j a_n & \text{if } 0 \leqslant j \leqslant n \\ 0 & \text{if } j > n \end{cases}$$

and x_n is given in (2). Thus for each $x \in l(r,s,t,p;\Delta^{(m)})$, $(a_n x_n)_n \in l_1$ if and only if $Cy \in l_1$, where $y \in l(p)$. Therefore $a = (a_n) \in [l(r,s,t,p;\Delta^{(m)})]^\alpha$ if and only if $C \in (l(p), l_1)$. By using Lemma 4 (i), there exists $L \in \mathbb{N}$ such that

$$\sup_F \sum_{j=0}^{\infty} \left| \sum_{n \in F} \sum_{k=j}^{n} (-1)^{k-j} \binom{m+n-k-1}{n-k} \frac{D_{k-j}^{(s)}}{t_k} r_j a_n L^{-1} \right|^{p'_j} < \infty.$$

Hence $[l(r,s,t,p;\Delta^{(m)})]^\alpha = H(p)$. This finishes the proof. □

We consider

$$\Gamma(p) = \bigcup_{L \in \mathbb{N}} \left\{ a = (a_k) \in w : \sup_{l \in \mathbb{N}_0} \sum_{n=0}^{\infty} |e_{ln} L^{-1}|^{p'_n} < \infty \right\},$$

where the matrix $E=(e_{ln})$ is defined by

$$e_{ln}=\begin{cases} r_n\left[\dfrac{a_n}{s_0t_n}+\displaystyle\sum_{k=n}^{n+1}(-1)^{k-n}\frac{D_{k-n}^{(s)}}{t_k}\sum_{j=n+1}^{l}\binom{m+j-k-1}{j-k}a_j\right. \\ \qquad\qquad \left.+\displaystyle\sum_{k=n+2}^{l}(-1)^{k-n}\frac{D_{k-n}^{(s)}}{t_k}\sum_{j=k}^{l}\binom{m+j-k-1}{j-k}a_j\right] & 0\leqslant n\leqslant l, \\ 0 & n>l. \end{cases} \tag{12}$$

Note we mean $\sum\limits_{j=n}^{l}=0$ if $n>l$.

THEOREM 6. *Let $p=(p_k)$ be a bounded sequence with $p_k>1$ for all k. Then $[l(r,s,t,p;\Delta^{(m)})]^{\gamma}=\Gamma(p)$.*

Proof. (a) Let $p_k>1$ for all k, $a=(a_k)\in w$, $x\in l(r,s,t,p;\Delta^{(m)})$ and $y\in l(p)$. Then by using (2), we have

$$\begin{aligned}\sum_{n=0}^{l}a_nx_n &= \sum_{n=0}^{l}\sum_{j=0}^{n}\sum_{k=j}^{n}(-1)^{k-j}\binom{m+n-k-1}{n-k}\frac{D_{k-j}^{(s)}}{t_k}r_ja_ny_j \\ &= \sum_{n=0}^{l-1}\sum_{j=0}^{n}\sum_{k=j}^{n}(-1)^{k-j}\binom{m+n-k-1}{n-k}\frac{D_{k-j}^{(s)}}{t_k}r_jy_ja_n \\ &\quad+\sum_{j=0}^{l}\sum_{k=j}^{l}(-1)^{k-j}\binom{m+l-k-1}{l-k}\frac{D_{k-j}^{(s)}}{t_k}r_jy_ja_l+\ldots+\frac{D_0^{(s)}}{t_l}a_lr_ly_l \\ &= \sum_{n=0}^{l}r_n\left[\frac{a_n}{s_0t_n}+\sum_{k=n}^{n+1}(-1)^{k-n}\frac{D_{k-n}^{(s)}}{t_k}\sum_{j=n+1}^{l}\binom{m+j-k-1}{j-k}a_j\right. \\ &\quad\left.+\sum_{k=n+2}^{l}(-1)^{k-n}\frac{D_{k-n}^{(s)}}{t_k}\sum_{j=k}^{l}\binom{m+j-k-1}{j-k}a_j\right]y_n \\ &= (Ey)_l, \end{aligned} \tag{13}$$

where the matrix E is defined in (12). Thus $a\in\left[l(r,s,t,p;\Delta^{(m)})\right]^{\gamma}$ if and only if $ax=(a_nx_n)\in bs$, where $x\in l(r,s,t,p;\Delta^{(m)})$ if and only if $\left(\sum\limits_{n=0}^{l}a_nx_n\right)_l\in l_\infty$, i.e., $Ey\in l_\infty$, where $y\in l(p)$. Hence by using Lemma 4 (iv),

$$\sup_{l\in\mathbb{N}_0}\sum_{n=0}^{\infty}|e_{ln}L^{-1}|^{p_n'}<\infty,$$

for some $L\in\mathbb{N}$. Hence $\left[l(r,s,t,p;\Delta^{(m)})\right]^{\gamma}=\Gamma(p)$. This completes the proof. $\square$

