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ASYMPTOTICS OF AN EULER–LAGRANGE EQUATION ASSOCIATED WITH EXTREMAL FUNCTIONS OF THE HARDY–SOBOLEV INEQUALITY

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Abstract. In this paper, we are concerned with the Hardy type integral equation

$$u(x)=\int_{R^n}\frac{u^p(y)dy}{|y|^l|x-y|^{n-\alpha}},\quad u>0\quad\text{in}\quad R^n,$$

where $n\geqslant 1$, $\alpha\in(0,n)$, $p>1$ and $l\geqslant\alpha$. Such an equation is related to the extremal functions of the Hardy-Sobolev inequality. If $l>\alpha$, then there is not any bounded positive solution for such an integral equation. If $l=\alpha$, the fast and the slow decay rates are discussed according to whether the nonnegative solutions possess some integrability.

1. Introduction

Recently, the following integral equation

$$u(x)=\int_{R^n}\frac{u^p(y)dy}{|x-y|^{n-\alpha}|y|^l},\quad u>0\quad\text{in}\quad R^n \tag{1.1}$$

was studied by many authors. Here $n\geqslant 1$, $p>1$, $0<\alpha<n$ and $l\in R$. When $l<0$, (1.1) is related to the *Hénon* model in the study of spherically symmetric clusters of stars (cf. [1], [2] and the references therein). When $l=0$, (1.1) is associated with the extremal functions of the Hardy-Littlewood-Sobolev inequality (cf. [3], [5], [8] and [10]). When $l\in(0,\alpha)$, it is related to the extremal functions of the Hardy-Sobolev inequality. For example, the Hardy-Sobolev inequality with $\alpha=2$ states that

$$\Lambda(\int_{R^n}|u|^{q+1}|x|^a dx)^{\frac{1}{q+1}}\leqslant(\int_{R^n}|\nabla u|^2dx)^{1/2},\quad\forall u\in\mathscr{D}^{1,2}(R^n), \tag{1.2}$$

where $0\leqslant -a<2<q+1=\frac{2(n+a)}{n-2}$, and $\mathscr{D}^{1,2}(R^n)$ is the homogeneous Sobolev space. Applying the symmetrization and the theories of ODE, one can obtain the sharp constant Λ (cf. [12]). To find the corresponding extremal function, we consider the minimization problem

$$\Lambda=\inf\{\|\nabla u\|_2^2;u\in\mathscr{D}^{1,2}(R^n),\int_{R^N}|u|^{q+1}|x|^a dx=1\}.$$

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To describe the shape of the extremal functions, Lu and Zhu [11] considered the Euler-Lagrange equation (1.1) and obtained that the positive solutions are radially symmetric and decreasing about the origin. Afterwards, [7] shows that (1.1) does not possess any positive solution as long as $p \leqslant \frac{n-l}{n-\alpha}$. When $p > \frac{n-l}{n-\alpha}$, $u(x)$ converges to zero either fast with $u(x) \simeq |x|^{\alpha-n}$ or slowly with $u(x) \simeq |x|^{-\frac{\alpha-l}{p-1}}$. Here $f(x) \simeq g(x)$ means that there exists $C > 0$ such that $C^{-1}g(x) \leqslant f(x) \leqslant Cg(x)$ when $|x| \to \infty$.

In this paper, we are concerned with the case of $l \geqslant \alpha$. When $\alpha = 2$, according to the properties of the Newton potential, (1.1) becomes the PDE

$$-\Delta u(x) = |x|^{-l}u^p(x), \quad u > 0 \quad \text{in} \quad R^n \setminus \{0\}. \tag{1.3}$$

Such an equation is associated with the study of the conformal scalar curvature problem (cf. [9]). It is showed that (1.3) has no bounded decaying positive solution in the case $l > 2$. When $l = 2$, each radial solution of (1.3) decays either with the fast rate $|x|^{2-n}$, or with the slow rate $(\log|x|)^{\frac{1}{1-p}}$ when $|x| \to \infty$. We expect to extend those results to the more general case $\alpha \in (0,n)$.

