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SOME NEW OPIAL DYNAMIC INEQUALITIES WITH WEIGHTED FUNCTIONS ON TIME SCALES

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Abstract. In this paper we prove some new dynamic inequalities with two weight functions and some new dynamic inequalities with two unknown functions of Opial type on time scales. The main results will be proved by employing Hölder's inequality, the chain rule and some basic algebraic inequalities.

1. Introduction

In 2001, Bohner and Kaymakçalan [1] proved some dynamic inequalities of Opial type on time scales. In particular, they proved that if $y:[0,a]\cap\mathbb{T}\longrightarrow\mathbb{R}$ is delta differentiable with $y(0)=0$, then

$$\int_0^a |y(t)+y^\sigma(t)|\left|y^\triangle(t)\right|\triangle t\leqslant a\int_0^a\left|y^\triangle(t)\right|^2\triangle t. \tag{1.1}$$

Also Bohner and Kaymakçalan in [1] proved that if p and q are positive $rd-$continuous functions on $[0,b]$, $\int_0^b(\triangle t/p(t))<\infty$, q is non-increasing and $y:[0,b]\cap\mathbb{T}\longrightarrow\mathbb{R}$ is delta differentiable with $y(0)=0$, then

$$\int_0^b q^\sigma(t)\left|(y(t)+y^\sigma(t))y^\triangle(t)\right|\triangle t\leqslant\int_0^b\frac{\triangle t}{p(t)}\int_0^b p(t)q(t)\left|y^\triangle(t)\right|^2\triangle t. \tag{1.2}$$

Karpuz et al. [6] replaced $q^\sigma(t)$ with $q(t)$ and proved an inequality similar to (1.2) of the form

$$\int_a^b q(t)\left|(y(t)+y^\sigma(t))y^\triangle(t)\right|\triangle t\leqslant K_q(a,b)\int_a^b\left|y^\triangle(t)\right|^2\triangle t, \tag{1.3}$$

where q is a positive $rd-$continuous function on $[a,b]_\mathbb{T}$, $y:[a,b]\cap\mathbb{T}\longrightarrow\mathbb{R}$ is delta differentiable with $y(a)=0$, and

$$K_q(a,b)=\left(2\int_a^b q^2(u)\,(\sigma(u)-a)\triangle u\right)^{1/2}. \tag{1.4}$$

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Wong et al. [11] and Srivastava et al. [10] proved that if q is a rd-continuous, positive and non-increasing function on $[a,b]\cap\mathbb{T}$, then

$$\int_a^b q(t)\,|y(t)|^{\lambda}\left|y^{\triangle}(t)\right|^{\gamma}\triangle t \leqslant \frac{\gamma}{\lambda+\gamma}(b-a)^{\lambda}\int_a^b q(t)\left|y^{\triangle}(t)\right|^{\lambda+\gamma}\triangle t, \tag{1.5}$$

where $y:[a,b]\cap\mathbb{T}\longrightarrow\mathbb{R}$ is delta differentiable with $y(a)=0$.

Saker [9] improved the conditions in (1.5) and proved some new dynamic inequalities with two weight functions p and q. In particular, he proved that if p and q are nonnegative rd-continuous function on $[a,b]_{\mathbb{T}}$ such that $\int_a^b p^{-1/(\lambda+\gamma-1)}\triangle t<\infty$, and $y:[a,b]\cap\mathbb{T}\to\mathbb{R}$ is delta differentiable with $y(a)=0$, then

$$\int_a^b q(t)\,|y(t)|^{\lambda}\left|y^{\triangle}(t)\right|^{\gamma}\triangle t \leqslant K_1(a,b,\lambda,\gamma)\int_a^b p(t)\left|y^{\triangle}(t)\right|^{\lambda+\gamma}\triangle t, \tag{1.6}$$

where

$$K_1(a,b,\lambda,\gamma)=\left(\frac{\gamma}{\lambda+\gamma}\right)^{\gamma/(\lambda+\gamma)}\left(\int_a^b \frac{(q(t))^{(\lambda+\gamma)/\lambda}}{(p(t))^{\gamma/\lambda}}\left(\int_a^t p^{\frac{-1}{(\lambda+\gamma-1)}}\triangle s\right)^{(\lambda+\gamma-1)}\triangle t\right)^{\gamma/(\lambda+\gamma)}. \tag{1.7}$$

