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LOGARITHMIC COMPLEMENTARY MEANS AND AN EXTENSION OF CARLSON'S LOG

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Abstract. The invariance equality $L\circ(\mathscr{M},\mathscr{N})=L$, where L is the logarithmic mean, and where the unsymmetric compound means $\mathscr{M}=A\circ(P_1,G)$, $\mathscr{N}=A\circ(P_2,G)$ are built with the arithmetic A, geometric G, and projective means P_1, P_2, is called *"Carlson's log"* and is important in iteration of means. In the present paper we present effective and simple 1-parameter families of unsymmetric means $M_t,N_t:(0,\infty)^2\to(0,\infty)$ such that, for all $t\in(-1,1)$,

$$L\circ(M_t,N_t)=L$$

and

$$\mathscr{M}=M_{\frac{1}{2}},\qquad \mathscr{N}=N_{\frac{1}{2}}.$$

Existence of elementary (simple) symmetric means M and N such that $L\circ(M,N)=L$ and $M\neq L$ is posed as an open problem.

1. Introduction

The logarithmic mean L, playing an important role in applications ([6, 7], see also [5, 1]), is *invariant* with respect to the mean-type mapping $(\mathscr{M},\mathscr{N})$, where

$$\mathscr{M}=A\circ(P_1,G),\qquad \mathscr{N}=A\circ(P_2,G);$$

here A is the arithmetic, G is the geometric mean, and P_1,P_2 denote the projective means; more precisely, we have

$$L\circ(\mathscr{M},\mathscr{N})=L.$$

This equality, referenced in [2] p. 248 as *"Carlson's log"* ([4]), meaning that L is the *Gaussian product* of the means $\mathscr{M}$ and $\mathscr{N}$ (see [3] p. 414), or, equivalently, that $\mathscr{M}$ and $\mathscr{N}$ are mutually *complementary with respect to* L ([9]), implies that the sequence $(\mathscr{M},\mathscr{N})^n$, $n\in\mathbb{N}$, of iterates of $(\mathscr{M},\mathscr{N})$ converges pointwise to the mean-type mapping (L,L).

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In the present paper we significantly extend this result. In particular we show that the 1-parameter families of means $M_t, N_t : (0,\infty)^2 \to (0,\infty)$, with $t \in (-1,1) \setminus \{0\}$, given by

$$M_t(x,y) = tx^t \frac{x-y}{x^t - y^t}, \quad N_t(x,y) = ty^t \frac{x-y}{x^t - y^t}, \qquad x \neq y,$$

satisfy the equalities

$$L \circ (M_t, N_t) = L$$

and

$$M_t = N_{-t}$$

for all $t \in (-1,1) \setminus \{0\}$. In particular,

$$(\mathscr{M}, \mathscr{N}) = \left(M_{\frac{1}{2}}, N_{\frac{1}{2}}\right) = \left(N_{-\frac{1}{2}}, M_{-\frac{1}{2}}\right),$$

that is for $t = \frac{1}{2}$ (and $t = -\frac{1}{2}$) we get the *Carlson log* equality (Theorem 2).

All means $M_t, N_t,\ t \in (-1,1) \setminus \{0\}$ are unsymmetric. Since

$$\lim_{t\to 0} M_t = L = \lim_{t\to 0} N_t$$

and

$$\lim_{t\to 1} M_t = P_1 = \lim_{t\to -1} N_t, \qquad \lim_{t\to -1} M_t = P_2 = \lim_{t\to 1} N_t,$$

we can extend the families of means $M_t, N_t,\ t \in (-1,1) \setminus \{0\}$ also for $t = 0$, $t = 1$ and $t = -1$. Thus, even in the extended families, there is no nontrivial pair (M_t, N_t) of different symmetric means for $t \in [-1,1]$.

In section 3 we note that if K is a symmetric, continuous and strictly increasing mean, then for every continuous mean M there exists a unique mean N such that K is (M,N)-invariant; moreover, if M is symmetric, then so is N. The means M and N are referred to as (mutually) complementary (or conjugate) with respect to K.

Thus there are a lot of nontrivial symmetric, complementary with respect to L, pairs of means. Since we do not know any simple (elementary) pair of symmetric different means M,N such that L is (M,N)-invariant, we end up with posing an open problem.