THEOREM 1.1. *Let* $n \geqslant 1$, $\alpha \in (0,n)$, $p > \frac{n}{n-\alpha}$ *and* $l \geqslant \alpha$.

(I) In the case of $l > \alpha$, *(1.1) has no locally bounded positive solution on* R^n.

(II) In the case of $l = \alpha$, *assume* u *is a locally bounded positive solution of (1.1) on* $R^n \setminus \{0\}$.

– *(i) If* $u \in L^{\frac{n(p-1)}{\alpha-\beta}}(R^n)$ *for some* $\beta \in [0,\alpha)$, *then* $u(x) \simeq |x|^{\alpha-n}$ *when* $|x| \to \infty$.

– *(ii) If* $u(x) \simeq (\log|x|)^\theta$, *then* $\theta = -\frac{1}{p-1}$.

When α is an even integer $2m (m \geqslant 1)$, by the properties of the Riesz potentials, (1.1) becomes

$$(-\Delta)^m u = |x|^{-l}u^p, \quad u > 0 \quad \text{in} \quad R^n \setminus \{0\}.$$

Thus, Theorem 1.1 is still true for this $2m$-order PDE.

Finally, we present a version of the weighted Hardy-Littlewood-Sobolev (WHLS) inequality which we will use in this paper.

LEMMA 1. *Let* $p,q > 1$, $\alpha \in (0,n)$, $0 \leqslant \beta_1 + \beta_2 \leqslant \alpha$, *and* $f \in L^p(R^n)$. *Define*

$$I_\alpha f(x) := \frac{1}{|x|^{\beta_1}} \int_{R^n} \frac{f(y)dy}{|y|^{\beta_2}|x-y|^{n-\alpha}},$$

then $I_\alpha f \in L^q(R^n)$, *where* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha-\beta_1-\beta_2}{n}$ *and* $\frac{1}{q} - \frac{n-\alpha}{n} < \frac{\beta_1}{n} < \frac{1}{q}$. *Moreover, there holds*

$$\|I_\alpha f\|_{L^q(R^n)} \leqslant C\|f\|_{L^p(R^n)}.$$

2. Proof of Theorem 1.1

Proof of (I). Assume u is a locally bounded positive solution of (1.1) with $l > \alpha$. When $|x| < 1$ and $y \in B_{|x|/2}(x)$, we have $|y| \leqslant 3|x|/2$ and $u(y) \geqslant c > 0$. Therefore,

$$u(x) \geqslant \int_{B_{|x|/2}(x)} \frac{u^p(y)dy}{|y|^l|x-y|^{n-\alpha}} \geqslant \frac{c}{|x|^l}\int_{B_{|x|/2}(x)} \frac{dy}{|x-y|^{n-\alpha}} \geqslant \frac{c}{|x|^{l-\alpha}}.$$

Let $|x| \to 0$, we see that $u \to \infty$. It contradicts with the boundedness. Thus, (I) is proved.

Proof of (II)–(i). Hereafter, we consider the decay rates in the case $l = \alpha$. Now, (1.1) becomes

$$u(x) = \int_{R^n} \frac{u^p(y)dy}{|y|^{\alpha}|x-y|^{n-\alpha}}, \qquad u > 0 \text{ in } R^n. \tag{2.1}$$

We assume that u is a locally bounded solution of (2.1) on $R^n \setminus \{0\}$.

Result (II)–(i) is the corollary of the following theorem.

THEOREM 2.1. *For the above solution u, the following three statements are equivalent:*

(R1) $u \in L^{\frac{n(p-1)}{\alpha-\beta}}(R^n)$ for some $\beta \in [0,\alpha)$;
(R2) $u \in L^s(R^n)$ for any $s > \frac{n}{n-\alpha}$;
(R3) u is bounded, and $\lim_{|x|\to\infty} |x|^{n-\alpha}u(x) = B$, where $B := \int_{R^n} \frac{u^p(y)dy}{|y|^{\alpha}}$ is a positive constant.