To find the β-dual of $l(r,s,t,p;\Delta^{(m)})$, we define the following sets:

$$B_1 = \Big\{a=(a_n)\in w : \sum_{j=n+1}^{\infty}\binom{m+j-k-1}{j-k}a_j \text{ exists for all } k\Big\},$$

$$B_2 = \Big\{a=(a_n)\in w : \sum_{k=n+2}^{\infty}(-1)^{k-n}\frac{D^{(s)}_{k-n}}{t_k}\sum_{j=k}^{\infty}\binom{m+j-k-1}{j-k}a_j \text{ exists for all } k\Big\},$$

$$B_3 = \Big\{a=(a_n)\in w : \Big(\frac{r_n a_n}{t_n}\Big)\in l_\infty(p)\Big\},$$

$$B_4 = \bigcup_{L\in\mathbb{N}}\Big\{a=(a_n)\in w : \sup_{l\in\mathbb{N}_0}\sum_{n=0}^{\infty}\Big|e_{ln}L^{-1}\Big|^{p'_n}<\infty\Big\},$$

$$B_5 = \Big\{a=(a_n)\in w : \exists(\alpha_n)\ \lim_{l\to\infty}e_{ln}=\alpha_n\ \forall\ n\Big\},$$

$$B_6 = \bigcap_{L\in\mathbb{N}}\Big\{a=(a_n)\in w : \exists(\alpha_n)\ \sup_{l\in\mathbb{N}_0}\sum_{n=0}^{\infty}\Big(|e_{ln}-\alpha_n|L\Big)^{p'_n}<\infty\Big\},$$

THEOREM 7. *(a) Let $p=(p_k)$ be a bounded sequence with $p_k>1$ for all k. Then* $[l(r,s,t,p;\Delta^{(m)})]^\beta = \bigcap_{k=1}^{6}B_k$.

Proof. (a) Let $p_k>1$ for all k. We have from (13)

$$\sum_{n=0}^{l}a_nx_n=(Ey)_l,$$

where the matrix E is defined in (12). Thus $a\in\big[l(r,s,t,p;\Delta^{(m)})\big]^\beta$ if and only if $ax=(a_nx_n)\in cs$, where $x\in l(r,s,t,p;\Delta^{(m)})$ if and only if $Ey\in c$, where $y\in l(p)$, i.e., $E\in(l(p),c)$. Hence by Lemma 4(iii),

$$\exists L\in\mathbb{N},\ \sup_{l\in\mathbb{N}_0}\sum_{n=0}^{\infty}\Big|e_{ln}L^{-1}\Big|^{p'_n}<\infty,$$

$$\exists(\alpha_n)\ \lim_{l\to\infty}e_{ln}=\alpha_n \text{ for all } n,$$

$$\exists(\alpha_n)\text{ and }\forall L\in\mathbb{N}\ \sup_{l\in\mathbb{N}_0}\sum_{n=0}^{\infty}\Big(|e_{ln}-\alpha_n|L\Big)^{p'_n}<\infty.$$

Therefore $[l(r,s,t,p;\Delta^{(m)})]^\beta=\bigcap_{k=1}^{6}B_k$ $\square$

4.2. Matrix mappings

THEOREM 8. *Let $\tilde{E}=(\tilde{e}_{ln})$ be the matrix which is same as the matrix $E=(e_{ln})$ defined in (12), where a_n and a_j are replaced by a_{ln} and a_{lj} respectively. Let $p_n>1$*

for all n. Then $A \in (l(r,s,t,p;\Delta^{(m)}), l_\infty)$ if and only if there exists $L \in \mathbb{N}$ such that

$$\sup_l \sum_n \left| \tilde{e}_{ln} L^{-1} \right|^{p_n'} < \infty \text{ and } (a_{ln})_n \in \bigcap_{k=1}^{6} B_k.$$

Proof. Let $p_n > 1$ for all n and $A \in (l(r,s,t,p;\Delta^{(m)}), l_\infty)$. Then $(a_{ln})_n \in \left[l(r,s,t,p;\Delta^{(m)})\right]^\beta$ for each fixed l. Thus Ax exists for all $x \in l(r,s,t,p;\Delta^{(m)})$. Now for each l, we have

$$\begin{aligned}\sum_{n=0}^{q} a_{ln} x_n &= \sum_{n=0}^{q} r_n \left[\frac{a_n}{s_0 t_n} + \sum_{k=n}^{n+1} (-1)^{k-n} \frac{D_{k-n}^{(s)}}{t_k} \sum_{j=n+1}^{l} \binom{m+j-k-1}{j-k} a_j \right. \\ &\quad \left. + \sum_{k=n+2}^{l} (-1)^{k-n} \frac{D_{k-n}^{(s)}}{t_k} \sum_{j=k}^{l} \binom{m+j-k-1}{j-k} a_j \right] y_n \\ &= (Ey)_l,\end{aligned}$$

Taking $q \to \infty$, we have

$$\sum_{n=0}^{\infty} a_{ln} x_n = \sum_{l=0}^{\infty} \tilde{e}_{ln} y_n \quad \text{for all } l.$$

We know that for any $T > 0$ and any complex numbers a, b

$$|ab| \leqslant T(|aT^{-1}|^{\gamma'} + |b|^{\gamma}) \tag{14}$$

where $\gamma > 1$ and $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$. From (14), we get

$$\sup_l \left| \sum_{n=0}^{\infty} a_{ln} x_n \right| \leqslant \sup_l \sum_{n=0}^{\infty} \left| \tilde{e}_{ln} \right| \left| y_n \right| \leqslant T \left[\sup_l \sum_{n=0}^{\infty} |\tilde{e}_{ln} T^{-1}|^{p_n'} + \sum_{n=0}^{\infty} |y_n|^{p_n} \right] < \infty.$$

Thus $Ax \in l_\infty$. This proves that $A \in (l(r,s,t,p;\Delta^{(m)}), l_\infty)$.

