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STEFFENSEN'S INEQUALITY FOR POSITIVE MEASURES

JULIJE JAKŠETIĆ AND JOSIP PEČARIĆ

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Abstract. We generalize Steffensen's inequality for positive measures. We obtain conditions for these inequalities which are invariant in form to Steffensen's inequality for absolutely continuous measures. Further, we produce linear functionals which generate exponential convexity and Cauchy means.

1. Introduction and preliminary results

In 1918 Steffensen proved the following inequality, [11]:

THEOREM 1. *Suppose that f is non-increasing and g is integrable on $[a,b]$ with $0 \leqslant g \leqslant 1$ and*

$$\lambda = \int_a^b g(t)dt. \tag{1}$$

Then we have

$$\int_{b-\lambda}^b f(t)dt \leqslant \int_a^b f(t)g(t)dt \leqslant \int_a^{a+\lambda} f(t)dt. \tag{2}$$

Using (2) we can define two linear functionals

$$L_1(f) = \int_{b-\lambda}^b f(t)dt - \int_a^b f(t)g(t)dt \tag{3}$$

and

$$L_2(f) = \int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt \tag{4}$$

that act on class of integrable functions. Steffensen's inequality tells us that $L_1(f) \geqslant 0$ and $L_2(f) \geqslant 0$ if f is non-decreasing function. In [6] it is showed how to produce Cauchy means using functionals L_1 and L_2 and, even more, how to construct nontrivial examples of exponentially convex functions.

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Our aim is to generalize Steffensen's inequality in measure theory settings, approach that is in already considered in papers [3], [4], [5] and book [9]. However, we will show that our approach to conditions under which these inequalities stand would give us certain invariance in form to basic Steffensen's inequality. Further, we define, similar to (3) and (4), linear functionals that produce exponential convexity and Cauchy means.

We now list definitions and properties of classes of functions that we will use in this paper.

DEFINITION 1. Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to (0,\infty)$ is log-convex in the Jensen sense if for all $x, y \in I$:

$$f^2\left(\frac{x+y}{2}\right) \leqslant f(x)f(y).$$

The following lemma, using quadratic form, gives a necessary and sufficient condition for the log-convexity of a function in the Jensen sense.

LEMMA 1. *Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to (0,\infty)$ is log-convex in the Jensen sense on I if for all $\xi_1,\ \xi_2 \in \mathbb{R}$ and all $x, y \in I$:*

$$\xi_1^2 f(x) + 2\xi_1\xi_2 f\left(\frac{x+y}{2}\right) + \xi_2^2 f(y) \geqslant 0.$$

DEFINITION 2. Let $I \subseteq \mathbb{R}$ be an open interval. A function $f : I \to (0,\infty)$ is log-convex on I if $\log f$ is convex on I, or equivalently for $x, y \in I$ and $\alpha \in [0,1]$

$$f(\alpha x + (1-\alpha)y) \leqslant f(x)^{\alpha} f(y)^{1-\alpha}.$$

The following subclass of convex functions is introduced by famous Bernstein in [2].

DEFINITION 3. A function $f : I \to \mathbb{R}$ is exponentially convex on I if it is continuous on I and

$$\sum_{i,j=1}^{n} \xi_i \xi_j f\left(\frac{x_i + x_j}{2}\right) \geqslant 0$$

for all $n \in \mathbb{N}$ and all choices $\xi_i, x_i \in \mathbb{R}$.

One of the most important properties of exponentially convex functions is their integral representation.

THEOREM 2. *The function $\psi : I \to \mathbb{R}$ is exponentially convex on I if and only if*

$$\psi(x) = \int_{-\infty}^{\infty} e^{tx} d\sigma(t), \quad x \in I$$

for some non-decreasing function $\sigma : \mathbb{R} \to \mathbb{R}$.

Proof. See [1, p. 211]. □

It is obvious that exponentially convex functions form convex cone. The product of finitely many exponentially convex functions on I is again exponentially convex function. Many other interesting properties and examples on exponential convexity can be found in [6].

The following families of functions will be useful in constructing exponentially convex functions.