Proof.

(R1)⇒(R2).
For $A > 0$, define

$$\begin{cases} u_A(x) = u(x), & \text{if } |x| > A, \\ u_A(x) = 0, & \text{if } |x| \leqslant A. \end{cases}$$

Let $s > \frac{n}{n-\alpha}$. For $f \in L^s(R^n)$, define

$$Tf(x) = \int_{R^n} \frac{u_A^{p-1}(y)f(y)dy}{|x-y|^{n-\alpha}|y|^{\alpha}}, \quad F(x) = \int_{R^n} \frac{(u-u_A)^p(y)dy}{|x-y|^{n-\alpha}|y|^{\alpha}}.$$

Clearly, u solves the operator equation

$$f = Tf + F.$$

Noting the definition of u_A, and applying the WHLS inequality introduced in Lemma 1, we get

$$\|Tf\|_s \leqslant C\||y|^{\beta-\alpha}u_A^{p-1}f\|_{\frac{ns}{n+(\alpha-\beta)s}} \leqslant CA^{\beta-\alpha}\|u_A\|^{p-1}_{\frac{n(p-1)}{\alpha-\beta}}\|f\|_s.$$

Here we also used the Hölder inequality $\|u^s f^t\|_r \leqslant \|u\|_p^s \|f\|_q^t$ with $\frac{1}{r} = \frac{s}{p} + \frac{t}{q}$. By $u \in L^{\frac{n(p-1)}{\alpha-\beta}}(R^n)$, if we take A suitably large, then $CA^{\beta-\alpha}\|u_A\|_{\frac{n(p-1)}{\alpha-\beta}}^{p-1} < 1$. Thus, T is a contraction map from $L^s(R^n)$ to itself as long as $s > \frac{n}{n-\alpha}$. Moreover, $p > \frac{n}{n-\alpha}$ implies $\frac{\alpha-\beta}{n(p-1)} \in (0, \frac{n-\alpha}{n})$. Thus, T is also a contraction map from $L^{n(p-1)/(\alpha-\beta)}(R^n)$ to itself.

Next, the WHLS inequality leads to

$$\|F\|_s \leqslant C\|u - u_A\|_{sp}^p.$$

By the definition of u_A, we have $F \in L^s(R^n)$ as long as $s > \frac{n}{n-\alpha}$.

Using the lifting lemma (Lemma 2.1 in [6]) on the regularity, we can obtain $u \in L^s(R^n)$ as long as $s > \frac{n}{n-\alpha}$.

REMARK. If $s \leqslant \frac{n}{n-\alpha}$, we claim $u \notin L^s(R^n)$. In fact, when $|x| > 2$,

$$u(x) \geqslant \int_{B_1(0)} \frac{u^p(y)dy}{|y|^\alpha |x-y|^{n-\alpha}} \geqslant \frac{c}{|x|^{n-\alpha}}.$$

Thus,

$$\int_{R^n} u^s(x)dx \geqslant c\int_2^\infty r^{n-s(n-\alpha)}\frac{dr}{r} = \infty.$$

(R2)⇒(R3).

Step 1. Clearly, $p > \frac{n}{n-\alpha}$ implies $\frac{n(p-1)}{\alpha} > \frac{n}{n-\alpha}$. According to Theorem 2 and Remark 2 in [4], (R2) shows that the positive solutions of (2.1) are radial symmetric and decreasing about the origin.

Step 2. We claim that u is bounded. Since u is monotone decreasing, for any $x \neq 0$,

$$\begin{aligned} u(x) &\geqslant \int_{B_{|x|/2}(x)\cap B_{|x|}(0)} \frac{u^p(y)dy}{|y|^\alpha |x-y|^{n-\alpha}} \\ &\geqslant \frac{cu^p(x)}{|x|^\alpha}\int_{B_{|x|/2}(x)\cap T}\frac{dy}{|x-y|^{n-\alpha}} \\ &\geqslant cu^p(x), \end{aligned}$$

where T is a cone with the corner angle $\pi/2$. This implies $u(x) \leqslant C$ for $x \neq 0$, where $C > 0$ is independent of x. Therefore, $\overline{\lim}_{|x|\to 0} u(x) \leqslant C$. Thus, we know that u is bounded.

