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WEIGHTED INEQUALITIES RELATED TO A MUCKENHOUPT AND WHEEDEN PROBLEM FOR ONE–SIDE SINGULAR INTEGRALS

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Abstract. In this paper we obtain for T^+, a one-sided singular integral given by a Calderón-Zygmund kernel with support in $(-\infty,0)$, a $L^p(w)$ bound when $w \in A_1^+$. In [A. K. Lerner, S. Ombrosi and C. Pérez, A_1 *Bounds for Calderón-Zygmund operators related to a problem of Muckenhoupt and Wheeden*, Math. Res. Lett. **16** (2009), no. 1, 149–156.], the authors proved that this bound is sharp with respect to $||w||_{A_1}$ and with respect to p. We also give a $L^{1,\infty}(w)$ estimate, for a related problem of Muckenhoupt and Wheeden for $w \in A_1^+$. We improve the classical results, for one-sided singular integrals, by putting in the inequalities a wider class of weights.

1. Introduction

Let M be the classical Hardy-Littlewood maximal operator and w a weight (i.e. $w \in L^1_{loc}(\mathbb{R}^n)$ and $w > 0$). C. Fefferman and E. M. Stein in [5] proved an extension of the classical weak-type $(1,1)$ estimate:

$$||Mf||_{L^{1,\infty}(w)} \leqslant C \int_{\mathbb{R}^n} |f(x)| Mw(x)\, dx, \tag{1.1}$$

where $C = C(n)$. This is a sort of duality for M. A consequence of this result, using an interpolation argument, is the following: if $1 < p < \infty$ and $p' = \frac{p}{p-1}$ then,

$$\int_{\mathbb{R}^n} (Mf(x))^p w(x)\, dx \leqslant C p' \int_{\mathbb{R}^n} |f(x)|^p Mw(x)\, dx,$$

where $C = C(n)$.

B. Muckenhoupt and R. Wheeden many years ago, in [18], conjectured that the analogue of (1.1) should hold for T, a singular integral operator, namely

$$\sup_{\lambda>0} \lambda w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leqslant C \int_{\mathbb{R}^n} |f(x)| Mw(x)\, dx, \tag{1.2}$$

where $C = C(n,T)$.

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The best result along this line was given by C. Pérez in [20], where M is replaced by the slightly larger operator $M_{L(logL)^{\varepsilon}}$, $\varepsilon > 0$,

$$||Tf||_{L^{1,\infty}(w)} \leqslant C2^{\frac{1}{\varepsilon}} \int_{\mathbb{R}^n} |f(x)| M_{L(\log L)^{\varepsilon}} w(x)\,dx,$$

where $C = C(n,T)$.

The one-sided version of this result was obtained in [12] by M. Lorente, J. M. Martell, C. Pérez and M. S. Riveros.

M. C. Reguera in [22] and M. C. Reguera and C. Thiele in [23] proved that the Muckenhout-Wheeden conjecture is false. In [22] the author give a first approach by putting in the right hand side the dyadic maximal operator. In [23] they disproved (1.2) for T the Hilbert transform.

On the other hand there is a variant of the conjecture (1.2) which has a lot of interest, namely the weak Muckenhoupt-Wheeden conjecture. The idea is to assume an a priori condition on the weight w. This condition can be read essentially from inequality (1.2): a weight $w \in A_1$ if there is a finite constant C such that $Mw(x) \leqslant Cw(x) \quad a.e. x \in \mathbb{R}^n$. Denote $||w||_{A_1}$ the smallest of these C. The conjecture is the following:

Let $w \in A_1$, then

$$\sup_{\lambda>0} \lambda w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leqslant C||w||_{A_1} \int_{\mathbb{R}^n} |f(x)|w(x)\,dx,$$

where $C = C(n,T)$.

In [11], A. K. Lerner, S. Ombrosy and C. Pérez, exhibit a logarithmic growth

$$||Tf||_{L^{1,\infty}(w)} \leqslant C||w||_{A_1} \log(e + ||w||_{A_1})||f||_{L^1(w)}, \tag{1.3}$$

where $C = C(n,T)$. It is an open problem if this result obtained by the authors in [11] is the best possible. Recently, F. Nazarov, A. Reznikov, V. Vasyunin, A. Volberg, proved that the weak Muckenhoupt-Wheeden conjecture is also false. See [19].

To prove this logarithmic growth result, they had to study first the corresponding weighted $L^p(w)$ estimate for $1 < p < \infty$ and $w \in A_1$,

$$||Tf||_{L^p(w)} \leqslant Cpp'||w||_{A_1}||f||_{L^p(w)}, \tag{1.4}$$

where $C = C(n,T)$, being the last inequality, this time, fully sharp. See [11]. As a consequence of (1.3) and applying the Rubio de Francia's algorithm they also get the following result, let $1 < p < \infty$, $w \in A_p$ and let T be a Calderón-Zygmund operator then

$$||T||_{L^{p,\infty}(w)} \leqslant C||w||_{A_p} \log(e + ||w||_{A_p})||f||_{L^p(w)}, \tag{1.5}$$

where $C = C(n,p,T)$.

The first result of this kind obtaining the precise constant dependence on the A_p norm of w of the operator norms of singular integrals, maximal functions, and other operators in $L^p(w)$ was obtained by S. M. Buckley in [2]. There, he proves that

$$||M||_{L^p(w)} \leqslant C||w||_{A_p}^{\frac{1}{p-1}},$$

where $C = C(n,p)$.

Recently T. Hytönen, C. Pérez and E. Rela in [9] improved this result by giving a sharp weighted bound for the Hardy-Littlewood maximal operator involving the Fujii-Wilson A_∞-constant. Also T. Hytönen and C. Pérez in [8] improved (1.3), (1.4) and several well known results, using for all these cases the Fujii-Wilson A_∞-constant.

In this paper we obtain similar results as the ones in (1.3), (1.4) and (1.5) for one-sided weights and one-sided singular integrals.

Now we will state the results obtained in this work. The definitions of Sawyer's weights and one-sided operators will appear in the next section.

THEOREM 1.1. *Let $1 < p < \infty$, $w \in A_1^+$ and T^+ be a one-sided singular integral, then,*

$$||T^+ f||_{L^p(w)} \leqslant C pp' ||w||_{A_1^+} ||f||_{L^p(w)}, \tag{1.6}$$

where $C = C(T^+)$.

THEOREM 1.2. *Let $w \in A_1^+$ and T^+ be a one-sided singular integral, then,*

$$||T^+ f||_{L^{1,\infty}(w)} \leqslant C ||w||_{A_1^+} \log(e + ||w||_{A_1^+}) ||f||_{L^1(w)}, \tag{1.7}$$

where $C = C(T^+)$.

COROLLARY 1.3. *Let $1 < p < \infty$, $w \in A_p^+$ and T^+ be a one-sided singular integral, then*

$$||T^+ f||_{L^{p,\infty}(w)} \leqslant C ||w||_{A_p^+} \log(e + ||w||_{A_p^+}) ||f||_{L^p(w)}, \tag{1.8}$$

where $C = C(T^+)$.

By a duality argument, Corollary 1.3 implies the following:

COROLLARY 1.4. *Let $1 < p < \infty$, $w \in A_p^-$ and T^- be a one-sided singular integral, then for any measurable set E*

$$||T^-(\sigma \chi_E)||_{L^p(w)} \leqslant C ||w||_{A_p^-}^{\frac{1}{p-1}} \log(e + ||w||_{A_p^-}) \sigma(E)^{\frac{1}{p}}, \tag{1.9}$$

where where $C = C(T^-)$ and $\sigma = w^{\frac{-1}{p-1}}$.

In these results $||w||_{A_1^+}$ is the best constant of the weight $w \in A_1^+$. Clearly, every theorem has a corresponding one, reversing the orientation of $\mathbb{R}$.

Theorems 1.1, 1.2 and Corollaries 1.3, 1.4, for one-sided singular integrals, improve the ones obtained in [11] by putting in the inequalities a wider class of weights (the Sawyer classes).

The article is organized as follows: in Section 2 we introduce notation, definitions and well known results. In Section 3 we prove some previous lemmas that will be essential to obtain the proofs of Theorems and Corollaries given in Section 4. In Section 5 we give a weaker version and a simplest proof of Lemma 3.2 of Section 3.