LEMMA 2. *For $p \in \mathbb{R}$ let $\phi_p : (0,\infty) \to \mathbb{R}$ be defined with*

$$\phi_p(x) = \begin{cases} \frac{x^p}{p}, & p \neq 0; \\ \log x, & p = 0. \end{cases}$$

Then $x \mapsto \phi_p(x)$ is increasing on $\mathbb{R}$ for each $p \in \mathbb{R}$ and $p \mapsto \phi_p(x)$ is exponentially convex on $(0,\infty)$, for each $x \in (0,\infty)$.

Proof. First part: follows from $\frac{d}{dx}(\phi_p(x)) = x^{p-1} > 0$ on $(0,\infty)$, for each $p \in \mathbb{R}$. *Second part*: $p \mapsto \frac{x^p}{p} = e^{p\log x} \cdot \frac{1}{p}$. Since $p \mapsto e^{p\log x}$ and $p \mapsto \frac{1}{p}$ are exponentially convex functions(see [6]), according to the above comment, conclusion follows. □

LEMMA 3. *For $p \in \mathbb{R}$ let $\varphi_p : \mathbb{R} \to [0,\infty)$ be defined with*

$$\varphi_p(x) = \begin{cases} \frac{e^{px}}{p}, & p \neq 0; \\ x, & p = 0. \end{cases}$$

Then $x \mapsto \varphi_p(x)$ is increasing on $\mathbb{R}$ for each $p \in R$, and $p \mapsto \varphi_p(x)$ is exponentially convex on $(0,\infty)$, for each $x \in \mathbb{R}$.

Proof. First part: follows from $\frac{d}{dx}(\varphi_p(x)) = e^{px} > 0$ on $\mathbb{R}$, for each $p \in \mathbb{R}$. *Second part*: follows from the fact $\frac{e^{px}}{p} = e^{px} \cdot \frac{1}{p}$. □

The following proposition is obvious.

PROPOSITION 1. Let $I \subseteq \mathbb{R}$ be an open interval. If $f : I \to (0,\infty)$ is exponentially convex on I then it is log-convex function on I.

The converse of the Proposition 1 is not true, in general (see example in [6]).

Using characterization of convexity by monotonicity of 1st order divided differences it follows (see [8, p. 4]):

THEOREM 3. *Let $I \subseteq \mathbb{R}$ be an open interval. Let $f : I \to (0,\infty)$ be log-convex, differentiable function on I and $M : I \times I \to (0,\infty)$ be defined with*

$$M(x,y) = \begin{cases} \left(\frac{f(x)}{f(y)}\right)^{\frac{1}{x-y}}, & x \neq y; \\ \exp\left(\frac{f'(x)}{f(x)}\right), & x = y. \end{cases}$$

If $x_1, x_2, y_1, y_2 \in I$ *such that* $x_1 \leqslant x_2,\ y_1 \leqslant y_2$ *then*

$$M(x_1, y_1) \leqslant M(x_2, y_2).$$

2. Main results

Denote by $\mathscr{B}([a,b])$ Borel σ-algebra generated on $[a,b]$.

THEOREM 4. *Let* μ *be a positive, finite, measure on* $\mathscr{B}([a,b])$ *and let* f *and* g *be measurable functions such that* f *is non-increasing and* $0 \leqslant g \leqslant 1$. *If there exists* $\lambda \in \mathbb{R}_+$ *such that*

$$\mu((b-\lambda, b]) = \int\limits_{[a,b]} g(t)d\mu(t), \tag{5}$$

then

$$\int\limits_{(b-\lambda,b]} f(t)d\mu(t) \leqslant \int\limits_{[a,b]} f(t)g(t)d\mu(t). \tag{6}$$

Proof.