Step 3. We claim that the improper integral $B := \int_{R^n} |y|^{-\alpha}u^p(y)dy < \infty$.

For $R > 0$, set

$$B_1 := \int_{B_R(0)} \frac{u^p(y)dy}{|y|^\alpha}, \quad B_2 := \int_{R^n\setminus B_R(0)} \frac{u^p(y)dy}{|y|^\alpha}.$$

By virtue of $u \in L^\infty(R^n)$, $B_1 < \infty$. Applying the Hölder inequality and taking $\frac{1}{t} = \frac{n-\alpha-\varepsilon}{n}p$ with $\varepsilon > 0$ sufficiently small, by (R2) we get

$$B_2 \leqslant C\|u\|_{tp}^p\left(\int_R^\infty r^{n-\frac{t\alpha}{t-1}}\frac{dr}{r}\right)^{1-1/t} < \infty.$$

Thus, $B < \infty$.

Step 4. For fixed $R > 0$, write

$$L_1 := \int_{B_R} \frac{u^p(y)}{|y|^\alpha}\Big(\frac{|x|^{n-\alpha}}{|x-y|^{n-\alpha}} - 1\Big)dy.$$

When $y \in B_R$ and $|x| \to \infty$,

$$\frac{u^p(y)}{|y|^\alpha}\Big|\frac{|x|^{n-\alpha}}{|x-y|^{n-\alpha}} - 1\Big| \leqslant 2\frac{u^p(y)}{|y|^\alpha}.$$

By virtue of Step 3, $\frac{u^p(y)}{|y|^\alpha} \in L^1(R^n)$. Using Lebesgue's dominated convergence theorem, we obtain $|L_1| \to 0$ as $|x| \to \infty$. This result leads to

$$\lim_{R\to\infty}\lim_{|x|\to\infty}\int_{B_R} \frac{u^p(y)}{|y|^\alpha}\frac{|x|^{n-\alpha}}{|x-y|^{n-\alpha}}dy = B.$$

Next, write

$$L_2 := \int_{(R^n\setminus B_R)\setminus B(x,|x|/2)} \frac{u^p(y)}{|y|^\alpha}\frac{|x|^{n-\alpha}}{|x-y|^{n-\alpha}}dy.$$

Clearly, $|x-y| \geqslant |x|/2$ when $y \in (R^n \setminus B_R)\setminus B(x,|x|/2)$. Therefore, when $R \to \infty$,

$$L_2 \leqslant C\int_{R^n\setminus B_R} \frac{u^p(y)}{|y|^\alpha}dy \to 0.$$

Finally, write

$$L_3 := \int_{B(x,|x|/2)} \frac{u^p(y)}{|y|^\alpha}\frac{|x|^{n-\alpha}}{|x-y|^{n-\alpha}}dy.$$

In view of Step 1, u is radially symmetric and decreasing about the origin. Therefore,

$$L_3 \leqslant C|x|^{n-2\alpha}u^p(x/2)\int_{B(x,|x|/2)} \frac{dy}{|x-y|^{n-\alpha}} \leqslant C|x|^{n-\alpha}u^p(x/2). \tag{2.2}$$

On the other hand, (R2) shows that $u \in L^s(R^n)$ with $\frac{1}{s} = \frac{n-\alpha-\varepsilon}{n}$. Here $\varepsilon > 0$ is sufficiently small. Noting the decreasing property of u, we obtain

$$u^s(x/2)|x|^n \leqslant C\int_{B(0,\frac{|x|}{2})\setminus B(0,\frac{|x|}{4})} u^s(y)dy \leqslant C.$$

This implies

$$u^p(x/2)|x|^{p(n-\alpha-\varepsilon)} \leqslant C.$$

Inserting this into (2.2), we get

$$L_3 \leqslant C|x|^{n-\alpha-p(n-\alpha-\varepsilon)} \to 0, \quad as \quad |x| \to \infty.$$

Combining all the estimates of L_1, L_2 and L_3 yields $\lim_{|x|\to\infty}|x|^{n-\alpha}u(x) = B$.