In this paper we prove some new integral inequalities involving two weight functions with a power k different from $\lambda+\gamma$ which appears in the right hand side of (1.6). We also prove some new integral inequalities involving two unknown functions g_1 and g_2 with two weight functions. The paper is divided into two sections. In Section 2, we introduce some preliminaries on time scales and in Section 3, we prove the main results.

2. Preliminaries on time scales

In this section, for completeness, we recall the following concepts related to the notion of time scales. For more details on time scale analysis, we refer the reader to the two books by Bohner and Peterson [2], [3] which summarize and organize much of time scale calculus. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by:

$$\sigma(t):=\inf\{s\in\mathbb{T}:\ s>t\},\quad \rho(t):=\sup\{s\in\mathbb{T}:\ s<t\},$$

where $\sup\emptyset=\inf\mathbb{T}$. A point $t\in\mathbb{T}$, is said to be left–dense if $\rho(t)=t$ and $t>\inf\mathbb{T}$, is right–dense if $\sigma(t)=t$, is left–scattered if $\rho(t)<t$ and right–scattered if $\sigma(t)>t$. A function $g:\mathbb{T}\to\mathbb{R}$ is said to be right–dense continuous (rd–continuous) provided g is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$.

The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t):=\sigma(t)-t$, and for any function $f:\mathbb{T}\to\mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. Fix $t\in\mathbb{T}$ and let

$f:\mathbb{T}\to\mathbb{R}$. Define $f^{\Delta}(t)$ to be the number (if it exists) with the property that given any $\varepsilon>0$ there is a neighborhood U of t with

$$|[f(\sigma(t))-f(s)]-f^{\Delta}(t)[\sigma(t)-s]|\leqslant\varepsilon|\sigma(t)-s|,\quad \text{for all } s\in U.$$

In this case, we say $f^{\Delta}(t)$ is the (delta) derivative of f at t and that f is (delta) differentiable at t. We will frequently use the following results due to Hilger [4]. Throughout the paper will assume that $f:\mathbb{T}\to\mathbb{R}$ and let $t\in\mathbb{T}$.

(i) If f is differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with $f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}$.

(iii) If f is differentiable and t is right-dense, then $f^{\Delta}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$.

(iv) If f is differentiable at t, then $f(\sigma(t))=f(t)+\mu(t)f^{\Delta}(t)$.

The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see Kac and Cheung [5]), that is, when $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$ and $\mathbb{T}=q^{\mathbb{N}_0}=\{q^t: t\in\mathbb{N}_0\}$ where $q>1$. Without loss of generality, we assume that $\sup\mathbb{T}=\infty$, and define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}}:=[a,b]\cap\mathbb{T}$. In this paper we will refer to the (delta) integral which we can define as follows. If $G^{\Delta}(t)=g(t)$, then the Cauchy (delta) integral of g is defined by $\int_a^t g(s)\Delta s:=G(t)-G(a)$. It can be shown (see [2]) that if $g\in C_{rd}(\mathbb{T})$, then the Cauchy integral $G(t):=\int_{t_0}^t g(s)\Delta s$ exists, $t_0\in\mathbb{T}$, and satisfies $G^{\Delta}(t)=g(t)$, $t\in\mathbb{T}$. An infinite integral is defined as $\int_a^{\infty}f(t)\Delta t=\lim_{b\to\infty}\int_a^b f(t)\Delta t$. The integration on discrete time scales is defined by $\int_a^b f(t)\triangle t=\sum_{t\in[a,b)}\mu(t)f(t)$. We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma}\neq 0$, here $g^{\sigma}=g\circ\sigma$) of two differentiable function f and g

$$(fg)^{\Delta}=f^{\Delta}g+f^{\sigma}g^{\Delta}=fg^{\Delta}+f^{\Delta}g^{\sigma},\ \text{and}\ \left(\frac{f}{g}\right)^{\Delta}=\frac{f^{\Delta}g-fg^{\Delta}}{gg^{\sigma}}. \tag{2.1}$$