Conversely, assume that $A \in (l(r,s,t,p;\Delta^{(m)}), l_\infty)$ and $p_n > 1$ for all n. Then Ax exists for each $x \in l(r,s,t,p;\Delta^{(m)})$, which implies that $(a_{ln})_n \in [l(r,s,t,p;\Delta^{(m)})]^\beta$ for each l. Thus $(a_{ln})_n \in \bigcap_{k=1}^{6} B_k$. Also from $\sum_{n=0}^{\infty} a_{ln} x_n = \sum_{n=0}^{\infty} \tilde{e}_{ln} y_n$, we have $\tilde{E} = (\tilde{e}_{ln}) \in (l(p), l_\infty)$. Therefore there exists some $L \in \mathbb{N}$, such that $\sup_l \sum_n \left| \tilde{e}_{ln} L^{-1} \right|^{p_n'} < \infty$. This completes the proof. □

THEOREM 9. *Let $p_n > 1$ for all n. Then $A \in (l(r,s,t,p;\Delta^{(m)}), l_1)$ if and only if*

$$\sup_F \sum_{n=0}^{\infty} \left| \sum_{l \in F} \tilde{e}_{ln} L^{-1} \right|^{p_n'} < \infty \text{ for some } L \in \mathbb{N} \text{ and } (a_{ln})_{n \in \mathbb{N}_0} \in \bigcap_{k=1}^{6} B_k$$

Proof. We omit the proof as it follows as routine. □

5. Compact operators on the space $l_p(r,s,t;\Delta^{(m)})$, $1 \leqslant p < \infty$

As the matrix transformations between BK spaces are continuous, it is natural to find necessary and sufficient conditions for a matrix mapping between BK spaces to be a compact operator. This can be achieved with the help of Hausdorff measure of noncompactness. Here, we give necessary and sufficient conditions for matrix operator to be a compact from the space $l_p(r,s,t;\Delta^{(m)})$ into c, c_0 and sufficient conditions in case of l_∞.

Recently several authors, namely, Malkowsky and Rakočević [25], Djolović et al. [15], Djolović [17], Mursaleen and Noman [29] Başarır et al. [9] have established some identities or estimates for the operator norms and the Hausdorff measure of noncompactness of matrix operators from an arbitrary BK space to arbitrary BK space. Let us recall some definitions and well-known results.

Let X, Y be two Banach spaces and S_X denotes the unit sphere in X. We denote by $\mathscr{B}(X,Y)$, the set of all bounded (continuous) linear operators $L : X \to Y$, which is a Banach space with the operator norm $\|L\| = \sup_{x\in S_X} \|L(x)\|_Y$ for all $L \in \mathscr{B}(X,Y)$. A linear operator $L : X \to Y$ is said to be compact if for every bounded sequence $(x_n) \in X$, the sequence $(L(x_n))$ has a convergent subsequence in Y. We denote by $\mathscr{C}(X,Y)$, the class of all compact operators in $\mathscr{B}(X,Y)$. If X is a BK space and $a = (a_k) \in w$, then we consider

$$\|a\|_X^* = \sup_{x\in S_X} \left| \sum_{k=0}^{\infty} a_k x_k \right|, \tag{15}$$

provided the expression on the right side exists and is finite which is the case whenever $a \in X^\beta$ [29].

Let (X,d) be a metric space and $\mathscr{M}_X$ be the class of all bounded subsets of X. Let $B(x,r) = \{y \in X : d(x,y) < r\}$ denotes the open ball of radius $r > 0$ with centre at x. The Hausdorff measure of noncompactness of a set $Q \in \mathscr{M}_X$, denoted by $\chi(Q)$, is defined as

$$\chi(Q) = \inf\left\{\varepsilon > 0 : Q \subset \bigcup_{i=0}^{n} B(x_i, r_i), x_i \in X, r_i < \varepsilon, n \in \mathbb{N}\right\}.$$

The function $\chi : \mathscr{M}_X \to [0,\infty)$ is called the Hausdorff measure of noncompactness. The basic properties of the Hausdorff measure of noncompactness can be found in ([16], [25], [15]).

Let ϕ denotes the set of all finite sequences. Throughout we denote p' as the conjugate of p for $1 \leqslant p < \infty$, i.e., $p' = \frac{p}{p-1}$ for $p > 1$ and $p' = \infty$ for $p = 1$. The following known results are fundamental for our investigation.

LEMMA 5. [29] *Let X denotes any of the sequence spaces c_0 or l_∞. If $A \in (X,c)$, then we have*

(i) $\alpha_k = \lim_{n\to\infty} a_{nk}$ *exists for all* $k \in \mathbb{N}_0$,

(ii) $\alpha = (\alpha_k) \in l_1$,

$$(iii)\ \sup_n \sum_{k=0}^{\infty} |a_{nk} - \alpha_k| < \infty,$$

$$(iv)\ \lim_{n\to\infty} A_n(x) = \sum_{k=0}^{\infty} \alpha_k x_k \textit{ for all } x = (x_k) \in X.$$

LEMMA 6. ([25], Theorem 1.29) *Let $1 \leqslant p < \infty$. Then we have $l_p^{\beta} = l_{p'}$ and $\|a\|_{l_p}^* = \|a\|_{l_{p'}}$ for all $a \in l_{p'}$.*