2. Preliminaries

In this section we give some definitions and well known results.

2.1. One-side singular integral operators and Sawyer's weights

DEFINITION 2.1. Let $f \in L^1_{loc}(\mathbb{R}^n)$. The one-sided maximal operators are defined as

$$M^+f(x) = \sup_{h>0} \frac{1}{h}\int_x^{x+h} |f(t)|\,dt, \quad M^-f(x) = \sup_{h>0} \frac{1}{h}\int_{x-h}^{x} |f(t)|\,dt.$$

The good weights for these operators are the Sawyer weights A_p^+ and A_p^-, see [25], [13], [14]. We recall the definition.

DEFINITION 2.2. Let w be a non-negative locally integrable function and $1 \leqslant p < \infty$. We say that $w \in A_p^+$ if there exists $C(p) < \infty$ such that for every $a < x < b$

$$\frac{1}{(b-a)^p}\left(\int_a^x w\right)\left(\int_x^b w^{\frac{-1}{p-1}}\right)^{p-1} \leqslant C(p), \tag{2.1}$$

when $1 < p < \infty$, and for $p = 1$,

$$M^-w(x) \leqslant C(1)\,w(x), \qquad \text{for a.e. } x \in (a,b), \tag{2.2}$$

finally $A_\infty^+ = \cup_{p\geqslant 1} A_p^+$, see [15].

The smallest possible $C(1)$ in (2.2) here is denoted by $||w||_{A_1^+}$ and the smallest possible $C(p)$ in (2.1) here is denoted by $||w||_{A_p^+}$.

The classes A_p^- for $1 \leqslant p \leqslant \infty$ are defined in a similar way.

We also define

$$M_r^+f(x) = \sup_{h>0}\left(\frac{1}{h}\int_x^{x+h} |f(t)|^r\,dt\right)^{\frac{1}{r}}, \quad M_r^-f(x) = \sup_{h>0}\left(\frac{1}{h}\int_{x-h}^{x} |f(t)|^r\,dt\right)^{\frac{1}{r}},$$

where $r \geqslant 1$. Observe that $M^+f \leqslant M_r^+f$ for all $r \geqslant 1$. Also, we will consider the following maximal operators introduced by F. J. Martín-Reyes, P. Ortega and A. de la Torre in [14],

$$M_g^+f(x) = \sup_{h>0}\int_x^{x+h} |f(t)|g(t)\,dt\left(\int_x^{x+h} g(t)\,dt\right)^{-1},$$

$$M_g^-f(x) = \sup_{h>0}\int_{x-h}^{x} |f(t)|g(t)\,dt\left(\int_{x-h}^{x} g(t)\,dt\right)^{-1},$$

where g is a positive locally integrable function on $\mathbb{R}$.

The classes $A_p^+(g)$, $1 \leqslant p \leqslant \infty$ are defined as follows, let w be non-negative locally integrable functions and let $1 \leqslant p < \infty$. We say that $w \in A_p^+(g)$ if there exists $C(p) < \infty$ such that for every $a < x < b$

$$\left(\int_a^x w\right)\left(\int_x^b g^{p'}\sigma\right)^{p-1} \leqslant C(p)\left(\int_a^b g\right)^p, \tag{2.3}$$

where $\sigma = w^{\frac{-1}{p-1}}$, $\frac{1}{p}+\frac{1}{p'}=1$, when $1<p<\infty$. For $p=1$,

$$M_g^-(g^{-1}w)(x) \leqslant C(1)\,g^{-1}w(x), \qquad \text{a.e. } x \in (a,b).$$

In [14] it was proved that $w \in A_p^+(g)$, if, and only if M_g^+ is bounded from $L^p(w)$ into $L^p(w)$, for $1<p<\infty$, and $w \in A_1^+(g)$, if, and only if M_g^+ maps $L^1(w)$ into $L^{1,\infty}(w)$. Observe that if $g \equiv 1$ then $A_p^+(g) = A_p^+$, for $1 \leqslant p \leqslant \infty$.

DEFINITION 2.3. We shall say that a function K in $L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ is a Calderón-Zygmund kernel if the following properties are satisfied:

- $||\widehat{K}||_\infty < C_1$
- $|K(x)| < \frac{C_2}{|x|^n}$
- $|K(x)-K(x-y)| < \frac{C_3|y|}{|x|^{n+1}}$, where $|y| < \frac{|x|}{2}$.

The Calderón-Zygmund singular integral operator associated to K is defined by

$$Tf(x) = p.v.(K * f)(x) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n/B_\varepsilon(0)} K(x-y)f(y)\,dy,$$

and the maximal operator associated with this kernel K is

$$T^*f(x) = \sup_{\varepsilon>0}\int_{\mathbb{R}^n/B_\varepsilon(0)} |K(x-y)||f(y)|\,dy.$$

A one-sided singular integral T^+ is a singular integral associated to a Calderón–Zygmund kernel with support in $(-\infty,0)$; therefore, in that case,

$$T^+f(x) = \lim_{\varepsilon\to 0^+}\int_{x+\varepsilon}^\infty K(x-y)f(y)\,dy.$$

Examples of such kernels are given by H. Aimar, L. Forzani and F. J. Martín-Reyes in [1]. The operator T^- is defined similarly.

REMARK 2.4.

1. In [1], it is proved that the one-sided singular integral T^+ is controlled by the one-sided maximal functions M^+ in the $L^p(w)$ norm if $w \in A_\infty^+$.

2. It is well known to that the classes A_p are included in A_p^+ and A_p^-; namely $A_p = A_p^- \cap A_p^+$. See [25], [13], [14].

3. The one-sided classes of weights satisfy the following factorization, $w \in A_p^+$ if only if $w = w_1 w_2^{1-p}$ with $w_1 \in A_1^+$ and $w_2 \in A_1^-$, and $||w||_{A_p^+} \leqslant ||w_1||_{A_1^+} ||w_2||_{A_1^-}^{p-1}$. See [25], [13], [14].

4. It is easy to check that $(M^- f)^\delta \in A_1^+$ for all $0 < \delta < 1$ with $||(M^- f)^\delta||_{A_1^+} \leqslant \frac{C}{1-\delta}$.

Finally, we recall some definition concerning Lorentz $L^{p,q}(\mu)$ spaces. Let f be a measurable function on a measure space $(X, \mathscr{M}, \mu)$. The non-increasing rearrangement $f^*(t)$ of f is defined as

$$f^*(t) = \inf\{\lambda > 0 : \mu(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}) \leqslant t\},$$

for all $0 < t < \infty$. The function f is said to belong to the Lorentz space $L^{p,q}(\mu)$ if the quantities

$$||f||^{p,q}(\mu) = \left(\frac{q}{p} \int_0^\infty \left[t^{1/p} f^*(t) \right]^q \frac{dt}{t} \right)^{1/q},$$

whenever $0 < p < \infty$ and $0 < q < \infty$, and

$$||f||^{p,\infty}(\mu) = \sup_{t>0} \left[t^{1/p} f^*(t) \right],$$

when $0 < p \leqslant \infty$, are finite. For more details see [26].

3. Previous Lemmas

To obtain Theorems 1.1 and 1.2 we need to prove some previous results: a sharp weak reverse Hölder's inequality for one-sided weights, a particular case of the Coifman-type estimate for one-sided singular integrals and one-sided maximal operator, and also build A_1^+ weights from duality with special control on the constant, based in the Rubio de Francia algorithm, see [6].