The chain rule formula on time scale is given by

$$(x^{\gamma}(t))^{\Delta}=\gamma\int_0^1[hx^{\sigma}+(1-h)x]^{\gamma-1}\,dhx^{\Delta}(t), \tag{2.2}$$

which is a simple consequence of Keller's chain rule [2, Theorem 1.90]. The integration by parts formula on time scales is given by

$$\int_a^b u(t)v^{\Delta}(t)\Delta t=[u(t)v(t)]_a^b-\int_a^b u^{\Delta}(t)v^{\sigma}(t)\Delta t. \tag{2.3}$$

Hölder's inequality [2, Theorem 6.13] states that for $u,v\in C_{rd}([a,\ b]_{\mathbb{T}},\mathbb{R})$, we have

$$\int_a^b|u(t)v(t)|\Delta t\leqslant\left[\int_a^b|u(t)|^q\Delta t\right]^{\frac{1}{q}}\left[\int_a^b|v(t)|^p\Delta t\right]^{\frac{1}{p}}, \tag{2.4}$$

where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

3. Main results

The results in this section generalize and extend the Opial inequality (1.6) and the inequality (1.5). Throughout this section (without mentioning) the integrals in the statements of the theorems are assumed to exist.

To prove the main results we will use of the chain rule (2.2), Hölder's inequality (2.4) and the inequality

$$a^{\lambda}+b^{\lambda} \leqslant (a+b)^{\lambda} \leqslant 2^{\lambda-1}(a^{\lambda}+b^{\lambda}), \text{ if } a,\ b \geqslant 0,\ \lambda \geqslant 1. \tag{3.1}$$

THEOREM 3.1. *Assume that $\mathbb{T}$ be a time scale with $\alpha,\ \tau \in \mathbb{T},\ k>1,\ \lambda>0$ and $0<\gamma<k$. Let $p(t), q(t)$ be non-negative rd-continuous functions on $[\alpha,\tau]_{\mathbb{T}}$ such that $\int_{\alpha}^{t}(p(s))^{-1/(k-1)}\triangle s<\infty$. If $x:[\alpha,\tau]_{\mathbb{T}} \longrightarrow \mathbb{R}$ is delta differentiable with $x(\alpha)=0$, then*

$$\int_{\alpha}^{\tau} q(t)\,|x(t)|^{\lambda}\left|x^{\triangle}(t)\right|^{\gamma}\triangle t \leqslant K_1(\lambda,\gamma,k)\left[\int_{\alpha}^{\tau} p(t)\left|x^{\triangle}(t)\right|^{k}\triangle t\right]^{(\lambda+\gamma)/k}, \tag{3.2}$$

where

$$K_1(\lambda,\gamma,k) := \left(\frac{\gamma}{\lambda+\gamma}\right)^{\gamma/k}\left[\int_{\alpha}^{\tau}\left(\frac{q^{k}(t)}{p^{\gamma}(t)}\right)^{\frac{1}{(k-\gamma)}}\left(\int_{\alpha}^{t} p^{\frac{-1}{k-1}}(s)\triangle s\right)^{\frac{\lambda(k-1)}{(k-\gamma)}}\triangle t\right]^{\frac{(k-\gamma)}{k}}. \tag{3.3}$$

Proof. Since $x(\alpha)=0$, we have that

$$|x(t)| \leqslant \int_{\alpha}^{t}\left|x^{\triangle}(s)\right|\triangle s = \int_{\alpha}^{t}(p(s))^{-1/k}(p(s))^{1/k}\left|x^{\triangle}(s)\right|\triangle s. \tag{3.4}$$