LEMMA 7. [29] *Let $X \supset \phi$ and Y be BK spaces. Then we have $(X,Y) \subset \mathscr{B}(X,Y)$, i.e., every matrix $A \in (X,Y)$ defines an operator $L_A \in \mathscr{B}(X,Y)$, where $L_A(x) = Ax$ for all $x \in X$.*

LEMMA 8. [17] *Let $X \supset \phi$ be a BK space and Y be any of the spaces c_0, c or l_∞. If $A \in (X,Y)$, then we have*

$$\|L_A\| = \|A\|_{(X,l_\infty)} = \sup_n \|A_n\|_X^* < \infty.$$

LEMMA 9. [25] *Let Q be a bounded subset of the normed space X, where $X = l_p$ for $1 \leqslant p < \infty$ and $X = c_0$ for $p = \infty$. Let $P_l : X \to X$ be the operator defined by $P_l(x) = (x_0, x_1, \cdots, x_l, 0, 0, \cdots)$ for all $x = (x_k) \in X$. Then*

$$\chi(Q) = \lim_{l\to\infty}\left(\sup_{x\in Q} \|(I - P_l)(x)\|\right),$$

where I is the identity operator on X.

Let $z = (z_n) \in c$. Then z has a unique representation $z = \hat{\ell} e + \sum_{n=0}^{\infty} (z_n - \hat{\ell}) e_n$, where $\hat{\ell} = \lim_{n\to\infty} z_n$. We now define the projections P_l^c $(l \in \mathbb{N}_0)$ from c onto the linear span of $\{e, e_0, e_1, \cdots, e_l\}$ as

$$P_l^c(z) = \hat{\ell} e + \sum_{n=0}^{l} (z_n - \hat{\ell}) e_n,$$

for all $z \in c$ and $\hat{\ell} = \lim_{n\to\infty} z_n$.

Then the following result gives an estimate for the Hausdorff measure of noncompactness in the BK space c.

LEMMA 10. [25] *Let $Q \in \mathscr{M}_c$ and $P_l^c : c \to c$ be the projector from c onto the linear span of $\{e, e_0, e_1, \ldots e_l\}$. Then we have*

$$\frac{1}{2} \lim_{l\to\infty}\left(\sup_{x\in Q} \|(I - P_l^c)(x)\|_\infty\right) \leqslant \chi(Q) \leqslant \lim_{l\to\infty}\left(\sup_{x\in Q} \|(I - P_l^c)(x)\|_\infty\right),$$

where I is the identity operator on c.

LEMMA 11. [25] *Let X,Y be two Banach spaces and $L \in \mathscr{B}(X,Y)$. Then*

$$\|L\|_\chi = \chi(L(S_X))$$

and

$$L \in \mathscr{C}(X,Y) \text{ if and only if } \|L\|_\chi = 0.$$

THEOREM 10. [14] (Pitt's compactness theorem) *Let $1 \leqslant q < p \leqslant \infty$ and put $X_p = l_p$ for $1 < p < \infty$ and $X_\infty = c_0$. Then every bounded linear operator from X_p into l_q is compact.*

We establish the following lemmas which are required to characterize the classes of compact operators with the help of Hausdorff measure of noncompactness.

LEMMA 12. *If $a = (a_k) \in [l_p(r,s,t;\Delta^{(m)})]^\beta$ then $\tilde{a} = (\tilde{a}_k) \in l_p^\beta = l_{p'}$ and the equality*

$$\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} \tilde{a}_k y_k$$

holds for every $x = (x_k) \in l_p(r,s,t;\Delta^{(m)})$ and $y = (y_k) \in l_p$, where $y = (A(r,s,t).\Delta^{(m)})x$. In addition

$$\begin{aligned} \tilde{a}_k = r_k \Bigg[\frac{a_k}{s_0 t_k} &+ \sum_{i=k}^{k+1} (-1)^{i-k} \frac{D_{i-k}^{(s)}}{t_i} \sum_{j=k+1}^{\infty} \binom{m+j-i-1}{j-i} a_j \\ &+ \sum_{l=2}^{\infty} (-1)^l \frac{D_l^{(s)}}{t_{l+k}} \sum_{j=k+l}^{\infty} \binom{m+j-k-l-1}{j-k-l} a_j \Bigg]. \end{aligned} \tag{16}$$

Proof. Let $a = (a_k) \in [l_p(r,s,t;\Delta^{(m)})]^\beta$. Then by ([26], Theorem 3.2), we have $R(a) = (R_k(a)) \in l_p^\beta = l_{p'}$ and also

$$\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} R_k(a) T_k(x) \qquad \text{for all } x \in l_p(r,s,t;\Delta^{(m)}),$$

where

$$\begin{aligned} R_k(a) &= \sum_{j=k}^{\infty} \sum_{i=k}^{j} (-1)^{i-k} \binom{m+j-i-1}{j-i} \frac{D_{i-k}^{(s)}}{t_i} r_k a_j \\ &= \frac{D_0^{(s)}}{t_k} r_k a_k + \sum_{j=k+1}^{\infty} \sum_{i=k}^{j} (-1)^{i-k} \binom{m+j-i-1}{j-i} \frac{D_{i-k}^{(s)}}{t_i} r_k a_j \\ &= r_k \Bigg[\frac{a_k}{s_0 t_k} + \sum_{i=k}^{k+1} (-1)^{i-k} \frac{D_{i-k}^{(s)}}{t_i} \sum_{j=k+1}^{\infty} \binom{m+j-i-1}{j-i} a_j \\ &\quad + \sum_{l=2}^{\infty} (-1)^l \frac{D_l^{(s)}}{t_{l+k}} \sum_{j=k+l}^{\infty} \binom{m+j-k-l-1}{j-k-l} a_j \Bigg] \end{aligned}$$