2. Means, complementary means, and iterates

Let $I \subset \mathbb{R}$ be an interval. A function $M : I^2 \to I$ is said to be a mean in I if, for all $x, y \in I$,

$$\min(x,y) \leqslant M(x,y) \leqslant \max(x,y).$$

A mean M is called *strict* if for $x \neq y$ these inequalities are sharp, and *symmetric* if $M(x,y) = M(y,x)$ for all $x, y \in I$. A mean M is called *homogeneous,* if $M(tx,ty) = tM(x,y)$ for all $x, y \in I$ and $t > 0$ such that $tx, ty \in I$. Every mean is a *reflexive function*, i.e. $M(x,x) = x$ for all $x \in I$.

REMARK 1. If a function $M : I^2 \to \mathbb{R}$ is (strictly) increasing with respect to each of the variables and reflexive, then it is a (strict) mean, and it is referred to as a (strictly) increasing mean. Moreover, a strictly increasing mean is a strict mean.

Note that all the quasi-arithmetic means, the Lagrangean and Cauchy means are strictly increasing.

DEFINITION 1. Let $K, M, N : I^2 \to I$ be means. The mean K is called *invariant* with respect to the mean-type mapping (M,N), briefly (M,N)-invariant, if $K \circ (M,N) = K$, that is, if

$$K(M(x,y), N(x,y)) = K(x,y), \qquad x, y \in I.$$

If K is (M,N)-invariant, then N is called complementary to M with respect to K, briefly, K-complementary to M (or contra-M-mean with respect to K).

Of course, N is K-complementary to M if, and only if, M is K-complementary to N. Thus the operation of taking the complementary mean is an involution ([9]).

Let us quote the following

THEOREM 1. *([11]) If $M, N : I^2 \to I$ are continuous means and*

$$\max(M(x,y), N(x,y)) - \min(M(x,y), N(x,y)) < \max(x,y) - \min(x,y),$$

for all $x, y \in I$ such that $x \neq y$, then there exists a unique mean K that is (M,N)-invariant; moreover the sequence of iterates of the mean-type mapping $(M,N) : I^2 \to I^2$ converges pointwise to the mean-type mapping (K,K).

(In [2], [8], [10] slightly stronger conditions are assumed).

Let us also quote the following

REMARK 2. ([9], Remark 1) Let $I \subset \mathbb{R}$ be an interval. Suppose that $K : I^2 \to I$ is a continuous, symmetric and strictly increasing mean. Then for every continuous mean $M : I^2 \to I$ there exists a unique mean $N : I^2 \to I$ such that K is (M,N)-invariant. Moreover, if M is symmetric, then so is N.

As an illustration consider some examples.

EXAMPLE 1. Let $A, G, H : (0,\infty)^2 \to (0,\infty)$ denote, respectively, the arithmetic, geometric and harmonic means:

$$A(x,y) = \frac{x+y}{2}, \quad G(x,y) = \sqrt{xy}, \quad H(x,y) = \frac{2xy}{x+y}.$$

Then $G \circ (A,H) = G$ (Pythagorean harmony proportion). So H is a complementary (or contra) mean to A with respect to G (as well as A is complementary to H with respect to G). In view of Theorem 1, we have

$$\lim_{n\to\infty} (A,H)^n = (G,G) \quad \text{pointwise in } (0,\infty)^2,$$

(here $(A,H)^n$ is the nth iterate of the mean-type mapping (A,H)).

EXAMPLE 2. The arithmetic mean A is invariant with respect to the mean-type mapping (H,N), where H is the harmonic mean and $N:(0,\infty)^2 \to (0,\infty)$ is given by

$$N(x,y) = \frac{x^2+y^2}{x+y},$$

that is we have $A \circ (H,N) = A$. Thus N is a complementary mean to H with respect to A. Traditionally N is referred to as the contra-harmonic mean (due to the relationship $N = 2A - H$). In view of Theorem 1, we have

$$\lim_{n\to\infty} (H,N)^n = (A,A) \quad \text{pointwise in} \quad (0,\infty)^2.$$

REMARK 3. Arithmetic mean A and harmonic mean H are increasing, while the contra-harmonic is not. So this example shows that the property "being increasing mean" is not inherited by the complementary mean.

3. An extension of *Carlson's log*

The logarithmic mean $L:(0,\infty)^2 \to (0,\infty)$,

$$L(x,y) := \begin{cases} \frac{x-y}{\log x - \log y} & \text{if } x \neq y \\ x & \text{if } x = y \end{cases},$$

is a homogeneous Lagrange mean. So it is continuous, symmetric and strictly increasing, but not quasi-arithmetic ([5]).