Applying the Hölder inequality (2.4) with indices $k/(k-1)$ and k on the right hand side of (3.4), we get

$$|x(t)| \leqslant \left(\int_{\alpha}^{t}(p(s))^{-1/(k-1)}\triangle s\right)^{(k-1)/k}\left(\int_{\alpha}^{t} p(s)\left|x^{\triangle}(s)\right|^{k}\triangle s\right)^{1/k}. \tag{3.5}$$

This implies that

$$|x(t)|^{\lambda} \leqslant \left(\int_{\alpha}^{t}(p(s))^{\frac{-1}{(k-1)}}\triangle s\right)^{\lambda(k-1)/k}\left(\int_{\alpha}^{t} p(s)\left|x^{\triangle}(s)\right|^{k}\triangle s\right)^{\lambda/k}. \tag{3.6}$$

Now, we define

$$y(t) := \int_{\alpha}^{t} p(s)\left|x^{\triangle}(s)\right|^{k}\triangle s. \tag{3.7}$$

This gives us

$$y^{\triangle}(t) = p(t)\left|x^{\triangle}(t)\right|^{k} > 0, \tag{3.8}$$

and hence

$$\left|x^{\triangle}(t)\right|^{\gamma}=(p(t))^{-\gamma/k}\left(y^{\triangle}(t)\right)^{\gamma/k}. \tag{3.9}$$

Since q is a non-negative function and $\lambda>0,$ we have from (3.6), (3.7) and (3.9) that

$$\begin{aligned} &q(t)\,|x(t)|^{\lambda}\left|x^{\triangle}(t)\right|^{\gamma} \\ &\leqslant q(t)(p(t))^{\frac{-\gamma}{k}}\left(\int_{\alpha}^{t}(p(s))^{\frac{-1}{k-1}}\triangle s\right)^{\lambda(k-1)/k}(y(t))^{\lambda/k}\left(y^{\triangle}(t)\right)^{\gamma/k}. \end{aligned} \tag{3.10}$$

Integrating (3.10) from α to τ, and applying Hölder's inequality (2.4) with indices $k/(k-\gamma)$ and $k/\gamma,$ we have that

$$\begin{aligned} &\int_{\alpha}^{\tau}q(t)\,|x(t)|^{\lambda}\left|x^{\triangle}(t)\right|^{\gamma}\triangle t \\ &\leqslant\left[\int_{\alpha}^{\tau}\left(\frac{q^{k}(t)}{p^{\gamma}(t)}\right)^{\frac{1}{(k-\gamma)}}\left(\int_{\alpha}^{t}(p(s))^{\frac{-1}{(k-1)}}\triangle s\right)^{\frac{\lambda(k-1)}{(k-\gamma)}}\triangle t\right]^{\frac{(k-\gamma)}{k}} \\ &\times\left(\int_{\alpha}^{\tau}(y(t))^{\lambda/\gamma}y^{\triangle}(t)\triangle t\right)^{\gamma/k}. \end{aligned} \tag{3.11}$$

From the chain rule (2.2), and the fact that $y^{\triangle}(t)>0,$ we see that

$$(y(t))^{\lambda/\gamma}y^{\triangle}(t)\leqslant\frac{\gamma}{\lambda+\gamma}(y^{(\lambda+\gamma)/\gamma}(t))^{\triangle}. \tag{3.12}$$

Substituting (3.12) into (3.11), we have

$$\begin{aligned} &\int_{\alpha}^{\tau}q(t)\,|x(t)|^{\lambda}\left|x^{\triangle}(t)\right|^{\gamma}\triangle t \\ &\leqslant\left[\int_{\alpha}^{\tau}\left(\frac{q^{k}(t)}{p^{\gamma}(t)}\right)^{\frac{1}{(k-\gamma)}}\left(\int_{\alpha}^{t}(p(s))^{\frac{-1}{(k-1)}}\triangle s\right)^{\frac{\lambda(k-1)}{(k-\gamma)}}\triangle t\right]^{\frac{(k-\gamma)}{k}} \\ &\times\left(\frac{\gamma}{\lambda+\gamma}\right)^{\gamma k}\left(\int_{\alpha}^{\tau}(y^{(\lambda+\gamma)/\gamma}(t))^{\triangle}\triangle t\right)^{\gamma/k} \\ &=\left[\int_{\alpha}^{\tau}\left(\frac{q^{k}(t)}{p^{\gamma}(t)}\right)^{\frac{1}{(k-\gamma)}}\left(\int_{\alpha}^{t}(p(s))^{-1/(k-1)}\triangle s\right)^{\frac{\lambda(k-1)}{(k-\gamma)}}\triangle t\right]^{\frac{(k-\gamma)}{k}} \\ &\times\left(\frac{\gamma}{\lambda+\gamma}\right)^{\gamma/k}(y(\tau))^{(\lambda+\gamma)/k}. \end{aligned}$$