and $y = T(x) = (A(r,s,t).\Delta^{(m)})x$. This completes the proof. $\square$

LEMMA 13. *Let* $1 \leqslant p < \infty$. *Then we have*

$$\|a\|^*_{l_p(r,s,t;\Delta^{(m)})} = \|\tilde{a}\|_{l_{p'}} = \begin{cases} \left(\sum_{k=0}^{\infty} |\tilde{a}_k|^{p'}\right)^{\frac{1}{p'}} & 1 < p < \infty \\ \sup_k |\tilde{a}_k| & p = 1 \end{cases}$$

for all $a = (a_k) \in [l_p(r,s,t;\Delta^{(m)})]^\beta$, *where* $\tilde{a} = (\tilde{a}_k)$ *is defined in (16).*

Proof. Let $a = (a_k) \in [l_p(r,s,t;\Delta^{(m)})]^\beta$. Then from Lemma 12, we have $\tilde{a} = (\tilde{a}_k) \in l_{p'}$. Since $l_p(r,s,t;\Delta^{(m)})$ is isomorphic to l_p, we have $x \in S_{l_p(r,s,t;\Delta^{(m)})}$ if and only if $y = T(x) \in S_{l_p}$ as $\|x\|_{l_p(r,s,t;\Delta^{(m)})} = \|y\|_{l_p}$. From (15), we have

$$\|a\|^*_{l_p(r,s,t;\Delta^{(m)})} = \sup_{x \in S_{l_p(r,s,t;\Delta^{(m)})}} \left|\sum_{k=0}^{\infty} a_k x_k\right| = \sup_{y \in S_{l_p}} \left|\sum_{k=0}^{\infty} \tilde{a}_k y_k\right| = \|\tilde{a}\|^*_{l_p}.$$

Using Lemma 6, we have $\|a\|^*_{l_p(r,s,t;\Delta^{(m)})} = \|\tilde{a}\|^*_{l_p} = \|\tilde{a}\|_{l_{p'}}$, which is finite as $\tilde{a} \in l_{p'}$. This completes the proof. $\square$

LEMMA 14. *Let* Y *be any sequence space,* $A = (a_{nk})_{n,k}$ *be an infinite matrix and* $1 \leqslant p < \infty$. *If* $A \in (l_p(r,s,t;\Delta^{(m)}),Y)$ *then* $\tilde{A} \in (l_p,Y)$ *such that* $Ax = \tilde{A}y$ *for all* $x \in l_p(r,s,t;\Delta^{(m)})$ *and* $y \in l_p$, *which are connected by* $y = (A(r,s,t).\Delta^{(m)})x$ *and* $\tilde{A} = (\tilde{a}_{nk})_{n,k}$ *is given by*

$$\tilde{a}_{nk} = r_k \left[\frac{a_{nk}}{s_0 t_k} + \sum_{i=k}^{k+1} (-1)^{i-k} \frac{D^{(s)}_{i-k}}{t_i} \sum_{j=k+1}^{\infty} \binom{m+j-i-1}{j-i} a_{nj} \right.$$
$$\left. + \sum_{l=2}^{\infty} (-1)^l \frac{D^{(s)}_l}{t_{l+k}} \sum_{j=k+l}^{\infty} \binom{m+j-k-l-1}{j-k-l} a_{nj} \right], \tag{17}$$

provided the series on the right side converges for all n,k.

Proof. We assume that $A \in (l_p(r,s,t;\Delta^{(m)}),Y)$, then $A_n \in [l_p(r,s,t;\Delta^{(m)})]^\beta$ for all n. Thus it follows from Lemma 12, we have $\tilde{A}_n \in l_p^\beta = l_{p'}$ for all n and $Ax = \tilde{A}y$ holds for every $x \in l_p(r,s,t;\Delta^{(m)})$, $y \in l_p$, which are connected by the relation $y = (A(r,s,t).\Delta^{(m)})x$. Hence $\tilde{A}y \in Y$. Since $x = (\Delta^{(m)})^{-1}(A(r,s,t))^{-1}y$, for every $y \in l_p$, we get some $x \in l_p(r,s,t;\Delta^{(m)})$ and hence $\tilde{A} \in (l_p,Y)$. This completes the proof. $\square$

LEMMA 15. *Let* $1 < p < \infty$, $A = (a_{nk})_{n,k}$ *be an infinite matrix and* $\tilde{A} = (\tilde{a}_{nk})_{n,k}$ *be the associate matrix defined in* (17). *If* $A \in (l_p(r,s,t;\Delta^{(m)}),Y)$, *where* $Y \in \{c_0, c, l_\infty\}$, *then*

$$\|L_A\| = \|A\|_{(l_p(r,s,t;\Delta^{(m)}),l_\infty)} = \sup_n \left(\sum_{k=0}^{\infty} |\tilde{a}_{nk}|^{p'}\right)^{\frac{1}{p'}} < \infty.$$

Proof. We write $X = l_p(r,s,t;\Delta^{(m)})$. Since X is a BK space, by using Lemma 8, we have

$$\|L_A\| = \|A\|_{(X,l_\infty)} = \sup_n \|A_n\|_X^* < \infty.$$

Now from Lemma 13, we have

$$\|A_n\|_X^* = \|\tilde{A}_n\|_{l_{p'}} = \Big(\sum_{k=0}^{\infty} |\tilde{a}_{nk}|^{p'}\Big)^{\frac{1}{p'}},$$

which is finite as $\tilde{A}_n \in l_{p'}$ for all n. This completes the proof. □

Now we prove the main results.