Denote by P_1, P_2 the projective means in $\mathbb{R}^2$:

$$P_1(x,y) = x, \qquad P_2(x,y) = y.$$

Defining the means $\mathscr{M}, \mathscr{N} : (0,\infty)^2 \to (0,\infty)$ as in the Introduction,

$$\mathscr{M} := A \circ (P_1, G), \qquad \mathscr{N} := A \circ (P_2, G),$$

where $A(x,y) = \frac{x+y}{2}$, $G(x,y) = \sqrt{xy}$, we have

$$\mathscr{M}(x,y) = \frac{x+\sqrt{xy}}{2}, \qquad \mathscr{N}(x,y) = \frac{y+\sqrt{xy}}{2}, \qquad x,y > 0.$$

Of course, $\mathscr{M}$ and $\mathscr{N}$ are strictly increasing, homogeneous and unsymmetric, and it is easy to check that

$$L \circ (\mathscr{M}, \mathscr{N}) = L,$$

i.e. the *Carlson log* holds true.

The main result of this paper reads as follows.

THEOREM 2. *For every $t \in (-1,1)$ define the functions $M_t : (0,\infty)^2 \to (0,\infty)$ and $N_t : (0,\infty)^2 \to (0,\infty)$ by*

$$M_t(x,y) := \begin{cases} tx^t \frac{x-y}{x^t - y^t} & \text{if } t \in (-1,1) \setminus \{0\} \\ L(x,y) & \text{if } \quad t = 0 \end{cases} \quad \text{for } x \neq y; \qquad M_t(x,x) := x;$$

and

$$N_t(x,y) := \begin{cases} ty^t \frac{x-y}{x^t-y^t} & \text{if } t \in (-1,1)\setminus\{0\} \\ L(x,y) & \text{if } \quad t = 0 \end{cases} \quad \text{for } x \neq y; \qquad N_t(x,x) := x.$$

Then

(i) for every $t \in (-1,1)$ the functions M_t and N_t are unsymmetric homogeneous strict means;

(ii) for every $t \in (-1,1)$ the means M_t and N_t are strictly increasing with respect to each variable;

(iii) for all $x,y \in (0,\infty)$, the functions

$$(-1,1) \ni t \mapsto M_t(x,y), \qquad (-1,1) \ni t \mapsto N_t(x,y)$$

are continuous;

(iv) for all $x,y \in (0,\infty)$,

$$\lim_{t\to 1} M_t(x,y) = P_1(x,y) = \lim_{t\to -1} N_t(x,y), \qquad \lim_{t\to -1} M_t(x,y) = P_2(x,y) = \lim_{t\to 1} N_t(x,y),$$

where P_1 and P_2 are the (non strict) projective means defined by

$$P_1(x,y) := x, \qquad P_2(x,y) := y;$$

(v) for every $t \in (-1,1)$ the logarithmic mean L is invariant with respect to the mean-type mapping $(M_t, N_t) : (0,\infty)^2 \to (0,\infty)^2$:

$$L \circ (M_t, N_t) = L,$$

that is the means M_t and N_t are conjugate with respect to the logarithmic mean L;

(vi) for every $t \in (-1,1)$ the sequence $((M_t,N_t)^n : n \in \mathbb{N})$ of the iterates of the mean-type mapping (M_t,N_t) is pointwise convergent to the mean-type mapping (L,L) in $(0,\infty)^2$, that is, for every $(x,y) \in (0,\infty)^2$,

$$\lim_{n\to\infty} (M_t,N_t)^n(x,y) = (L(x,y), L(x,y)).$$

Proof. Clearly, the functions M_t and N_t are unsymmetrical and homogeneous. Denote by $E = \{E(s,r) : s,r \in \mathbb{R}\}$ the family of Stolarsky means and, by G_t, the geometric mean of the weight t. Since for $t \in (0,1)$ and $x,y \in (0,\infty)$, $x \neq y$,

$$t\frac{x-y}{x^t-y^t} = [E(t,1)(x,y)]^{1-t},$$

we have

$$tx^t \frac{x-y}{x^t-y^t} = x^t [E(t,1)(x,y)]^{1-t}.$$

that is

$$M_t = G_t \circ (P_1, E(t,1)). \tag{1}$$

Similarly, for all $t \in (0,1)$, we get

$$N_t = G_t \circ (P_2, E(t,1)). \tag{2}$$

Identities (1) and (2) imply that M_t and N_t are strict means for every $t \in (0,1)$. Hence, taking into account the obvious identities

$$M_t = N_{-t} \quad \text{and} \quad N_t = M_{-t}, \tag{3}$$

we conclude that M_t and N_t are strict means for every $t \in (-1,0)$.