$$\begin{aligned}
&\int_{[a,b]} f(t)g(t)d\mu(t) - \int_{(b-\lambda,b]} f(t)d\mu(t) \\
&= \int_{[a,b-\lambda]} f(t)g(t)d\mu(t) - \int_{(b-\lambda,b]} f(t)(1-g(t))d\mu(t) \\
&\geqslant \int_{[a,b-\lambda]} f(t)g(t)d\mu(t) - f(b-\lambda)\int_{(b-\lambda,b]} (1-g(t))d\mu(t) \qquad (7) \\
&= \int_{[a,b-\lambda]} f(t)g(t)d\mu(t) - f(b-\lambda)\int_{[a,b-\lambda]} g(t)d\mu(t) \qquad (8) \\
&= \int_{[a,b-\lambda]} (f(t)-f(b-\lambda))g(t)d\mu(t) \geqslant 0. \quad \square
\end{aligned}$$

THEOREM 5. *Let* μ *be a positive, finite, measure on* $\mathscr{B}([a,b])$ *and let* f *and* g *be measurable functions such that* f *is non-increasing, non-negative function, and* $0 \leqslant g \leqslant 1$. *If there exists* $\lambda \in \mathbb{R}_+$ *that satisfies*

$$\mu((b, b-\lambda]) \leqslant \int\limits_{[a,b]} g(t)d\mu(t), \tag{9}$$

then (6) *holds.*

Proof. We re-adjust proof of Theorem 4: condition (9) together with $f(b-\lambda) > 0$ ensures us transition from line (7) to (8). $\square$

THEOREM 6. *Let μ be a positive, finite, measure on $\mathscr{B}([a,b])$ and let f and g be measurable functions such that f is non-increasing and $0 \leqslant g \leqslant 1$. If there exists $\lambda \in \mathbb{R}_+$ such that*

$$\mu([a,a+\lambda]) = \int_{[a,b]} g(t)d\mu(t), \tag{10}$$

then

$$\int_{[a,a+\lambda]} f(t)d\mu(t) \geqslant \int_{[a,b]} f(t)g(t)d\mu(t). \tag{11}$$

Proof.

$$\begin{aligned}
&\int_{[a,a+\lambda]} f(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \\
&= \int_{[a,a+\lambda]} f(t)(1-g(t))d\mu(t) - \int_{(a+\lambda,b]} f(t)g(t)d\mu(t) \\
&\geqslant f(a+\lambda)\int_{[a,a+\lambda]} (1-g(t))d\mu(t) - \int_{(a+\lambda,b]} f(t)g(t)d\mu(t) \\
&= f(a+\lambda)\int_{(a+\lambda,b]} g(t)d\mu(t) - \int_{(a+\lambda,b]} f(t)g(t)d\mu(t) \\
&= \int_{(a+\lambda,b]} (f(a+\lambda)-f(t))g(t)d\mu(t) \geqslant 0. \quad \square
\end{aligned}$$

THEOREM 7. *Let μ be a positive, finite, measure on $\mathscr{B}([a,b])$ and let f and g be measurable functions such that f is non-increasing, non-negative function, and $0 \leqslant g \leqslant 1$. If there exists $\lambda \in \mathbb{R}_+$ such that*

$$\mu([a,a+\lambda]) \geqslant \int_{[a,b]} g(t)d\mu(t), \tag{12}$$

then (11) *holds.*

Proof. Similar to proof of Theorem 5. $\square$

REMARK 1. Some comments on Theorems 4-7 are needed.

(i) If we consider the Lebesgue measure in conditions (5) and (10), we get standard constant λ in Steffensen's inequality given in (1).

(ii) If $\mu \ll \nu$ and $h = \frac{d\mu}{d\nu}$ then condition (9) becomes

$$\int_{(b-\lambda,b]} h(t)d\nu(t) \leqslant \int_{[a,b]} h(t)g(t)d\nu(t),$$

while condition (12) becomes

$$\int_{[a,a+\lambda]} h(t)d\nu(t) \geqslant \int_{[a,b]} h(t)g(t)d\nu(t).$$

Clearly, both conditions have the same form as Steffensen's inequality altough monotonicity request on the function h is dropped.