THEOREM 11. *Let $1 < p < \infty$, $A = (a_{nk})_{n,k}$ be an infinite matrix and $\tilde{A} = (\tilde{a}_{nk})_{n,k}$ be the associate matrix defined in (17). We have*

(a) if $A \in (l_p(r,s,t;\Delta^{(m)}), c_0)$ then

$$\|L_A\|_\chi = \limsup_{n\to\infty} \Big(\sum_{k=0}^{\infty} |\tilde{a}_{nk}|^{p'}\Big)^{\frac{1}{p'}}. \tag{18}$$

(b) if $A \in (l_p(r,s,t;\Delta^{(m)}), c)$ then

$$\frac{1}{2}\limsup_{n\to\infty} \Big(\sum_{k=0}^{\infty} |\tilde{a}_{nk} - \tilde{\alpha}_k|^{p'}\Big)^{\frac{1}{p'}} \leqslant \|L_A\|_\chi \leqslant \limsup_{n\to\infty} \Big(\sum_{k=0}^{\infty} |\tilde{a}_{nk} - \tilde{\alpha}_k|^{p'}\Big)^{\frac{1}{p'}}, \tag{19}$$

where $\tilde{\alpha}_k = \lim_{n\to\infty} \tilde{a}_{nk}$ for all k.

(c) if $A \in (l_p(r,s,t;\Delta^{(m)}), l_\infty)$ then

$$0 \leqslant \|L_A\|_\chi \leqslant \limsup_{n\to\infty} \Big(\sum_{k=0}^{\infty} |\tilde{a}_{nk}|^{p'}\Big)^{\frac{1}{p'}}. \tag{20}$$

Proof. (a) Let us first observe that the expressions in (18) and in (20) exist by Lemma 15. Also by using Lemma 14 and Lemma 5, we can deduce that the expressions in (19) exists.

We write $S = S_{l_p(r,s,t;\Delta^{(m)})}$ in short. Then by Lemma 11, we have $\|L_A\|_\chi = \chi(AS)$. Since $l_p(r,s,t;\Delta^{(m)})$ and c_0 are BK spaces, A induces a continuous map L_A from $l_p(r,s,t;\Delta^{(m)})$ to c_0 by Lemma 7. Thus AS is bounded in c_0, i.e., $AS \in \mathscr{M}_{c_0}$. Now by Lemma 9,

$$\chi(AS) = \lim_{l\to\infty}\Big(\sup_{x\in S}\|(I-P_l)(Ax)\|_\infty\Big),$$

where the projection $P_l : c_0 \to c_0$ is defined by $P_l(x) = (x_0, x_1, \cdots, x_l, 0, 0, \cdots)$ for all $x = (x_k) \in c_0$ and $l \in \mathbb{N}_0$. Therefore $\|(I-P_l)(Ax)\|_\infty = \sup_{n>l} |A_n(x)|$ for all $x \in l_p(r,s,t;\Delta^{(m)})$. Using (15) and Lemma 13, we have

$$\begin{aligned}\sup_{x\in S}\|(I-P_l)(Ax)\|_\infty &= \sup_{n>l}\|A_n\|^*_{l_p(r,s,t;\Delta^{(m)})}\\ &= \sup_{n>l}\|\tilde{A}_n\|_{l_{p'}}\end{aligned}$$

Therefore $\chi(AS) = \lim_{l\to\infty}\left(\sup_{n>l}\|\tilde{A}_n\|_{l_{p'}}\right) = \limsup_{n\to\infty}\|\tilde{A}_n\|_{l_{p'}} = \limsup_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk}|^{p'}\right)^{\frac{1}{p'}}$. This completes the proof.

(b) We have $AS \in \mathscr{M}_c$. Let $P_l^c : c \to c$ be the projection from c onto the span of $\{e, e_0, e_1, \cdots, e_l\}$ defined as

$$P_l^c(z) = \hat{\ell}e + \sum_{k=0}^{l}(z_k - \hat{\ell})e_k,$$

where $\hat{\ell} = \lim_{k\to\infty} z_k$. Thus for every $l \in \mathbb{N}_0$, we have

$$(I - P_l^c)(z) = \sum_{k=l+1}^{\infty}(z_k - \hat{\ell})e_k.$$

Therefore $\|(I - P_l^c)(z)\|_\infty = \sup_{k>l}|z_k - \hat{\ell}|$ for all $z = (z_k) \in c$. Applying Lemma 10, we have

$$\frac{1}{2}\lim_{l\to\infty}\left(\sup_{x\in S}\|(I - P_l^c)(Ax)\|_\infty\right) \leqslant \|L_A\|_\chi \leqslant \lim_{l\to\infty}\left(\sup_{x\in S}\|(I - P_l^c)(Ax)\|_\infty\right). \tag{21}$$