Since, by the definition, $M_0 = N_0 = L = E(0,1)$, the proof of (i) is completed.

Result (ii) follows from (1), (2) and the fact that the Stolarsky means are increasing [12].

Result (iii) follows from (1), (2) and the continuity of the function

$$\mathbb{R}^2 \ni (s,r) \longmapsto E(s,r).$$

Part (iv) is obvious.

To verify (v) note that, by the definitions of L, M_t, N_t, for all $t \in (-1,1)$ and $x,y > 0$, $x \neq y$, we have

$$\begin{aligned} L(M_t(x,y), N_t(x,y)) &= \frac{tx^t \frac{x-y}{x^t-y^t} - ty^t \frac{x-y}{x^t-y^t}}{\log(x^t \frac{x-y}{x^t-y^t}) - \log(y^t \frac{x-y}{x^t-y^t})} = \frac{t \frac{x-y}{x^t-y^t}(x^t - y^t)}{t(\log x - \log y)} \\ &= \frac{x-y}{\log x - \log y} = L(x,y), \end{aligned}$$

so L is invariant with respect to the mean-type mapping (M_t, N_t).

Result (vi) is a consequence of Theorem 1. □

REMARK 4. Clearly, the geometric mean G is invariant with respect to the mean-type mapping (L, L^*) where

$$L^*(x,y) = \begin{cases} xy\frac{\log x - \log y}{x-y} & \text{if } x \neq y \\ x & \text{if } x = y \end{cases},$$

whence, applying the above Theorem 2, we conclude that L^* is invariant with respect to the mean-type mapping (M^*, N^*) where

$$M_t^*(x,y) := \begin{cases} \frac{xy^{1-t}}{t}\frac{x^t-y^t}{x-y} & \text{if } t \in (-1,1)\setminus\{0\} \\ L^*(x,y) & \text{if } \quad t = 0 \end{cases} \quad \text{for } x \neq y;$$

$$N_t^*(x,y) := \begin{cases} \frac{x^{1-t}y}{t}\frac{x^t-y^t}{x-y} & \text{if } t \in (-1,1)\setminus\{0\} \\ L^*(x,y) & \text{if } \quad t = 0 \end{cases} \quad \text{for } x \neq y.$$

4. Symmetric L-complementary means and an open problem

Since L is a symmetric, continuous and strictly increasing mean, applying Remark 2, we obtain the following corollaries:

COROLLARY 1. *There exists a unique function mean* $A_L^* : (0,\infty)^2 \to (0,\infty)$, *complementary to the arithmetic mean A with respect to the logarithmic mean L, that is such that L is (A, A_L^*)-invariant. Moreover A_L^* is symmetric, continuous and homogeneous.*

COROLLARY 2. *There exists a unique function mean* $G_L^* : (0,\infty)^2 \to (0,\infty)$, *complementary to the geometric mean G with respect to the logarithmic mean L, that is such that L is (G, G_L^*)-invariant. Moreover G_L^* is symmetric, continuous and homogeneous.*

COROLLARY 3. *There exists a unique function mean* $H_L^* : (0,\infty)^2 \to (0,\infty)$, *complementary to the harmonic mean H with respect to the logarithmic mean L, that is such that L is (H, H_L^*)-invariant. Moreover H_L^* is symmetric, continuous and homogeneous.*

One could formulate many corollaries of that type to get the existence of symmetric means M_L^* satisfying the invariance equality

$$L \circ (M, M_L^*) = L.$$

Note that in Theorem 2, for every $t \in [0,1] \setminus \{0\}$, the means M_t and N_t are unsymmetric. Since we do not know the form of an effective (elementary) nontrivial pair of mutually L-complementary *symmetric* means, we pose the following

PROBLEM 1. Find two symmetric means M and N, both elementary, such that $L \circ (M, N) = L$ and $M \neq L$.

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