From (3.7), and the last inequality, we have

$$\int_{\alpha}^{\tau}q(t)\,|x(t)|^{\lambda}\left|x^{\triangle}(t)\right|^{\gamma}\triangle t\leqslant K_{1}(\lambda,\gamma,k)\left[\int_{\alpha}^{\tau}p(t)\left|x^{\triangle}(t)\right|^{k}\triangle t\right]^{(\lambda+\gamma)/k},$$

which is the desired inequality (3.2). The proof is complete. □

Here, we only state the following theorem, since its proof is similar to that of Theorem 3.1 with $[\alpha,\ \tau]_{\mathbb{T}}$ replaced by $[\tau,\ \beta]_{\mathbb{T}}$.

THEOREM 3.2. *Assume that $\mathbb{T}$ be a time scale with $\tau,\ \beta \in \mathbb{T},\ k>1,\ \lambda>0$ and $0<\gamma<k$. Let $p(t), q(t)$ be non-negative rd-continuous functions on $[\tau,\beta]_{\mathbb{T}}$ such that $\int_t^{\beta}(p(s))^{-1/(k-1)}\triangle s<\infty$. If $x:[\tau,\beta]_{\mathbb{T}} \longrightarrow \mathbb{R}$ is delta differentiable with $x(\beta)=0$, then*

$$\int_{\tau}^{\beta} q(t)\,|x(t)|^{\lambda}\left|x^{\triangle}(t)\right|^{\gamma}\triangle t \leqslant K_2(\lambda,\gamma,k)\left[\int_{\tau}^{\beta} p(t)\left|x^{\triangle}(t)\right|^{k}\triangle t\right]^{(\lambda+\gamma)/k}, \tag{3.13}$$

where

$$K_2(\lambda,\gamma,k) := \left(\frac{\gamma}{\lambda+\gamma}\right)^{\gamma/k}\left[\int_{\tau}^{\beta}\left(\frac{q^k(t)}{p^{\gamma}(t)}\right)^{\frac{1}{(k-\gamma)}}\left(\int_t^{\beta}(p(s))^{\frac{-1}{(k-1)}}\triangle s\right)^{\frac{\lambda(k-1)}{(k-\gamma)}}\triangle t\right]^{\frac{(k-\gamma)}{k}}. \tag{3.14}$$

REMARK 3.1. As a special case when we take $k=\lambda+\gamma$, we see that inequality (3.2) becomes inequality (1.6).

If we put $p(t)=q(t)$ in (3.2), we obtain the following special case from Theorem 3.1, which is a generalization of inequality (1.5).

COROLLARY 3.1. *Assume that $\mathbb{T}$ be a time scale with $\alpha,\ \tau \in \mathbb{T},\ k>1,\ \lambda>0$ and $0<\gamma<k$. Let $p(t)$ be non-negative rd-continuous functions on $[\alpha,\tau]_{\mathbb{T}}$ such that $\int_{\alpha}^{t}(p(s))^{-1/(k-1)}\triangle s<\infty$. If $x:[\alpha,\tau]_{\mathbb{T}} \longrightarrow \mathbb{R}$ is delta differentiable with $x(\alpha)=0$, then*