Since $A \in (l_p(r,s,t;\Delta^{(m)}), c)$, we have by Lemma 14, $\tilde{A} \in (l_p, c)$ and $Ax = \tilde{A}y$ for every $x \in l_p(r,s,t;\Delta^{(m)})$ and $y \in l_p$, which are connected by $y = (A(r,s,t).\Delta^{(m)})x$. Using Lemma 5, we have $\tilde{\alpha}_k = \lim_{n\to\infty}\tilde{a}_{nk}$ exists for all k, $\tilde{\alpha} = (\tilde{\alpha}_k) \in l_p^\beta = l_{p'}$ and $\lim_{n\to\infty}\tilde{A}_n(y) = \sum_{k=0}^{\infty}\tilde{\alpha}_k y_k$. Since $\|(I - P_l^c)(z)\|_\infty = \sup_{k>l}|z_k - \hat{\ell}|$, we have

$$\begin{aligned}\|(I - P_l^c)(Ax)\|_\infty &= \|(I - P_l^c)(\tilde{A}y)\|_\infty \\ &= \sup_{n>l}\left|\tilde{A}_n(y) - \sum_{k=0}^{\infty}\tilde{\alpha}_k y_k\right| \\ &= \sup_{n>l}\left|\sum_{k=0}^{\infty}(\tilde{a}_{nk} - \tilde{\alpha}_k)y_k\right|.\end{aligned}$$

Also we know that $x \in S = S_{l_p(r,s,t;\Delta^{(m)})}$ if and only if $y \in S_{l_p}$. From (15) and Lemma 6, we deduce

$$\sup_{x\in S}\|(I - P_l^c)(Ax)\|_\infty = \sup_{n>l}\left(\sup_{y\in S_X}\left|\sum_{k=0}^{\infty}(\tilde{a}_{nk} - \tilde{\alpha}_k)y_k\right|\right) = \sup_{n>l}\|\tilde{A}_n - \tilde{\alpha}\|_{l_p}^* = \sup_{n>l}\|\tilde{A}_n - \tilde{\alpha}\|_{l_{p'}}.$$

Hence from (21), we have

$$\tfrac{1}{2}\limsup_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk} - \tilde{\alpha}_k|^{p'}\right)^{\frac{1}{p'}} \leqslant \|L_A\|_\chi \leqslant \limsup_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk} - \tilde{\alpha}_k|^{p'}\right)^{\frac{1}{p'}}.$$

(c) Since the proof is trivial, we omit the proof. □

COROLLARY 2. *Let* $1 < p < \infty$.
(*a*) *If* $A \in (l_p(r,s,t;\Delta^{(m)}), c_0)$, *then* L_A *is compact if and only if*

$$\lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk}|^{p'}\right)^{\frac{1}{p'}} = 0.$$

(*b*) *If* $A \in (l_p(r,s,t,\Delta^{(m)}), c)$ *then* L_A *is compact if and only if*

$$\lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk}-\tilde{\alpha}_k|^{p'}\right)^{\frac{1}{p'}} = 0, \text{ where } \tilde{\alpha}_k = \lim_{n\to\infty}\tilde{a}_{nk} \text{ for all } k.$$

(*c*) *If* $A \in (l_p(r,s,t,\Delta^{(m)}), l_\infty)$ *then* L_A *is compact if* $\lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk}|^{p'}\right)^{\frac{1}{p'}} = 0.$

Proof. The proof is immediate from Theorem 11. □

REMARK 1. The condition on the matrix $\tilde{A} = (\tilde{a}_{nk})$ in Corollary 2 (c) is only the sufficient condition for the operator L_A to be compact. For example, define a matrix $A = (a_{nk})$ by $a_{n0} = \frac{s_0 t_0}{r_0}$ and $a_{nk} = 0$ for $k \geqslant 1$. Then for every $x = (x_k)_{k=0}^{\infty} \in l_p(r,s,t;\Delta^{(m)})$, $Ax = \frac{s_0 t_0}{r_0} x_0 e$. Therefore $A \in (l_p(r,s,t,\Delta^{(m)}), l_\infty)$. Clearly the operator L_A induced by the matrix A is a finite rank operator and hence L_A is compact. But for the choice of the matrix $A = (a_{nk})$, we have from (17) $(\tilde{a}_{nk})_{k=0}^{\infty} = e_0$ and hence $\lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk}|^{p'}\right)^{\frac{1}{p'}} = 1$.

THEOREM 12. *Let* $1 \leqslant q < p < \infty$. *If* $A \in (l_p(r,s,t;\Delta^{(m)}), l_q)$ *then* $\|L_A\|_\chi = 0$.

Proof. Let $1 \leqslant q < p < \infty$. Suppose that $A \in (l_p(r,s,t;\Delta^{(m)}), l_q)$. Then by Lemma 14, we have $\tilde{A} \in (l_p, l_q)$, where $Ax = \tilde{A}y$ for all $x \in l_p(r,s,t;\Delta^{(m)})$ and $y \in l_p$ and $\tilde{A} = (\tilde{a}_{nk})$ is defined in (17). Since l_p, l_q are BK spaces, so $\tilde{A}$ induces a bounded linear operator $L_{\tilde{A}} : l_p \to l_q$, $1 \leqslant q < p < \infty$. Now by Theorem 10, we have $L_{\tilde{A}}$ is compact. Again $l_p(r,s,t;\Delta^{(m)}) \cong l_p$. Let $T : l_p(r,s,t;\Delta^{(m)}) \to l_p$ be a linear isomorphism. Then the operator L_A induced by the matrix A is nothing but $L_{\tilde{A}} \circ T$, i.e., $L_A = L_{\tilde{A}} \circ T$: $l_p(r,s,t;\Delta^{(m)}) \to l_q$ which is a compact operator as $L_{\tilde{A}}$ is so. Hence by Lemma 11, $\|L_A\|_\chi = 0$. This completes the proof. □