3.1. Sharp weak reverse Hölder's inequality

F. J. Martín-Reyes proved a weak reverse Hölder's inequality (see [13] Lemma 5).

LEMMA 3.1. [13] (Sharp weak reverse Hölder's inequality) *Let* $1 \leqslant p < \infty$ *and* $w \in A_p^+$. *There exist positive numbers* δ *and* C *such that*

$$\int_a^b w^{1+\delta} \leqslant C M^-(w\chi_{(a,b)})(b)^\delta \int_a^b w, \tag{3.1}$$

for every bounded interval (a,b)*, and therefore*

$$M^-_{1+\delta}(w\chi_{(a,b)})(b) \leqslant CM^-(w\chi_{(a,b)})(b),$$

and

$$M^-_{1+\delta}(w)(b) \leqslant CM^-(w)(b).$$

The constant C *depends only on* δ *and the constant of the* A_p^+ *condition.*

Here we will need to be more precise in the constants. In the proof of Lemma 3.1 the constat C depends of a constant β and the constant δ. If we chose $\beta = (4^p||w||_{A_p^+})^{-1}$ and $\delta = \frac{1}{4^{p+2}e^{\frac{1}{e}}||w||_{A_p^+}}$, when $1 < p < \infty$, and $\beta = (||w||_{A_1^+})^{-1}$ and $\delta = \frac{1}{16e^{\frac{1}{e}}||w||_{A_1^+}}$, when $p = 1$, following the same steps of the proof, we obtain that $C \leqslant 2$. Then we can rewrite the equation (3.1) as

$$\int_a^b w^{r_w} \leqslant 2M^-(w\chi_{(a,b)})(b)^{r_w-1}\int_a^b w, \tag{3.2}$$

for every bounded interval (a,b), and therefore

$$M^-_{r_w}(w\chi_{(a,b)})(b) \leqslant 2M^-(w\chi_{(a,b)})(b),$$

and

$$M^-_{r_w}(w)(b) \leqslant 2M^-(w)(b), \tag{3.3}$$

where $r_w = 1 + \frac{1}{4^{p+2}e^{\frac{1}{e}}||w||_{A_p^+}}$, when $p > 1$, and $r_w = 1 + \frac{1}{16e^{\frac{1}{e}}||w||_{A_1^+}}$ when $p = 1$.

The following Lemma will be necessary to prove good-λ result in the next section.

LEMMA 3.2. *Let* $1 < p < \infty$*,* $w \in A_p^-$*,* $a < b < c$ *and* $E \subseteq (b,c)$ *a measurable set. For all* $\varepsilon > 0$*, there exists* $C = C(\varepsilon, p)$ *such that if* $|E| < e^{-C||w||_{A_p^-}}(b-a)$ *then* $w(E) < \varepsilon w(a,c)$*.*

Proof. Let $w \in A_p^-$. Let apply the analogous to equation (3.2), i.e.

$$\int_b^c w^r \leqslant 2M^+(w\chi_{(b,c)})(b)^{r-1}\int_b^c w.$$

This last inequality implies

$$(M_w^+(w^{r-1}\chi_{(b,c)})(b))^{\frac{1}{r-1}} \leqslant 2M^+(w\chi_{(b,c)})(b),$$

where we take $r = r_w = 1 + \frac{1}{4^{p+2}e^{\frac{1}{e}}||w||_{A_p^-}}$, when $p > 1$ and $r = r_w = 1 + \frac{1}{16e^{\frac{1}{e}}||w||_{A_1^-}}$, when $p = 1$.

Using the definition of M_g^+, with $g = w$, we have that for all $x \in (a,b)$

$$\left(\frac{1}{w(a,c)}\int_b^c w^{r-1}w\right)^{\frac{1}{r-1}} \leqslant (M_w^+(w^{r-1}\chi_{(x,c)})(x))^{\frac{1}{r-1}} \leqslant 2M^+(w\chi_{(x,c)})(x) \leqslant 2M^+(w\chi_{(a,c)})(x),$$

then

$$(a,b) \subseteq \left\{x : M^+(w\chi_{(a,c)})(x) > \frac{1}{4}\left(\frac{1}{w(a,c)}\int_b^c w^r\right)^{\frac{1}{r-1}}\right\}.$$

Recalling that M^+ is of weak type $(1,1)$ with respect to the Lebesgue measure, we get

$$b-a < C_1 w(a,c)^{\frac{1}{r-1}}\left(\int_b^c w^r\right)^{\frac{-1}{r-1}} w(a,c),$$

where C_1 does not depend on the weight w. This last inequality says that $1 \in A_{r'}^+(w)$ (where r' is such that $\frac{1}{r}+\frac{1}{r'} = 1$), with constant C_1, see (2.3). Let $x \in (a,b)$, by hypothesis $E \subset (b,c)$ then

$$M_w^+(\chi_E(x)) \geqslant \frac{1}{w(x,c)}\int_x^c \chi_E(t)w(t)\,dt \geqslant \frac{w(E)}{w(a,c)},$$

obtaining that the interval $(a,b) \subset \{x : M_w^+(\chi_E(x)) > \frac{w(E)}{2w(a,c)}\}$. Observe that M_w^+ is of weak type (r',r') with respect to the Lebesgue measure with constant $||M_w^+||_{L^{r'}\to L^{r',\infty}} = C_2$ (see [14]), then

$$b-a \leqslant \left|\left\{x : M_w^+(\chi_E(x)) > \frac{w(E)}{2w(a,c)}\right\}\right| \leqslant C_2\left(\frac{2w(a,c)}{w(E)}\right)^{r'}|E|.$$

Taking into account that $1 < r < 2$ and $C_2^{\frac{1}{r'}} \leqslant C_3$, where C_3 does not depend on p or $||w||_{A_p^-}$ (see [14]), then

$$\frac{w(E)}{w(a,c)} < \left(C_2\frac{|E|}{b-a}\right)^{\frac{1}{r'}} < C_3 e^{\frac{-C||w||_{A_p^-}}{r'}} < \varepsilon, \tag{3.4}$$

where the last inequality holds by choosing an appropriate C depending only on p and ε. □

3.2. The Coifman-type estimate

Now we give a particular case of the Coifman-type estimate. In order to do this we need a kind of good-λ inequality result. We will use the next result due to S. M. Buckley in [2].

LEMMA 3.3. [2] *Let $g \in L^\infty(I)$ and T be an operator for which*

$$|\{x : T\phi(x) > \alpha\}| \leqslant \left(\frac{Cp||\phi||_p}{\alpha}\right),$$

for all $\phi \in L^p(\mathbb{R})$, sufficiently large p and α and C being a constant independent of p.

Then,

$$|\{x : Tg(x) > \alpha\}| \leqslant Ce^{\frac{\alpha}{e||g||_\infty}}|I|.$$

LEMMA 3.4. *Let $1 \leqslant p < \infty$, $w \in A_p^-$, T^- be a one-sided singular integral and T^* the maximal operator related to T^-. Then, there exist positive constants c_1, c_2, $\gamma_0 > 0$ such that for every $0 < \gamma < \gamma_0$*

$$\left|\{x \in \mathbb{R} : T^*f(x) > 2\lambda, M^-f(x) < \gamma\lambda\}\right| < c_1 e^{-\frac{c_2}{\gamma}} |\{T^*f(x) > \lambda\}|$$

holds for $f \in L^1(\mathbb{R})$, $\lambda > 0$. Also, for all $\varepsilon > 0$, there exists c' depending on ε, γ_0 and p such that

$$w\left(\left\{x \in \mathbb{R} : T^*f(x) > 2\lambda, M^-f(x) < \frac{c'\lambda}{||w||_{A_p^-}}\right\}\right) < \varepsilon w(\{T^*f(x) > \lambda\}).$$

Proof. Since $\{x : T^*f(x) > \lambda\}$ is an open set and it has finite measure for $f \in L^1(\mathbb{R})$, it can be written as a disjoint countable union of open intervals. Let $J = (a,b)$ be such an interval. It is enough to prove that there exist c_1, c_2, c' and γ_0 such that

$$\left|\{x \in J : T^*f(x) > 2\lambda, M^-f(x) < \gamma\lambda\}\right| < c_1 e^{-\frac{c_2}{\gamma}} |J|,$$

and

$$w\left(\left\{x \in J : T^*f(x) > 2\lambda, M^-f(x) < \frac{c'\lambda}{||w||_{A_p^-}}\right\}\right) < \varepsilon w(J),$$

for every $0 < \gamma < \gamma_0$ and every $\lambda > 0$. Let us take a sequence $\{x_i\}_{i=0}^\infty$ in $J = (a,b)$ in such a way that $x_0 = b$ and $x_{i-1} - x_i = x_i - a$ for every $i > 0$. Observe that we only need to prove that

$$\left|\{x \in (x_{i+1}, x_i) : T^*f(x) > 2\lambda, M^-f(x) < \gamma\lambda\}\right| < c_1 e^{-\frac{c_2}{\gamma}} (x_{i+1} - x_{i+2}). \tag{3.5}$$