3. Applications

Similar to (3) and (4) we can produce two linear functionals from Theorems 4 and 6:

$$\mathfrak{L}_1(f) = \int_{(b-\lambda,b]} f(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \tag{13}$$

and

$$\mathfrak{L}_2(f) = \int_{[a,b]} f(t)g(t)d\mu(t) - \int_{[a,a+\lambda]} f(t)d\mu(t). \tag{14}$$

THEOREM 8. *Let* $f \mapsto \mathfrak{L}_i(f)$, $i = 1,2$, *be linear functionals defined with* (13) *and* (14) *and let* $F_i : (0,\infty) \to \mathbb{R}$, $i = 1,2$, *be defined with*

$$F_i(p) = \mathfrak{L}_i(\phi_p)$$

where ϕ_p *is defined in Lemma* 2. *Then the following statements hold for every* $i = 1,2$.

(i) The function F_i *is continuous on* $\mathbb{R}$.

(ii) If $n \in \mathbb{N}$ *and* $p_1,\ldots,p_n \in \mathbb{R}$ *are arbitrary, then the matrix*

$$\left[F_i\left(\frac{p_j+p_k}{2}\right)\right]_{j,k=1}^{n}$$

is positive semidefinite. Particularly,

$$det\left[F_i\left(\frac{p_j+p_k}{2}\right)\right]_{j,k=1}^{n} \geqslant 0.$$

(iii) The function F_i *is exponentially convex on* $\mathbb{R}$.

(iv) The function F_i *is log-convex on* $\mathbb{R}$.

(v) If $p,q,r \in \mathbb{R}$ *are such that* $p<q<r$, *then*

$$F_i(q)^{r-p} \leqslant F_i(p)^{r-q} F_i(r)^{q-p}. \tag{15}$$

Proof. (i) Continuity of the function $p \mapsto F_i(p)$ is obvious for $p \in \mathbb{R}\setminus\{0\}$. For $p = 0$ it is directly checked using Heine characterization.

(ii) Let $n \in \mathbb{N}$, $p_i \in \mathbb{R}$ $(i = 1,\ldots,n)$ be arbitrary and define auxiliary function $\psi : (0,\infty) \to \mathbb{R}$ by

$$\psi(x) = \sum_{j,k=1}^{n} \xi_j \xi_k \phi_{\frac{p_j+p_k}{2}}(x).$$

Now

$$\psi'(x) = \left(\sum_{j=1}^{n} \xi_j x^{\frac{p_j-1}{2}} \right)^2 \geqslant 0$$

implies that ψ is non-increasing function on $(0,\infty)$ and then

$$\mathfrak{L}_i(\psi) \geqslant 0, \; i = 1,2.$$

This means that the matrix

$$\left[F_i\left(\frac{p_i + p_j}{2} \right) \right]_{j,k=1}^{n}$$

is positive semi-definite.

(iii), (iv), (v) are simple consequences of (i) and (ii). □

With similar arguments we deduce the next theorem.

THEOREM 9. *Theorem 8 is still valid for φ_p given in Lemma 3.*

We now use mean value theorems to produce Cauchy means.

THEOREM 10. *Let $f \mapsto \mathfrak{L}_i(f)$, $i = 1,2$ be linear functionals defined with (13) and (14) and $\psi \in C^1[a,b]$. Then there exist $\xi_i \in [a,b]$, $i = 1,2$ such that*

$$\mathfrak{L}_i(\psi) = \psi'(\xi_i)\mathfrak{L}_i(id),$$

where $id(x) = x$.

Proof. Since $\psi \in C^1[a,b]$ there exist $m = \min_{x\in[a,b]} \psi'(x)$ and $M = \max_{x\in[a,b]} \psi'(x)$. Denote $h_1(x) = Mx - \psi(x)$ and $h_2(x) = \psi(x) - mx$. Then

$$h_1'(x) = M - \psi'(x) \geqslant 0$$
$$h_2'(x) = \psi'(x) - m \geqslant 0$$

which means that $\mathfrak{L}_i(h_1)$, $\mathfrak{L}_i(h_2) \geqslant 0$, $i = 1,2$ i.e.

$$m\mathfrak{L}_i(id) \leqslant \mathfrak{L}_i(\psi) \leqslant M\mathfrak{L}_i(id).$$

If $\mathfrak{L}_i(id) = 0$, the proof is complete. If $\mathfrak{L}_i(id) > 0$, then

$$m \leqslant \frac{\mathfrak{L}_i(\psi)}{\mathfrak{L}_i(id)} \leqslant M$$

and the existence of $\xi_i \in [a,b]$ follows. □

Using, standard, Cauchy type mean value theorem we get the next corollary.