$$\int_{\alpha}^{\tau} p(t)\,|x(t)|^{\lambda}\left|x^{\triangle}(t)\right|^{\gamma}\triangle t \leqslant K_1^*(\lambda,\gamma,k)\left[\int_{\alpha}^{\tau} p(t)\left|x^{\triangle}(t)\right|^{k}\triangle t\right]^{(\lambda+\gamma)/k}, \tag{3.15}$$

where

$$K_1^*(\lambda,\gamma,k) := \left(\frac{\gamma}{\lambda+\gamma}\right)^{\gamma/k}\left[\int_{\alpha}^{\tau} p(t)\left(\int_{\alpha}^{t}(p(s))^{\frac{-1}{(k-1)}}\triangle s\right)^{\frac{\lambda(k-1)}{(k-\gamma)}}\triangle t\right]^{\frac{(k-\gamma)}{k}}. \tag{3.16}$$

On a time scale $\mathbb{T}$, we note as a consequence of the chain rule (2.2) that

$$\begin{aligned}\left((t-\alpha)^{\frac{\lambda(k-1)}{(k-\gamma)}+1}\right)^{\triangle} &= \left(\frac{\lambda(k-1)}{(k-\gamma)}+1\right)\int_0^1\left[h(\sigma(t)-\alpha)+(1-h)(t-\alpha)\right]^{\frac{\lambda(k-1)}{(k-\gamma)}}dh\\ &\geqslant \left(\frac{\lambda(k-1)}{(k-\gamma)}+1\right)\int_0^1\left[h(t-\alpha)+(1-h)(t-\alpha)\right]^{\frac{\lambda(k-1)}{(k-\gamma)}}dh\\ &= \left(\frac{\lambda(k-1)}{(k-\gamma)}+1\right)(t-\alpha)^{\frac{\lambda(k-1)}{(k-\gamma)}}. \end{aligned} \tag{3.17}$$

This implies that

$$\int_{\alpha}^{\tau}(t-\alpha)^{\frac{\lambda(k-1)}{(k-\gamma)}}\triangle t \leqslant \frac{(\tau-\alpha)^{\Lambda}}{\Lambda}, \tag{3.18}$$

where $\Lambda=\frac{\lambda(k-1)}{(k-\gamma)}+1$. From (3.18) and (3.16) with $p(t)=1$, one obtains that

$$\begin{aligned} K_1^*(\lambda,\gamma,k) &= \left(\frac{\gamma}{\lambda+\gamma}\right)^{\gamma/k}\left[\int_{\alpha}^{\tau}(t-\alpha)^{\frac{\lambda(k-1)}{(k-\gamma)}}\triangle t\right]^{\frac{(k-\gamma)}{k}} \\ &\leqslant \left(\frac{\gamma}{\lambda+\gamma}\right)^{\gamma/k}\left[\frac{(\tau-\alpha)^{\Lambda}}{\Lambda}\right]^{\frac{(k-\gamma)}{k}}. \end{aligned} \tag{3.19}$$

Setting $p(t)=1$ in (3.15) and using (3.19), we have the following inequality.

COROLLARY 3.2. *Let $\mathbb{T}$ be a time scale with α, $\tau\in\mathbb{T}$, $k>1$, $\lambda>0$ and $0<\gamma<k$. If $x:[\alpha,\tau]_{\mathbb{T}}\longrightarrow\mathbb{R}$ is delta differentiable with $x(\alpha)=0$, then*

$$\int_{\alpha}^{\tau}|x(t)|^{\lambda}\left|x^{\triangle}(t)\right|^{\gamma}\triangle t \leqslant \left(\frac{\gamma}{\lambda+\gamma}\right)^{\gamma/k}\left[\frac{(\tau-\alpha)^{\Lambda}}{\Lambda}\right]^{\frac{(k-\gamma)}{k}}\left[\int_{\alpha}^{\tau}\left|x^{\triangle}(t)\right|^{k}\triangle t\right]^{(\lambda+\gamma)/k}, \tag{3.20}$$

where $\Lambda=\frac{\lambda(k-1)}{(k-\gamma)}+1$.