By Lemma 3.2 there exists c' depending on $\varepsilon, \gamma_0, p, c_1, c_2$, such that

$$w\left(\left\{x \in (x_{i+1}, x_i) : T^*f(x) > 2\lambda, M^-f(x) < \frac{c'\lambda}{||w||_{A_p^-}}\right\}\right) < \varepsilon w(x_i, x_{i+2}). \tag{3.6}$$

Let us show (3.5). Let $i\in\mathbb{N}$, if $\{x\in(x_{i+1},x_i): T^*f(x)>2\lambda, M^-f(x)<\gamma\lambda\}=\emptyset$ there is nothing to prove. We choose $\overline{a}<a$ such that $x_i-a=a-\overline{a}$ and

$$\xi=\sup\{x\in(x_{i+1},x_i): M^-f(x)\leqslant\gamma\lambda\}.$$

Let us write $f=f_1+f_2$ with $f_1=f\chi(\overline{a},\xi)$ then

$$\{x\in(x_{i+1},x_i): T^*f(x)>2\lambda, M^-f(x)<\gamma\lambda\}\subset A\cup B,$$

where

$$A=\{x\in(x_{i+1},\xi): T^*f_1(x)>\frac{1}{2}\lambda, M^-f(x)<\gamma\lambda\},$$
$$B=\left\{x\in(x_{i+1},\xi): T^*f_2(x)>\frac{3}{2}\lambda, M^-f(x)<\gamma\lambda\right\}.$$

The second set B is essentially empty for γ small enough. By standard estimation (see [1]), we get that for $x\in(x_{i+1},\xi)$, $T^*f_2(x)\leqslant\frac{3}{2}\lambda$ then

$$\left\{x\in(x_{i+1},\xi): T^*f_2(x)>\frac{3}{2}\lambda, M^-f(x)<\gamma\lambda\right\}=\emptyset,$$

for $0<\gamma<\gamma_0$ small enough.

Now we work with set A. Let $\Omega=\{x\in(x_{i+1},\xi): M^-f_1(x)>3\gamma\lambda\}$, observe that

$$\int_{\mathbb{R}} f_1(t)\,dt\leqslant 4\gamma\lambda(x_i-x_{i+1}).$$

The last inequality implies that $\Omega\subset(\overline{a},\widetilde{a})$ with $\widetilde{a}-\xi=\frac{4}{3}(x_i-x_{i+1})$. Let us write $\Omega=\bigcup I_j$ where $I_j=(a_j,b_j)$ are disjoint maximal intervals. Then

$$\frac{1}{|I_j|}\int_{I_j} f_1(t)\,dt=3\gamma\lambda.$$

We define $I_j^+=(b_j,c_j)$, $\left|I_j^+\right|=2\left|I_j\right|$, $\widetilde{\Omega}=\bigcup(I_j^+\cup I_j)=\bigcup\widetilde{I_j}$ and $f_1=g+h$ with

$$g=f_1\chi_{\mathbb{R}/\Omega}+\sum_j 3\gamma\lambda\chi_{I_j},\qquad h=\sum_j h_j=\sum_j(f_1-3\gamma\lambda)\chi_{I_j}.$$

Observe that $g\leqslant 3\gamma\lambda$ and g has support in $(\overline{a},\widetilde{a})$. Then using Lemma 3.3 we have

$$\left|\left\{x: T^*g(x)>\frac{\lambda}{4}\right\}\right|\leqslant e^{\frac{-c}{\gamma}}(\widetilde{a}-\overline{a})\leqslant\frac{32}{3}e^{\frac{-c}{\gamma}}(x_{i+1}-x_{i+2}).$$

Now let us study T^*h for $x \notin \widetilde{\Omega}$,

$$\begin{aligned}
|T^*h(x)| &\leqslant \sum_j \int_{I_j} \left| h_j(y)(K(x-y)-K(x-b_j)) \right| dy \\
&\leqslant C \sum_j \int_{I_j} \left| h_j(y) \right| \frac{y-b_j}{(x-b_j)^2}\, dy \\
&\leqslant \frac{3}{2} C \sum_j \frac{\delta_j}{\delta_j^2+(x-b_j)^2} \int_{I_j} \left| h_j(y) \right| dy \\
&\leqslant 9C\gamma\lambda \sum_j \frac{\delta_j}{\delta_j^2+(x-b_j)^2} |I_j| \\
&\leqslant C\gamma\lambda \sum_j \frac{\delta_j^2}{\delta_j^2+(x-b_j)^2},
\end{aligned}$$

where $\delta_j = c_j - a_j$. We write $\Delta(x) = \sum_j \frac{\delta_j^2}{\delta_j^2+(x-b_j)^2}$.

Observe that if $x \in \widetilde{\Omega}$ then $M^- f(x) \geqslant \gamma\lambda$. In fact, if $x \in I_j$, for some j, then by definition of Ω we have that $3\gamma\lambda < M^- f_1(x) < M^- f(x)$. If $x \in I_j^+$ then

$$3\gamma\lambda = \frac{1}{|I_j|} \int_{I_j} f_1(t)\, dt = \frac{x-a_j}{(x-a_j)|I_j|} \int_{a_j}^{x} f(t)\, dt \leqslant 3M^- f(x).$$

By the exponential Carleson's estimation, (see [3]), we have

$$\left| \left\{ x \in (x_{i+1}, \xi) : \Delta(x) > \frac{c}{\gamma} \right\} \right| < Ce^{\frac{-c}{\gamma}} \left| (x_{i+1}, \xi) \right| \leqslant 2Ce^{\frac{-c}{\gamma}} (x_{i+1} - x_{i+2}),$$

therefore

$$\left| \left\{ x \in (x_{i+1}, \xi) : T^*h(x) > \frac{1}{4}\lambda, M^- f(x) < \gamma\lambda \right\} \right| \leqslant Ce^{\frac{-c}{\gamma}} (x_{i+1} - x_{i+2}).$$

Putting together this last estimate with the ones for f_2 and g, we obtain the desired result. $\square$

LEMMA 3.5. *Let $p \geqslant 1$, $w \in A_p^-$ and let T^- be a one-sided singular integral. Then there exists a constant $C = C(p, T^-)$, such that*

$$||T^- f||_{L^1(w)} \leqslant C ||w||_{A_p^-} ||M^- f||_{L^1(w)}.$$

Proof. By Lemma 3.4, for $\varepsilon = \frac{1}{4}$ exists c' such that

$$w\left(\left\{ x \in \mathbb{R} : T^* f(x) > 2\lambda, M^- f(x) < \frac{c'\lambda}{||w||_{A_p^-}} \right\} \right) < \frac{1}{4} w(\{T^* f(x) > \lambda\}).$$

Observe that

$$\int_0^N w(\{T^- f > \lambda\})\,d\lambda \leqslant 2\int_0^{\frac{N}{2}} w(\{T^* f > 2\lambda\})\,d\lambda \leqslant B_1 + B_2,$$

where

$$B_1 = 2\int_0^{\frac{N}{2}} w\left(\left\{T^* f > 2\lambda, M^- f < \frac{c'\lambda}{||w||_{A_p^-}}\right\}\right) d\lambda,$$

$$B_2 = 2\int_0^{\frac{N}{2}} w\left(\left\{M^- f \geqslant \frac{c'\lambda}{||w||_{A_p^-}}\right\}\right) d\lambda.$$

For B_1, we obtain

$$B_1 = 2\int_0^{\frac{N}{2}} w\left(\left\{T^* f > 2\lambda, M^- f < \frac{c'\lambda}{||w||_{A_p^-}}\right\}\right) d\lambda \leqslant \frac{1}{2}\int_0^N w(\{T^* f > \lambda\})\,d\lambda.$$

It is easy to see that

$$\frac{1}{2}\int_0^N w(\{T^* f > \lambda\})\,d\lambda \leqslant \frac{2||w||_{A_p^-}}{c'}\int_0^{\frac{Nc'}{2||w||_{A_p^-}}} w(M^- f \geqslant \lambda\})\,d\lambda,$$

then

$$||T^- f||_{L^1(w)} \leqslant \frac{4||w||_{A_p^-}}{c'}||M^- f||_{L^1(w)},$$

obtaining the desired result. □

For the next result we shall need the following Lemma due to A. K. Lerner, S. Ombrosi and C. Pérez in [11].