COROLLARY 1. *Let $f \mapsto \mathfrak{L}_i(f)$, $i=1,2$ be linear functionals defined with (13) and (14) and $\psi_1,\ \psi_2 \in C^1[a,b]$ such that $\psi_2'(x)$ does not vanish for any value of $x \in [a,b]$, then there exist $\xi_i \in [a,b]$, $i=1,2$ such that*

$$\frac{\psi_1'(\xi_i)}{\psi_2'(\xi_i)} = \frac{\mathfrak{L}_i(\psi_1)}{\mathfrak{L}_i(\psi_2)}, \tag{16}$$

provided that the denominator on right side is non-zero.

REMARK 2. If the inverse of ψ_1'/ψ_2' exists then various kinds of means can be defined by (16). That is

$$\xi_i = \left(\frac{\psi_1'}{\psi_2'}\right)^{-1}\left(\frac{\mathfrak{L}_i(\psi_1)}{\mathfrak{L}_i(\psi_2)}\right), \quad i=1,2. \tag{17}$$

Particularly, if we substitute $\psi_1(x) = \phi_p(x)$, $\psi_2(x) = \phi_q(x)$ in (17) and use continuous extension, the following expressions are obtained ($i=1,2$).

$$M_i(p,q) = \begin{cases} \left(\frac{\mathfrak{L}_i(\phi_p)}{\mathfrak{L}_i(\phi_q)}\right)^{1/(p-q)}, & p\neq q;\\ \exp\left(-\frac{1}{p} + \frac{\mathfrak{L}_i(\phi_0\phi_p)}{\mathfrak{L}_i(\phi_p)}\right), & p=q\neq 0;\\ \exp\left(\frac{\mathfrak{L}_i(\phi_0^2)}{2\mathfrak{L}_i(\phi_0)}\right), & p=q=0 \end{cases}$$

By Theorem 3, if $p,q,u,v \in \mathbb{R}$ such that $p \leqslant u$, $q \leqslant v$ then,

$$M_i(p,q) \leqslant M_i(u,v).$$

REMARK 3. Similarly, if we substitute $\psi_1(x) = \varphi_p(x)$, $\psi_2(x) = \varphi_q(x)$ in (17) and use continuous extension, the following expressions are obtained ($i=1,2$).

$$\overline{M}_i(p,q) = \begin{cases} \left(\frac{\mathfrak{L}_i(\varphi_p)}{\mathfrak{L}_i(\varphi_q)}\right)^{1/(p-q)}, & p\neq q;\\ \exp\left(-\frac{1}{p} + \frac{\mathfrak{L}_i(\varphi_0\varphi_p)}{\mathfrak{L}_i(\varphi_p)}\right), & p=q\neq 0;\\ \exp\left(\frac{\mathfrak{L}_i(\varphi_0^2)}{2\mathfrak{L}_i(\varphi_0)}\right), & p=q=0 \end{cases}$$

Again, using Theorem 3, if $p,q,u,v \in \mathbb{R}$ such that $p \leqslant u$, $q \leqslant v$ then,

$$\overline{M}_i(p,q) \leqslant \overline{M}_i(u,v).$$

We can now further refine obtained results by dropping some of analytical properties of families of functions from Lemmas 2 and 3. Therefore we define

$$\mathscr{C} = \{\psi_p :\ \psi_p : [a,b] \to \mathbb{R},\ p \in J\},$$

a family of functions from $C([a,b])$ such that $p \mapsto [x_0,\ x_1; \psi_p]$ is log-convex in the Jensen sense on J for every choice of two distinct points $x_0, x_1 \in [a,b]$.

THEOREM 11. *Let $f \mapsto \mathfrak{L}_i(f)$, $i = 1,2$ be linear functionals defined with (13) and (14) and let $G_i : J \to \mathbb{R}$, be defined with*

$$G_i(p) = \mathfrak{L}_i(\psi_p)$$

where $\psi_p \in \mathscr{C}$. Then the following statements hold, for every $i = 1,2$.