REMARK 3.2. As a special case when $k=\lambda+\gamma$, inequality (3.20) becomes

$$\int_{\alpha}^{\tau}|x(t)|^{\lambda}\left|x^{\triangle}(t)\right|^{\gamma}\triangle t \leqslant \frac{\gamma^{\gamma/(\lambda+\gamma)}}{\lambda+\gamma}(\tau-\alpha)^{\lambda}\int_{\alpha}^{\tau}\left|x^{\triangle}(t)\right|^{\lambda+\gamma}\triangle t. \tag{3.21}$$

REMARK 3.3. Note that when $\mathbb{T}=\mathbb{R}$, inequality (3.21) becomes

$$\int_{\alpha}^{\tau}|x(t)|^{\lambda}\left|x'(t)\right|^{\gamma}dt \leqslant \frac{\gamma^{\gamma/(\lambda+\gamma)}}{\lambda+\gamma}(\tau-\alpha)^{\lambda}\int_{\alpha}^{\tau}\left|x'(t)\right|^{\lambda+\gamma}dt. \tag{3.22}$$

REMARK 3.4. Note that when $\lambda=\gamma=1$, we have the following result

$$\int_{\alpha}^{\tau}|x(t)|\left|x'(t)\right|dt \leqslant \left(\frac{\tau-\alpha}{2}\right)\int_{\alpha}^{\tau}\left|x'(t)\right|^{2}dt. \tag{3.23}$$

THEOREM 3.3. *Assume that $\mathbb{T}$ be a time scale with α, $\tau\in\mathbb{T}$, $\lambda\geqslant 0$ and $\gamma\geqslant 1$. Let $f(t)$ be positive on $[\alpha,\tau]_{\mathbb{T}}$ with $\int_{\alpha}^{\tau}(f(t))^{-\gamma}\triangle t<\infty$, and let $g(t)$ be a positive non-increasing and rd-continuous function on $[\alpha,\tau]_{\mathbb{T}}$. If $x:[\alpha,\tau]_{\mathbb{T}}\longrightarrow\mathbb{R}$ is delta differentiable with $x(\alpha)=0$, then*

$$\begin{aligned} &\int_{\alpha}^{\tau}(f(t))^{\gamma(\gamma-1)}g^{\gamma}(t)\,|x(t)|^{\lambda}\left|x^{\triangle}(t)\right|^{\gamma}\triangle t \\ &\leqslant \left(\frac{\gamma}{\lambda+\gamma}\right)\left(\int_{\alpha}^{\tau}(f(t))^{-\gamma}\triangle t\right)^{\lambda}\int_{\alpha}^{\tau}(f(t))^{\gamma(\lambda+\gamma-1)}g^{\gamma}(t)\left|x^{\triangle}(t)\right|^{\lambda+\gamma}\triangle t. \end{aligned} \tag{3.24}$$

Proof. Define

$$z(t) = \int_{\alpha}^{t} (f(s))^{\gamma(\gamma-1)} (g(s))^{\frac{\gamma^2}{(\lambda+\gamma)}} \left|x^{\triangle}(s)\right|^{\gamma} \triangle s, \text{ for } t \in [\alpha,\tau]_{\mathbb{T}}. \tag{3.25}$$

Hence

$$z^{\triangle}(t) = (f(t))^{\gamma(\gamma-1)} (g(t))^{\gamma^2/(\lambda+\gamma)} \left|x^{\triangle}(t)\right|^{\gamma}, \tag{3.26}$$

and then

$$\left|x^{\triangle}(t)\right|^{\gamma} = (f(t))^{-\gamma(\gamma-1)} (g(t))^{-\gamma^2/(\lambda+\gamma)} z^{\triangle}(t). \tag{3.27}$$

Since $x(\alpha) = 0$, we see that

$$|x(t)| \leqslant \int_{\alpha}^{t} \left|x^{\triangle}(s)\right| \triangle s = \int_{\alpha}^{t} (f(s))^{-(\gamma-1)} (f(s))^{\gamma-1} \left|x^{\triangle}(s)\right| \triangle s.$$