LEMMA 3.6. [11] *Let $1 < s < \infty$ and v be a weight. There exists an operator R in $L^s(v)$ such that*

- $h \leqslant R(h)$
- $||R(h)||_{L^s(v)} \leqslant 2||h||_{L^s(v)}$
- $R(h)(v)^{\frac{1}{s}} \in A_1$ *with* $||R(h)(v)^{\frac{1}{s}}||_{A_1} \leqslant cs'$.

It is known that the weight $(M_r^- w)^{1-p'}$ belongs to the A_∞^- class with the corresponding constants independent of w. Hence the next Lemma is a particular case of the Coifman-type estimate.

LEMMA 3.7. *Let T^- be a one-sided singular integral, $p, r \geqslant 1$. Then there exists $C = C(T^-)$ such that*

$$\left|\left|\frac{T^- f}{M_r^- w}\right|\right|_{L^{p'}(M_r^- w)} \leqslant C p' \left|\left|\frac{M^- f}{M_r^- w}\right|\right|_{L^{p'}(M_r^- w)}. \qquad (3.7)$$

Proof. By duality we have

$$\left|\left|\frac{T^- f}{M_r^- w}\right|\right|_{L^{p'}(M_r^- w)} = \sup_{||h||_{L^p(M_r^- w)}=1} \int_{\mathbb{R}} \left|T^- f\right| h\, dx.$$

Choosing $s = p$ and $v = M_r^- w$, by Lemma 3.6, there exists an operator R such that $R(h)(M_r^- w)^{\frac{1}{p}} \in A_1$ with $||R(h)(M_r^- w)^{\frac{1}{p}}||_{A_1} \leqslant cp'$, then by Remark 2.4, item (2), $R(h)(M_r^- w)^{\frac{1}{p}} \in A_1^-$ with $||R(h)(M_r^- w)^{\frac{1}{p}}||_{A_1^-} \leqslant cp'$.

Now using the Remark 2.4, item (3) and item (4), we have

$$\begin{aligned} ||R(h)||_{A_3^-} &= ||R(h)(M_r^- w)^{\frac{1}{p}}[(M_r^- w)^{\frac{1}{2p}}]^{-2}||_{A_3^+} \\ &\leqslant ||R(h)(M_r^- w)^{\frac{1}{p}}||_{A_1^-}||(M_r^- w)^{\frac{1}{2p}}||_{A_1^+}^2 \\ &\leqslant cp' \left(\frac{c}{1-\frac{1}{2pr}}\right)^2 \leqslant Cp'. \end{aligned}$$

Finally by Lemma 3.5,

$$\begin{aligned} \int_{\mathbb{R}} \left|T^- f\right| h\, dx &\leqslant \int_{\mathbb{R}} \left|T^- f\right| R(h)\, dx \leqslant C||R(h)||_{A_3^-} \int_{\mathbb{R}} M^-(f) R(h)\, dx \\ &\leqslant Cp' \int_{\mathbb{R}} \frac{M^- f}{M_r^- w} R(h) M_r^- w\, dx \leqslant Cp' \left|\left|\frac{M^- f}{M_r^- w}\right|\right|_{L^{p'}(M_r^- w)} ||R(h)||_{L^p(M_r^- w)}. \end{aligned}$$

As $||h||_{L^p(M_r^- w)} = 1$ we have

$$\left|\left|\frac{T^- f}{M_r^- w}\right|\right|_{L^{p'}(M_r^- w)} \leqslant Cp' \left|\left|\frac{M^- f}{M_r^- w}\right|\right|_{L^{p'}(M_r^- w)}. \quad \square$$

4. Proof of the results

4.1. Proof of the Theorems

In order to prove Theorem 1.1 we first need to show the following result:

THEOREM 4.1. *Let $1 < p < \infty$, $1 < r < 2$, w a weight and T^+ be a one-sided singular integral. Then*

$$||T^+ f||_{L^p(w)} \leqslant C pp'(r')^{\frac{1}{p'}} ||f||_{L_p(M_r^- w)}, \tag{4.1}$$

where $C = C(T^+)$.

Proof. Observe that T^- is the adjoint operator of T^+, with kernel supported in $(0,\infty)$. Also observe that as $(M_r^- w) \in A_1^+ \subset A_p^+$, then $(M_r^- w)^{1-p'} \in A_{p'}^- \subset A_\infty^-$. Therefore (4.1) is equivalent to prove

$$\left|\left|\frac{T^- f}{M_r^- w}\right|\right|_{L^{p'}(M_r^- w)} \leqslant Cpp'(r')^{\frac{1}{p'}} \left|\left|\frac{f}{w}\right|\right|_{L^{p'}(w)}.$$

By Hölder's inequality

$$\frac{1}{b-a}\int_a^b f w^{-\frac{1}{p}} w^{\frac{1}{p}} \leqslant \left(\frac{1}{b-a}\int_a^b w^r\right)^{\frac{1}{pr}} \left(\frac{1}{b-a}\int_a^b \left(f w^{-\frac{1}{p}}\right)^{(pr)'}\right)^{\frac{1}{(pr)'}},$$

and taking supremum we get

$$(M^- f(b))^{p'} \leqslant (M_r^- w(b))^{p'-1} (M_{(pr)'}^- (f w^{-\frac{1}{p}})(b))^{p'},$$

then

$$\left|\left|\frac{M^- f}{M_r^- w}\right|\right|_{L^{p'}(M_r^- w)} \leqslant \left|\left|M_{(pr)'}^- \left(f w^{-\frac{1}{p}}\right)\right|\right|_{L^{p'}}.$$

Now using that $||M_k^- g||_{L^s} \leqslant C\left(\frac{s}{k}\right)'^{\frac{1}{k}} ||g||_{L^s}$, for $g = f w^{\frac{1}{p}}$, $k = (pr)'$ and $s = p'$ we get

$$\left|\left|\frac{M^- f}{M_r^- w}\right|\right|_{L^{p'}(M_r^- w)} \leqslant C\left(\frac{rp-1}{r-1}\right)^{1-\frac{1}{pr}} \left|\left|\frac{f}{w}\right|\right|_{L^{p'}(w)} \leqslant Cp\left(\frac{1}{r-1}\right)^{1-\frac{1}{pr}} \left|\left|\frac{f}{w}\right|\right|_{L^{p'}(w)}.$$

Observe that $t^{\frac{1}{t}} \leqslant 2$ for $t \geqslant 1$, then

$$\left(\frac{1}{r-1}\right)^{1-\frac{1}{pr}} \leqslant (r')^{1-\frac{1}{p+1}+\frac{1}{pr'}} \leqslant 2(r')^{\frac{1}{p'}}.$$

Finally applying Lemma 3.7 we get,

$$\left|\left|\frac{T^- f}{M_r^- w}\right|\right|_{L^{p'}(M_r^- w)} \leqslant Cp'\left|\left|\frac{M^- f}{M_r^- w}\right|\right|_{L^{p'}(M_r^- w)} \leqslant Cpp'(r')^{\frac{1}{p'}} \left|\left|\frac{f}{w}\right|\right|_{L^{p'}(w)}. \quad \square$$

Proof of Theorem 1.1. This result is a consequence of Theorem 4.1. Using the equation (3.3), we observe that $r'_w \lesssim ||w||_{A_1^+}$ and $M_{r_w}^-(w)(x) \leqslant 2M^-(w)(x) \leqslant 2||w||_{A_1^+} w(x)$ $a.e.\ x$. Then,

$$\begin{aligned}
||T^+ f||_{L^p(w)} &\leqslant Cpp'(r'_w)^{\frac{1}{p'}} \left(\int_{\mathbb{R}} |f|^p(x) M_{r_w}^-(w)(x)\,dx\right)^{\frac{1}{p}} \\
&\leqslant Cpp'(||w||_{A_1^+})^{\frac{1}{p'}} ||w||_{A_1^+}^{\frac{1}{p}} ||f||_{L^p(w)} \\
&\leqslant Cpp'||w||_{A_1^+} ||f||_{L^p(w)}. \quad \square
\end{aligned}$$