THEOREM 13. *Let* $1 \leqslant p < \infty$. *If* $A \in (l_1(r,s,t;\Delta^{(m)}), l_p)$, *then*

$$\|L_A\|_\chi = \lim_{l\to\infty}\left(\sup_k\left(\sum_{n=l+1}^{\infty}|\tilde{a}_{nk}|^p\right)^{\frac{1}{p}}\right).$$

and

$$L_A \text{ is compact if and only if } \lim_{l\to\infty}\left(\sup_k\left(\sum_{n=l+1}^{\infty}|\tilde{a}_{nk}|^p\right)^{\frac{1}{p}}\right) = 0.$$

Proof. We write $S = S_{l_1(r,s,t;\Delta^{(m)})}$. Since L_A is a bounded linear operator from $l_1(r,s,t;\Delta^{(m)})$ to l_p, $AS \in \mathscr{M}_{l_p}$. By Lemma 11 and Lemma 9, we have

$$\|L_A\|_\chi = \chi(AS) = \lim_{l\to\infty}\left(\sup_k\left(\sum_{n=l+1}^{\infty}|\tilde{a}_{nk}|^p\right)^{\frac{1}{p}}\right).$$

Now

$$\begin{aligned}\|(I-P_l)(Ax)\|_{l_p} = \|(I-P_l)(\tilde{A}y)\|_{l_p} &= \left(\sum_{n=l+1}^{\infty}|\tilde{A}_n(y)|^p\right)^{\frac{1}{p}}\\ &\leqslant \sum_{k=0}^{\infty}\left(\sum_{n=l+1}^{\infty}|\tilde{a}_{nk}y_k|^p\right)^{\frac{1}{p}}\\ &\leqslant \sup_k\left(\sum_{n=l+1}^{\infty}|\tilde{a}_{nk}|^p\right)^{\frac{1}{p}}\|y\|_1\\ &\leqslant \sup_k\left(\sum_{n=l+1}^{\infty}|\tilde{a}_{nk}|^p\right)^{\frac{1}{p}}\|x\|_{l_1(r,s,t;\Delta^{(m)})}.\end{aligned}$$

Thus

$$\|L_A\|_\chi \leqslant \lim_{l\to\infty}\left(\sup_k\left(\sum_{n=l+1}^{\infty}|\tilde{a}_{nk}|^p\right)^{\frac{1}{p}}\right).$$

Conversely let $b^{(k)}, k = 0,1,2,\ldots$ be a basis of $l_1(r,s,t;\Delta^{(m)})$ such that $A(r,s,t;\Delta^{(m)})b^{(k)} = e_k$ for all k and $Ab^{(k)} = \tilde{A}e_k$. Let $E = \{b^{(k)} : k \in \mathbb{N}\}$. Then $E \subset S$ and hence $AE \subset AS$. Therefore $\chi(AE) \leqslant \chi(AS) = \|L_A\|_\chi$. Again

$$\begin{aligned}\chi(AE) &= \lim_{l\to\infty}\left(\sup_k\|(I-P_l)Ab^{(k)}\|_p\right)\\ &= \lim_{l\to\infty}\left(\sup_k\|(I-P_l)\tilde{A}e^{(k)}\|_p\right)\\ &= \lim_{l\to\infty}\left(\sup_k\left(\sum_{n=l+1}^{\infty}|\tilde{a}_{nk}|^p\right)^{\frac{1}{p}}\right) \leqslant \|L_A\|_\chi.\end{aligned}$$

Hence

$$\|L_A\|_\chi = \lim_{l\to\infty}\left(\sup_k\left(\sum_{n=l+1}^{\infty}|\tilde{a}_{nk}|^p\right)^{\frac{1}{p}}\right).$$

The second part of this theorem is immediate from the first part. □

EXAMPLE. Let $A = (a_{nk}) = A(r,s,t).\Delta^{(m)}$ be an infinite matrix. Since $l_1(r,s,t;\Delta^{(m)})$ is the matrix domain of $A(r,s,t;\Delta^{(m)})$ in l_1, we have $A \in (l_1(r,s,t;\Delta^{(m)}), l_1)$ and hence $A \in (l_1(r,s,t;\Delta^{(m)}), l_p)$, $1 \leqslant p < \infty$. With this choice of the matrix $A = (a_{nk})$, the associate matrix $\tilde{A} = (\tilde{a}_{nk})$ becomes the identity matrix, i.e., $\tilde{a}_{nn} = 1$ and $\tilde{a}_{nk} = 0$ for $n \neq k$. Therefore for each $k \in \mathbb{N}_0$, we have

$$\sum_{n=l}^{\infty}|\tilde{a}_{nk}|^p = \begin{cases}1 & k \geqslant n,\\ 0 & k < n.\end{cases}$$

Thus $\sup_k \left(\sum_{n=l}^{\infty} |\tilde{a}_{nk}|^p \right)^{\frac{1}{p}} = 1$ and hence $\|L_A\|_\chi = 1$. So by Theorem 13, the bounded linear operator $L_A : l_1(r,s,t;\Delta^{(m)}) \to l_p$ for $1 \leqslant p < \infty$ is not compact.

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