(i) G_i is log-convex in the Jensen sense on J.

(ii) If G_i is continuous on J, then it is log-convex on J and for $p,q,r \in J$ such that $p < q < r$, we have

$$G_i(q)^{r-p} \leqslant G_i(p)^{r-q} G_i(r)^{q-p}.$$

(iii) If G_i is positive and differentiable on J, then for every $p,q,u,v \in J$ such that $p \leqslant u$, $q \leqslant v$, we have

$$\widetilde{M}_i(p,q) \leqslant \widetilde{M}_i(u,v) \tag{18}$$

where $\widetilde{M}_i(p,q)$ is defined with

$$\widetilde{M}_i(p,q) = \begin{cases} \left(\frac{G_i(p)}{G_i(q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{\frac{d}{dp}(G_i(p))}{G_i(p)}\right), & p = q. \end{cases} \tag{19}$$

Proof. (i) We prove our claim for the case $i = 1$, second case is treated similarly. Choose any two distinct points $x_0, x_1 \in [a,b]$, any ξ_1, $\xi_2 \in \mathbb{R}$ and any p, $q \in J$. Define auxiliary function $\psi : [a,b] \to \mathbb{R}$ by

$$\psi(x) = \xi_1^2 \psi_p(x) + 2\xi_1\xi_2 \psi_{\frac{p+q}{2}}(x) + \xi_2^2 \psi_q(x), \tag{20}$$

where $\psi_p, \psi_{\frac{p+q}{2}}$ and ψ_q are from class C_1. Then

$$\begin{aligned}[x_0,\, x_1; \psi] =& \xi_1^2 [x_0,\, x_1; \psi_p] + 2\xi_1\xi_2 [x_0,\, x_1; \psi_{\frac{p+q}{2}}] \\ &+ \xi_2^2 [x_0,\, x_1,\, x_2; \psi_q] \geqslant 0\end{aligned}$$

by definition of $\mathscr{C}$ and characterization of log-convexity. This implies that ψ is nondecreasing function on $[a,b]$. Hence $\mathfrak{L}_1(\psi) \geqslant 0$ which is equivalent to

$$\xi_1^2 G_1(p) + 2\xi_1\xi_2 G_1\left(\frac{p+q}{2}\right) + \xi_2^2 G_1(q) \geqslant 0.$$

This proves that G_1 is log-convex in the Jensen sense on J.

(ii) Since G_i is continuous on J, then it is log-convex.

(iii) This is a simple consequence of Theorem 3. □

Let us introduce the following family of functions which will be used in the next theorem.

$$\mathscr{D} = \{\psi_p : \ \psi_p : [a,b] \to \mathbb{R}, \ p \in J\},$$

a family of functions from $C([a,b])$ *such that* $p \mapsto [x_0, \ x_1; \psi_p]$ *is exponentially convex on* J *for every choice of two distinct points* $x_0, x_1 \in [a,b]$.

THEOREM 12. *Let* $f \mapsto \mathfrak{L}_i(f), \ i = 1,2$ *be linear functionals defined with* (13) *and* (14) *and let* $H_i : J \to \mathbb{R}$, *be defined with*

$$H_i(p) = \mathfrak{L}_i(\psi_p) \tag{21}$$

where $\psi_p \in \mathscr{D}$. *Then the following statements hold for every* $i = 1,2$.

(i) If $n \in \mathbb{N}$ *and* $p_1, \ldots, p_n \in \mathbb{R}$ *are arbitrary, then the matrix*

$$\left[H_i\left(\frac{p_k + p_m}{2}\right)\right]_{k,m=1}^{n}$$

is positive semidefinite. Particularly,

$$det\left[H_i\left(\frac{p_k + p_m}{2}\right)\right]_{k,m=1}^{n} \geqslant 0.$$

(ii) If the function H_i *is continuous on* J, *then* H_i *is exponentially convex on* J.