Proof of Theorem 1.2. Without loss of generality we assume that $0 \leqslant f \in L_c^\infty(\mathbb{R})$. Let

$$\Omega = \{x \in \mathbb{R} : M^+ f(x) > \lambda\} = \bigcup_j I_j = \bigcup_j (a_j, b_j),$$

where $I_j = (a_j, b_j)$ are the connected component of Ω and they satisfy

$$\frac{1}{|I_j|} \int_{I_j} f(y)\,dy = \lambda.$$

Note that if $x \notin \Omega$, then for all $h \geqslant 0$

$$\frac{1}{h} \int_x^{x+h} f(y)\,dy \leqslant \lambda.$$

Therefore $f(x) \leqslant \lambda$ for a.e $x \in \mathbb{R} \setminus \Omega$. Let $I_j^- = (c_j, a_j)$ with c_j chosen so that $|I_j^-| = 2|I_j|$ and set

$$\tilde{\Omega} = \bigcup_j (I_j^- \cup I_j) = \bigcup_j \tilde{I}_j.$$

We write $f = g + h$ where

$$g = f\chi_{\mathbb{R}\setminus\Omega} + \sum_{j=1}^{\infty} \lambda \chi_{I_j}, \qquad h = \sum_{j=1}^{\infty} h_j = \sum_{j=1}^{\infty} (f - \lambda)\chi_{I_j}.$$

Observe that $0 \leqslant g(x) \leqslant \lambda$ for a.e. x and also that h_j has vanishing integral. Then

$$\begin{aligned} w(\{x : |T^+ f(x)| > \lambda\}) &\leqslant w(\widetilde{\Omega}) + w\left(\left\{x \in \mathbb{R} \setminus \widetilde{\Omega} : |T^+ h(x)| > \frac{\lambda}{2}\right\}\right) \\ &\quad + w\left(\left\{x \in \mathbb{R} \setminus \widetilde{\Omega} : |T^+ g(x)| > \frac{\lambda}{2}\right\}\right) = I + II + III. \end{aligned}$$

We estimate I:

$$I = w(\widetilde{\Omega}) \leqslant \sum_j (w(I_j^-) + w(I_j)),$$

for each j

$$\begin{aligned} w(I_j^-) &= \frac{w(I_j^-)}{|I_j|}|I_j| = \frac{w(I_j^-)}{|I_j|} \frac{1}{\lambda} \int_{I_j} f(x)\,dx \\ &= \frac{1}{\lambda} \int_{I_j} \frac{1}{|I_j|} \int_{I_j^-} w(t)\,dt\, f(x)\,dx \leqslant \frac{3}{\lambda} \int_{I_j} \frac{1}{(x - c_j)} \int_{c_j}^x w(t)\,dt\, f(x)\,dx \\ &\leqslant \frac{3}{\lambda} \int_{I_j} f(x) M^- w(x)\,dx. \end{aligned}$$

On the other hand, $(w,M^-w)\in A_1^+$ then M^+ is weak type $(1,1)$ with respect to this pair of weights, then

$$\sum_j w(I_j)=w(\{x:M^+f(x)>\lambda\})<\frac{4}{\lambda}\int_{\mathbb{R}} f(t)M^-w(t)\,dt,$$

therefore

$$I=w(\widetilde{\Omega})\leqslant \frac{7}{\lambda}\int_{\mathbb{R}} f(t)M^-w(t)\,dt\leqslant \frac{7}{\lambda}||w||_{A_1^+}\int_{\mathbb{R}} f(t)w(t)\,dt.$$

To estimate II, let $r_j=|I_j|=|I_j^-|/2$. Now we use that h_j is supported in I_j, $\int_{I_j}h_j=0$, and that K is supported in $(-\infty,0)$:

$$\begin{aligned}II&=w\left(\left\{x\in\mathbb{R}\setminus\widetilde{\Omega}:\left|T^+h(x)\right|>\frac{\lambda}{2}\right\}\right)\leqslant\frac{2}{\lambda}\int_{\mathbb{R}\setminus\widetilde{\Omega}}\left|T^+h(t)\right|w(t)\,dt\\&\leqslant\frac{2}{\lambda}\sum_j\int_{I_j}\left|h_j(y)\right|\int_{\mathbb{R}\setminus\widetilde{I}_j}\left|K(t-y)-K(t-a_j)\right|w(t)\,dt\,dy\\&=\frac{2}{\lambda}\sum_j\int_{I_j}\left|h_j(y)\right|\int_{-\infty}^{c_j}\left|K(t-y)-K(t-a_j)\right|w(t)\,dt\,dy.\end{aligned}$$

Observe that it is suffice to obtain that for all $y\in I_j$,

$$\int_{-\infty}^{c_j}\left|K(t-y)-K(t-a_j)\right|w(t)\,dt\leqslant C\operatorname*{ess\,inf}_{I_j}M^-(w\chi_{\mathbb{R}\setminus\widetilde{I}_j}).$$

To see this we use the condition of the kernel K,

$$\begin{aligned}\int_{-\infty}^{c_j}\left|K(t-y)-K(t-a_j)\right|w(t)\,dt&=\sum_{k=1}^{\infty}\int_{a_j-2^{k+1}r_j}^{a_j-2^kr_j}\left|K(t-y)-K(t-a_j)\right|w(t)\,dt\\&\leqslant C\sum_{k=1}^{\infty}\int_{a_j-2^{k+1}r_j}^{a_j-2^kr_j}\left|\frac{y-a_j}{(t-a_j)^2}\right|w(t)\,dt\\&\leqslant C\sum_{k=1}^{\infty}\frac{y-a_j}{(2^kr_j)^2}\int_{a_j-2^{k+1}r_j}^{a_j-2^kr_j}w(t)\chi_{(a_j-2^kr_j,a_j-2^{k+1}r_j)}\,dt\\&\leqslant C\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{1}{(2^kr_j)}\int_{a_j-2^{k+1}r_j}^{a_j-2^kr_j}w(t)\chi_{(a_j-2^kr_j,a_j-2^{k+1}r_j)}\,dt,\end{aligned}$$

where $C=C(T^+)$. If $x\in I_j$

$$\begin{aligned}&\int_{-\infty}^{c_j}\left|K(t-y)-K(t-a_j)\right|w(t)\,dt\\&\leqslant C\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{x-a_j+2^{k+1}r_j}{2^kr_j}\frac{1}{x-a_j+2^{k+1}r_j}\int_{a_j-2^{k+1}r_j}^{x}w(t)\chi_{(a_j-2^kr_j,a_j-2^{k+1}r_j)}\,dt\\&\leqslant CM^-w\chi_{\mathbb{R}\setminus\widetilde{I}_j}(x)\sum_{k=1}^{\infty}\frac{1}{2^k}\left(\frac{x-a_j}{2^kr_j}+\frac{2^{k+1}r_j}{2^kr_j}\right)\leqslant CM^-(w\chi_{\mathbb{R}\setminus\widetilde{I}_j})(x),\end{aligned}$$

therefore

$$\begin{aligned} II &\leqslant \frac{C}{\lambda}\sum_j \operatorname*{ess\,inf}_{I_j} M^-(w\chi_{\mathbb{R}\setminus\widetilde{I}_j})\int_{I_j}\left|h_j(y)\right|dy \\ &\leqslant \frac{C}{\lambda}\sum_j\int_{I_j}\left|h_j(y)\right|M^-(w\chi_{\mathbb{R}\setminus\widetilde{I}_j})(y)\,dy \\ &\leqslant \frac{C}{\lambda}\left[\sum_j\int_{I_j} f(y)M^-(w\chi_{\mathbb{R}\setminus\widetilde{I}_j})(y)\,dy+\sum_j\int_{I_j}|g(y)|M^-(w\chi_{\mathbb{R}\setminus\widetilde{I}_j})(y)\,dy\right] \\ &= \frac{C}{\lambda}(A+B). \end{aligned}$$

For A there is nothing to prove. To work with B we need to prove the following inequality

$$M^-(w\chi_{\mathbb{R}\setminus\widetilde{I}_j})(y)\leqslant\frac{3}{2}\operatorname*{ess\,inf}_{z\in I_j}\ M^-(w\chi_{\mathbb{R}\setminus\widetilde{I}_j})(z), \tag{4.2}$$

for all $y\in I_j$. In fact for $y,z\in I_j$,

$$\begin{aligned} M^-(w\chi_{\mathbb{R}\setminus\widetilde{I}_j})(y) &= \sup_{t<y}\frac{1}{y-t}\int_t^y w(s)\chi_{\mathbb{R}\setminus\widetilde{I}_j}(s)\,ds=\sup_{t<c_j}\frac{1}{y-t}\int_t^{c_j} w(s)\chi_{\mathbb{R}\setminus\widetilde{I}_j}(s)\,ds \\ &\leqslant \sup_{t<c_j}\frac{3}{2}\frac{1}{z-t}\int_t^{c_j} w(s)\chi_{\mathbb{R}\setminus\widetilde{I}_j}(s)\,ds\leqslant\frac{3}{2}M^-(w\chi_{\mathbb{R}\setminus\widetilde{I}_j})(z). \end{aligned}$$