(R3)⇒(R1). By virtue of $p > \frac{n}{n-\alpha}$, it follows $n < \frac{n(p-1)}{\alpha-\beta}(n-\alpha)$. Therefore,

$$\int_{R^n} u^{\frac{n(p-1)}{\alpha-\beta}}(x)dx = \int_{B_R(0)} u^{\frac{n(p-1)}{\alpha-\beta}}(x)dx + \int_{R^n \setminus B_R(0)} u^{\frac{n(p-1)}{\alpha-\beta}}(x)dx$$
$$\leqslant C + C\int_R^\infty r^{n-\frac{n(p-1)}{\alpha-\beta}(n-\alpha)} \frac{dr}{r} < \infty.$$

Thus, the proof of Theorem 2.1 is complete, and hence (II)–(i) is proved. □

To prove (II)–(ii), we first give two results.

THEOREM 2.2. *Let u be a locally bounded solution of (2.1). If there exists $c > 0$ such that*

$$u(x) \geqslant c(\log|x|)^{\frac{1}{1-p}+\varepsilon} \tag{2.3}$$

with $\varepsilon \geqslant 0$ for large $|x|$, then $\varepsilon = 0$.

Proof. When $|y| > 2|x|$, we see $|x-y| \leqslant |x|+|y| \leqslant 3|y|/2$. Therefore,

$$\begin{aligned} u(x) &\geqslant \int_{R^n \setminus B_{2|x|}(0)} \frac{u^p(y)dy}{|y|^\alpha |x-y|^{n-\alpha}} \\ &\geqslant c\int_{R^n \setminus B_{2|x|}(0)} \frac{dy}{|y|^n (\log|y|)^{\frac{p}{p-1}-p\varepsilon}} \\ &= c\int_{2|x|}^\infty \frac{1}{(\log r)^{\frac{p}{p-1}-p\varepsilon}} \frac{dr}{r} \\ &= c\int_{2|x|}^\infty \frac{d\log r}{(\log r)^{\frac{p}{p-1}-p\varepsilon}} \\ &= \frac{c}{(\log|x|)^{\frac{1}{p-1}-p\varepsilon}}. \end{aligned}$$

Replacing (2.3) by this new estimate and repeating the process above, we get

$$u(x) \geqslant c(\log|x|)^{\frac{1}{1-p}+p^2\varepsilon}.$$

By induction, for any $j = 1, 2, \cdots,$ we have

$$u(x) \geqslant c(\log|x|)^{\frac{1}{1-p}+p^j\varepsilon}.$$

Suppose $\varepsilon > 0$. Letting $j \to \infty$, we see that $u(x)$ blows up. It contradicts the local boundedness and hence $\varepsilon = 0$. □

THEOREM 2.3. *Assume $u \in L^\infty_{loc}(R^n) \setminus L^{\frac{n(p-1)}{\alpha-\beta}}(R^n)$ for all $\beta \in [0,\alpha)$ solves of (2.1) with $p > \frac{n-\beta}{n-\alpha}$. If there exists $C > 0$ such that*

$$u(x) \leqslant C(\log|x|)^{\frac{1}{1-p}-\varepsilon} \tag{2.4}$$

with $\varepsilon \geqslant 0$ for large $|x|$, then $\varepsilon = 0$.