(iii) If H_i *is positive and differentiable on* J, *then for every* $p,q,u,v \in J$ *such that* $p \leqslant u$, $q \leqslant v$, *we have*

$$\widehat{M}_i(p,q) \leqslant \widehat{M}_i(u,v)$$

where $\widehat{M}_i(p,q)$ *is defined with*

$$\widehat{M}_i(p,q) = \begin{cases} \left(\frac{H_i(p)}{H_i(q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{\frac{d}{dp}(H_i(p))}{H_i(p)}\right), & p = q. \end{cases}$$

Proof. (i) We prove our claim for the case $i = 1$, second case is treated similarly. Let $n \in \mathbb{N}$, $p_1, \ldots p_n \in \mathbb{R}$ be arbitrary and define auxiliary function $\psi : [a,b] \to \mathbb{R}$ by

$$\psi(x) = \sum_{k,m=1}^{n} \xi_k \xi_m \psi_{\frac{p_k+p_m}{2}}(x).$$

Then

$$[x_0, \ x_1; \psi] = \sum_{k,m=1}^{n} \xi_k \xi_m [x_0, \ x_1; \psi_{\frac{p_k+p_m}{2}}] \geqslant 0$$

by definition of $\mathscr{D}$ and exponential convexity. This implies that ψ is non-decreasing function on $[a,b]$ and then $\mathfrak{L}_1(\psi) \geqslant 0$ which is equivalent to

$$\sum_{k,m=1}^{n} \xi_i \xi_j H_1\left(\frac{p_k+p_m}{2}\right) \geqslant 0.$$

(ii) Follows from (i).

(iii) This is a simple consequence of Theorem 3. □

4. Concluding remarks

Families of exponentially convex functions similar to families given in Lemmas 2 and 3 can be easily constructed because of application of Theorem 2.

EXAMPLE 1. Consider a family of functions $h_p : (0,\infty) \to (0,\infty), p > 0,$ defined with

$$h_p(x) = \begin{cases} -\frac{p^{-x}}{\log p}, & p \neq 1; \\ x, & p = 1. \end{cases}$$

Since $p \mapsto \frac{d}{dx}(h_p(x)) = p^{-x}$ is the Laplace transform of a non-negative function (see [10] p. 210), it is exponentially convex according to Theorem 2.

Obviously $x \mapsto h_p(x)$ are non-decreasing functions for every $p > 0$. It is easy to prove that the function $p \mapsto [x_0,x_1;h_p]$ is also exponentially convex for arbitrary positive x_0,x_1 (see also [6]). Using Theorem 12 it follows that for linear functionals $f \mapsto \mathfrak{L}_i(f)$, $i=1,2$ defined with (13) and (14) we have that $p \mapsto \mathfrak{L}_i(h_p)$ is exponentially convex (it is easy to verify that it is continuous), for $i=1,2$.

Using further Theorem 12 we conclude that

$$R_i(p,q) = \begin{cases} \left(\frac{\mathfrak{L}_i(h_p)}{\mathfrak{L}_i(h_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\frac{\mathfrak{L}_i(h_1 \cdot h_p)}{p\mathfrak{L}_i(h_p)} - \frac{1}{p}\log p\right), & p = q \neq 1; \\ \exp\left(-\frac{\mathfrak{L}_i(h_1^2)}{2\mathfrak{L}_i(h_1)}\right), & p = q \neq 1; \end{cases}$$

satisfies

$$R_i(p,q) \leqslant R_i(u,v).$$

for $p,q,u,v \in \mathbb{R}$ such that $p \leqslant u,\ q \leqslant v$.

REMARK 4. From Example 1 and Theorem 12 it is clear that we presented a new way how to generate exponentially convex functions, aside from Laplace transform and Theorem 2.

REMARK 5. Notion of exponential convexity can be even further refined. For details see [7].

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Julije Jakšetić
Faculty of Mechanical Engineering and Naval Architecture
University of Zagreb
Ivana Lučića 5
10000 Zagreb, Croatia
e-mail: julije@math.hr

Josip Pečarić
Faculty of Textile Technology
University of Zagreb
Prilaz baruna Filipovića 28a
10000 Zagreb, Croatia
e-mail: pecaric@element.hr