Then

$$\begin{aligned} B &= \sum_j\int_{I_j}|g(y)|M^-(w\chi_{\mathbb{R}\setminus\widetilde{I}_j})(y)\,dy=\sum_j\int_{I_j}\lambda M^-(w\chi_{\mathbb{R}\setminus\widetilde{I}_j})(y)\,dy \\ &\leqslant \sum_j\int_{I_j} f(t)\,dt\,\frac{1}{|I_j|}\int_{I_j} M^-(w\chi_{\mathbb{R}\setminus\widetilde{I}_j})(y)\,dy \\ &\leqslant \frac{3}{2}\sum_j\int_{I_j} f(t)\,dt\ \operatorname*{ess\,inf}_{I_j} M^-(w\chi_{\mathbb{R}\setminus\widetilde{I}_j})\leqslant\frac{3}{2}\sum_j\int_{I_j} f(t)M^-(w\chi_{\mathbb{R}\setminus\widetilde{I}_j})(t)\,dt. \end{aligned}$$

So

$$II\leqslant\frac{C}{\lambda}\sum_j\int_{I_j} f(t)M^-(w\chi_{\mathbb{R}\setminus\widetilde{I}_j})(t)\,dt\leqslant\frac{C}{\lambda}||w||_{A_1^+}\int_{\mathbb{R}} f(t)w(t)\,dt.$$

Finally we estimate III. First observe that doing the same proof that in (4.2) we obtain

$$M^-(w\chi_{\mathbb{R}\setminus\widetilde{\Omega}})(y)\leqslant\frac{3}{2}\operatorname*{ess\,inf}_{z\in I_j}\ M^-(w\chi_{\mathbb{R}\setminus\widetilde{\Omega}})(z), \tag{4.3}$$

for all $y\in I_j$.

By Chevichef's inequality, using the fact that $g \leqslant \lambda$ and choosing $r = r_w = 1 + \frac{1}{16e^{\frac{1}{e}}||w||_{A_1^+}}$ in order to apply Theorem 4.1 and equation (3.3), we get that

$$\begin{aligned} III &= w\left(\left\{x \in \mathbb{R} \setminus \widetilde{\Omega} : \left|T^+ g\right|(x) > \frac{\lambda}{2}\right\}\right) \\ &\leqslant \frac{2^p}{\lambda^p} \int_{\mathbb{R}} (\left|T^+ g\right|(x))^p w(x) \chi_{(\mathbb{R}\setminus\widetilde{\Omega})}(x)\, dx \\ &\leqslant \frac{2^p}{\lambda^p} (Cpp'(r')^{\frac{1}{p'}})^p \int_{\mathbb{R}} (|g|(x))^p M_r^-(w\chi_{(\mathbb{R}\setminus\widetilde{\Omega})})(x)\, dx \\ &\leqslant \frac{2^{p+1}}{\lambda} (Cpp'(r')^{\frac{1}{p'}})^p \int_{\mathbb{R}} (|g|(x)) M^-(w\chi_{(\mathbb{R}\setminus\widetilde{\Omega})})(x)\, dx \\ &\leqslant \frac{2^{p+1}}{\lambda} (Cpp'((r')^{\frac{1}{p'}})^p \left[\int_{\mathbb{R}/\Omega} |g(x)| M^-(w\chi_{\mathbb{R}\setminus\widetilde{\Omega}})(x) dx + \int_{\Omega} |g(x)| M^-(w\chi_{\mathbb{R}\setminus\widetilde{\Omega}})(x) dx\right]. \end{aligned}$$

Recalling that $f(x) = g(x)$ for all $x \in \mathbb{R} \setminus \Omega$ and arguing as in B for the integral of g in Ω, this time using (4.3), we obtain

$$\int_{\mathbb{R}} |g(x)| M^-(w\chi_{\mathbb{R}\setminus\widetilde{\Omega}})(x)\, dx \leqslant \frac{3}{2} \int_{\mathbb{R}} f(x) M^-(w)(x)\, dx.$$

Now the fact that $w \in A_1^+$ and $r' = r'_w \leqslant C||w||_{A_1^+}$ implies

$$\begin{aligned} III &\leqslant \frac{2^{p+1}}{\lambda} (Cpp'((r')^{\frac{1}{p'}})^p \int_{\mathbb{R}} f(x) M^- w(x)\, dx \\ &\leqslant \frac{2^{p+1}}{\lambda} (Cpp'(||w||_{A_1^+})^{\frac{1}{p'}})^p ||w||_{A_1^+} \int_{\mathbb{R}} f(x) w(x)\, dx \\ &\leqslant \frac{C^p 2^{p+1}}{\lambda} [pp'||w||_{A_1^+}]^p \int_{\mathbb{R}} f(x) w(x)\, dx. \end{aligned}$$

We take $p = 1 + \frac{1}{\log(e + ||w||_{A_1^+})}$ and observing that $t^{(\log(e+t))^{-1}}$ and $t^{t^{-1}}$ are bounded for $t > 1$ we have

$$[pp'||w||_{A_1^+}]^p \leqslant C \log(e + ||w||_{A_1^+}) ||w||_{A_1^+},$$

and as $1 < p < 2$ we obtain

$$III \leqslant \frac{C}{\lambda} \log(e + ||w||_{A_1^+}) ||w||_{A_1^+} \int_{\mathbb{R}} f(x) w(x)\, dx.$$

Combining this estimate with I and II completes the proof. □

REMARK 4.2. The choice of p, in proof of Theorem 1.2, is similar to the one given in [11]. The conjecture or goal was to find a linear dependence of the constant $||w||_{A_1}$. In [19] the authors proved that this is not possible. The nearest one is a $t \log t$

dependence of the constant, by the kind of steps followed to approach to the result. Maybe it can be shown that a $t\log^{\varepsilon} t$, dependence is possible, for some $0<\varepsilon<1$. To prove this result, a better estimate in Theorem 4.1 it should be obtain. C. Pérez in [21] conjectures that a $t\log^{\varepsilon} t$ dependence is not possible.

4.2. Proof of the Corollaries

To prove the Corollary 1.3, we need to build A_1^+ weights with special control on the constant, based in the Rubio de Francia algorithm. In order to do this we need the one-sided version of the Buckley result sharp estimate in norm L^p of the Hardy-Littlewood maximal operator M respect to weights $w\in A_p$. This Theorem was proved by F. J. Martín-Reyes and A. de la Torre in [17].

THEOREM 4.3. [17] *If $w\in A_p^-$ then*

$$||M^-||_{L^p(w)} \leqslant Cp'2^{p'}||w||_{A_p^-}^{\frac{1}{p-1}}. \tag{4.4}$$

The equivalent of the following Lemma for weights in A_p, is proven in [4].

LEMMA 4.4. *Let $1<q<\infty$ and let $w\in A_q^+$. Then there exists a nonnegative sublinear operator D bounded on $L^{q'}$ such that for any nonnegative $h\in L^{q'}(w)$:*

1. *$h\leqslant D(h)$;*
2. *$||D(h)||_{L^{q'}(w)}\leqslant 2||h||_{L^{q'}(w)}$;*
3. *$D(h).w\in A_1^+$ with $||D(h).w||_{A_1^+}\leqslant Cq2^q||w||_{A_q^+}$,*

where the constant C not depend on $||w||_{A_1^+}$ and q.