Proof. For large $|x|$, first we have

$$\int_{B_1(0)} \frac{u^p(y)dy}{|y|^\alpha |x-y|^{n-\alpha}} \leqslant \frac{C\|u\|_\infty^p}{|x|^{n-\alpha}} \int_0^1 r^{n-\alpha} \frac{dr}{r} \leqslant \frac{C}{|x|^{n-\alpha}}.$$

Second, there holds $|x|/2 \leqslant |y| \leqslant 3|x|/2$ as $y \in B_{|x|/2}(x)$. Thus,

$$\begin{aligned}&\int_{B_{|x|/2}(x)} \frac{u^p(y)dy}{|y|^\alpha |x-y|^{n-\alpha}} \\ &\leqslant C|x|^{-\alpha}(\log|x|)^{\frac{p}{1-p}-p\varepsilon} \int_{B_{|x|/2}(x)} \frac{dy}{|x-y|^{n-\alpha}} \\ &\leqslant \frac{C}{(\log|x|)^{\frac{p}{p-1}+p\varepsilon}}.\end{aligned}$$

Third, since $\mathscr{F}(r) = r^{n-\alpha}(\log r)^{\frac{p}{1-p}-p\varepsilon}$ is an increasing function as long as r is suitably large, we have

$$\begin{aligned}&\int_{(B_{2|x|}(0)\setminus B_1(0))\setminus B_{|x|/2}(x)} \frac{u^p(y)dy}{|y|^\alpha |x-y|^{n-\alpha}} \\ &\leqslant \frac{C}{|x|^{n-\alpha}} \int_{B_{2|x|}(0)\setminus B_1(0)} \frac{u^p(y)dy}{|y|^\alpha} \\ &\leqslant \frac{C}{|x|^{n-\alpha}} \int_1^{2|x|} \frac{r^{n-\alpha}}{(\log r)^{\frac{p}{p-1}+p\varepsilon}} \frac{dr}{r} \\ &\leqslant \frac{C}{(\log|x|)^{\frac{p}{p-1}+p\varepsilon}} \int_1^{2|x|} \frac{dr}{r} \\ &\leqslant \frac{C}{(\log|x|)^{\frac{1}{p-1}+p\varepsilon}}.\end{aligned}$$

Finally, $|y| \geqslant 2|x|$ implies $|x-y| \geqslant |y|-|x| \geqslant |y|/2$. Thus, similar to the calculation in the proof of Theorem 2.2,

$$\begin{aligned}\int_{R^n\setminus B_{2|x|}(0)} \frac{u^p(y)dy}{|y|^\alpha |x-y|^{n-\alpha}} &\leqslant \int_{R^n\setminus B_{2|x|}(0)} \frac{Cdy}{|y|^n(\log|y|)^{\frac{p}{p-1}+p\varepsilon}} \\ &= \frac{C}{(\log|x|)^{\frac{1}{p-1}+p\varepsilon}}.\end{aligned}$$

Combining the four results above together yields

$$u(x) \leqslant \frac{C}{|x|^{n-\alpha}} + \frac{C}{(\log|x|)^{\frac{1}{p-1}+p\varepsilon}}.$$

This implies $u(x) \leqslant C(\log|x|)^{\frac{1}{1-p}-p\varepsilon}$ for large $|x|$. Replacing (2.4) by this new estimate and repeating the process above, we get for any $j = 1,2,\cdots$,

$$u(x) \leqslant \frac{C}{|x|^{n-\alpha}} + \frac{C}{(\log|x|)^{\frac{1}{p-1}+p^j\varepsilon}}.$$

Suppose $\varepsilon > 0$. Letting $j \to \infty$, we see $u(x) \leqslant \frac{C}{|x|^{n-\alpha}}$. According to Theorem 2.1, it follows that $u \in L^{\frac{n(p-1)}{\alpha-\beta}}(R^n)$ for some $\beta \in [0,\alpha)$. It contradicts the condition of Theorem 2.3. Thus, $\varepsilon = 0$. $\square$

Proof of (II)–(ii). Since $u(x) \simeq (\log|x|)^\theta$ when $|x| \to \infty$, it is easy to verify $u \notin L^{\frac{n(p-1)}{\alpha-\beta}}(R^n)$ for all $\beta \in [0,\alpha)$. Combining Theorem 2.2 with Theorem 2.3, we can complete the proof of (II)–(ii). $\square$

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