Proof. We define the operator $S(h)=w^{-1}M^-(|h|w)$, then S is bounded in $L^{q'}(w)$, moreover, $||S||_{L^{q'}(w)}\leqslant Cq2^q||w||_{A_q^+}$, indeed using equation (4.4) we get,

$$||Sh||_{L^{q'}(w)} = \left(\int_{\mathbb{R}} (w^{-1}M^-(|h|w))^{q'} w\,dx\right)^{\frac{1}{q'}} = \left(\int_{\mathbb{R}} (M^-(|h|w))^{q'} w^{1-q'}\,dx\right)^{\frac{1}{q'}}$$
$$\leqslant ||M^-||_{L^{q'}(w^{1-q'})}|||h|w||_{L^{q'}(w^{1-q'})} \leqslant Cq2^q||w^{1-q'}||_{A_{q'}^-}^{\frac{1}{q'-1}}||h||_{L^{q'}(w)}.$$

Recalling that $w\in A_q^+$ implies $w^{1-q'}\in A_{q'}^-$ and that $||w^{1-q'}||_{A_{q'}^-}=||w||_{A_q^+}^{\frac{1}{q-1}}$, we get $||S||_{L^{q'}(w)}\leqslant Cq2^q||w||_{A_q^+}$, as claimed.

Now we define the operator D via the following convergent Neumann series:

$$D(h)=\sum_{k=0}^{\infty}\frac{S^k(h)}{2^k||S||^k}, \qquad \text{where} \quad ||S||=||S||_{L^{q'}(w)}.$$

Then (1) and (2) are clearly satisfied.

(3) It follows from the definition of D and the sublinearity of S that

$$S(D(h)) \leqslant 2||S||(D(h)-h) \leqslant 2||S||D(h),$$

therefore

$$\begin{aligned} M^-(D(h)w) = M^-(D(h)w)w^{-1}w = S(D(h))w \leqslant 2||S||D(h)w \\ \leqslant cq2^q||w||_{A_q^+}D(h)w. \quad \square \end{aligned}$$

Proof of Corollary 1.3. For $\alpha > 0$ we set $\Omega_\alpha = \{x \in \mathbb{R} : |T^+ f(x)| > \alpha\}$ and let $\varphi(t) = t\log(e+t)$. Applying Lemma 4.4 with $q = p$, we get a sublinear operator D bounded on $L^{p'}$ satisfying properties (1), (2), and (3). Using these properties and Theorem 1.2, we obtain

$$\begin{aligned} \int_{\Omega_\alpha} hw\,dx &\leqslant \int_{\Omega_\alpha} D(h)w\,dx \leqslant \frac{C}{\alpha}\varphi(||D(h).w||_{A_1^+})||f||_{L^1(D(h).w)} \\ &\leqslant \frac{C}{\alpha}\varphi(Cp2^p||w||_{A_p^+})\int_{\mathbb{R}}|f|D(h)w\,dx \\ &\leqslant \frac{C}{\alpha}2\varphi(Cp2^p)\varphi(||w||_{A_p^+})\left(\int_{\mathbb{R}}|f|^p w\,dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}}D(h)^{p'}w\,dx\right)^{\frac{1}{p'}} \\ &\leqslant \frac{C}{\alpha}\varphi(||w||_{A_p^+})||f||_{L^p(w)}||h||_{L^{p'}(w)}. \end{aligned}$$

The proof is completed by taking the supremum over all h with $||h||_{L^{p'}(w)} = 1$. $\square$

Proof of Corollary 1.4. Given a one-sided singular operator T^-, its adjoint operator is T^+. Let $w \in A_p^-$ then $\sigma = w^{1-p'} \in A_{p'}^+$ with $||\sigma||_{A_{p'}^+} = ||w||_{A_p^-}^{\frac{1}{p-1}}$. Applying Corollary 1.3 to the one-sided singular operator T^+ and the weight σ we get

$$\begin{aligned} ||T^+||_{L^{p',\infty}(\sigma)} &\leqslant C||w||_{A_p^-}^{\frac{1}{p-1}}\log\left(e+||w||_{A_p^-}^{\frac{1}{p-1}}\right)||f||_{L^{p'}(\sigma)} \\ &\leqslant C||w||_{A_p^-}^{\frac{1}{p-1}}\log(e+||w||_{A_p^-})||f||_{L^{p'}(\sigma)}. \end{aligned}$$

From this, by duality we obtain

$$||T^-||_{L^p(w)} \leqslant C||w||_{A_p^-}^{\frac{1}{p-1}}\log(e+||w||_{A_p^-})\left|\left|\frac{f}{\sigma}\right|\right|_{L^{p,1}(\sigma)},$$

where $L^{p,1}(\sigma)$ is the standard weighted Lorentz space. Setting here $f = \sigma\chi_E$, where E is any measurable set, completes the proof. $\square$

5. Appendix

We will give an easier proof of a slight weak version of Lemma 3.2.

In [24] M. S. Riveros and A. de la Torre, using Lemma 5 in [13], obtained another version of weak reverse Hölder's inequality. If we use the equation (3.2), following the same steps of the proof in [24], we obtain a new result with special control on the constant.

LEMMA 5.1. [24] (One-sided RHI) *Let* $1 \leqslant p < \infty$, $w \in A_p^+$ *and* $a < b < c$ *with* $b-a = 2(c-b)$. *If* $r = 1 + \frac{1}{4^{p+2} e^{\frac{1}{e}} ||w||_{A_p^+}}$, *for* $p > 1$ *and* $r = 1 + \frac{1}{16 e^{\frac{1}{e}} ||w||_{A_1^+}}$, *for* $p = 1$ *then*

$$\frac{1}{b-a}\int_a^b w^r \leqslant C\left(\frac{1}{c-a}\int_a^c w\right)^r,$$

where C *does not depends on the weight* w.

The following Lemma is a slight weak version of Lemma 3.2.

LEMMA 5.2. *Let* $p \geqslant 1$, $w \in A_p^-$, $a < b < c$ *such that* $2(b-a) = (c-b)$ *and* $E \subseteq (b,c)$ *a measurable set. Then for every* $\varepsilon > 0$ *there exists* $C = C(\varepsilon, p)$ *such that if* $|E| < e^{-C||w||_{A_p^-}}(b-a)$ *then* $w(E) < \varepsilon w(a,c)$.

Proof. We will use the analogous to Lemma 5.1 for A_p^- weights.

$$\begin{aligned} w(E) &= \frac{1}{c-b}\int_b^c w\chi_E\,(c-b) \leqslant (c-b)\left(\frac{1}{c-b}\int_b^c w^r\right)^{\frac{1}{r}}\left(\frac{1}{c-b}\int_b^c \chi_E^{r'}\right)^{\frac{1}{r'}} \\ &= \left(\frac{|E|}{c-b}\right)^{\frac{1}{r'}}(c-b)C\frac{1}{c-a}\int_a^c w \leqslant \left(\frac{|E|}{b-a}\right)^{\frac{1}{r'}} C\int_a^c w \leqslant \varepsilon w(a,c), \end{aligned}$$

where the last inequality is obtained by following the same steps as in (3.4). □

As a Corollary of Lemma 5.1 we obtain another proof of Proposition 3 in [13], this is

COROLLARY 5.3. *Let* $1 < p < \infty$ *and* $w \in A_p^+$. *Then* $w \in A_{p-\varepsilon}^+$, *with* $p - \varepsilon = \frac{p-1}{r(\sigma)} + 1$ *where* $\sigma = w^{1-p'}$ *and* $r(\sigma)$ *is the one obtained in the analogous version of Lemma 5.1 for a weight in* $A_{p'}^-$.

Proof. In [24] it is proved that $w \in A_p^+$ if, and only if there exists $C > 0$ such that

$$\sup_{a,b,c,d} \frac{1}{(b-a)^p}\left(\int_a^b w\right)\left(\int_c^d w^{\frac{-1}{p-1}}\right)^{p-1} < C. \tag{5.1}$$

where the supremum is taken over all a,b,c,d such that $a<b<c<d$ and $2(b-a)=2(d-c)=c-b$.

Let $r=r(\sigma)$ be the one of Lemma 5.1 and a,b,c,d as in the previous line, then

$$\left(\frac{1}{b-a}\int_a^b w\right)\left(\frac{1}{d-c}\int_c^d w^{\frac{-1}{p-\varepsilon-1}}\right)^{p-\varepsilon-1} \leqslant \left(\frac{1}{b-a}\int_a^b w\right)\left(\frac{1}{d-c}\int_c^d \sigma^r\right)^{\frac{p-1}{r}}$$
$$\leqslant \left(\frac{1}{b-a}\int_a^b w\right)\left(\frac{1}{d-b}C\int_b^d \sigma\right)^{p-1}$$
$$\leqslant (C)^{p-1}\,||w||_{A_p^+},$$

where C does not depend on p nor w. □

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