Applying Hölder's inequality (2.4) with indices $\gamma/(\gamma-1)$ and γ, we have

$$|x(t)| \leqslant \left(\int_{\alpha}^{t} (f(s))^{-\gamma} \triangle s\right)^{(\gamma-1)/\gamma} \left(\int_{\alpha}^{t} (f(s))^{\gamma(\gamma-1)} \left|x^{\triangle}(s)\right|^{\gamma} \triangle s\right)^{1/\gamma}.$$

Then for $\alpha \leqslant t \leqslant \tau$, we get that

$$|x(t)|^{\lambda} \leqslant \left(\int_{\alpha}^{t} (f(s))^{-\gamma} \triangle s\right)^{\lambda(\gamma-1)/\gamma} \left(\int_{\alpha}^{t} (f(s))^{\gamma(\gamma-1)} \left|x^{\triangle}(s)\right|^{\gamma} \triangle s\right)^{\lambda/\gamma}. \tag{3.28}$$

Thus, if $g(t)$ is positive and non-increasing, we have from (3.27) and (3.28) that

$$\begin{aligned} &(f(t))^{\gamma(\gamma-1)} g^{\gamma}(t) |x(t)|^{\lambda} \left|x^{\triangle}(t)\right|^{\gamma} \triangle t \\ &\leqslant (g(t))^{\frac{\lambda\gamma}{(\lambda+\gamma)}} \left(\int_{\alpha}^{t} (f(s))^{-\gamma} \triangle s\right)^{\frac{\lambda(\gamma-1)}{\gamma}} \left(\int_{\alpha}^{t} (f(s))^{\gamma(\gamma-1)} \left|x^{\triangle}(s)\right|^{\gamma} \triangle s\right)^{\frac{\lambda}{\gamma}} z^{\triangle}(t) \\ &= (F(t))^{\frac{\lambda(\gamma-1)}{\gamma}} z^{\frac{\lambda}{\gamma}}(t) z^{\triangle}(t), \end{aligned} \tag{3.29}$$

where $F(t) := \int_{\alpha}^{t} (f(s))^{-\gamma} \triangle s$. Now, since $F^{\Delta}(t) = (f(t))^{-\gamma} > 0$, we see that $F(t) \leqslant F(\tau)$. This and (3.29) implies that

$$\begin{aligned} &\int_{\alpha}^{\tau} (f(t))^{\gamma(\gamma-1)} g^{\gamma}(t) |x(t)|^{\lambda} \left|x^{\triangle}(t)\right|^{\gamma} \triangle t \\ &\leqslant \int_{\alpha}^{\tau} (F(\tau))^{\lambda(\gamma-1)/\gamma} z^{\lambda/\gamma}(t) z^{\triangle}(t) \triangle t \\ &= \left(\int_{\alpha}^{\tau} (f(s))^{-\gamma} \triangle s\right)^{\lambda(\gamma-1)/\gamma} \int_{\alpha}^{\tau} z^{\lambda/\gamma}(t) z^{\triangle}(t) \triangle t. \end{aligned} \tag{3.30}$$

From the chain rule (2.2), and the fact that $z^{\triangle}(t) > 0,$ we obtain

$$\begin{aligned}\left(z^{(\lambda+\gamma)/\gamma}(t)\right)^{\triangle} &= \left(\frac{\lambda+\gamma}{\gamma}\right)\int_0^1 [hz^{\sigma}+(1-h)z]^{\lambda/\gamma}\,dhz^{\triangle}(t)\\ &\geqslant \left(\frac{\lambda+\gamma}{\gamma}\right)\int_0^1 [hz+(1-h)z]^{\lambda/\gamma}\,dhz^{\triangle}(t)\\ &= \left(\frac{\lambda+\gamma}{\gamma}\right) z^{\lambda/\gamma}(t)z^{\triangle}(t).\end{aligned} \tag{3.31}$$

Substituting (3.31) into (3.30), and using the fact that $z(\alpha) = 0,$ we have

$$\begin{aligned}&\int_{\alpha}^{\tau}(f(t))^{\gamma(\gamma-1)}g^{\gamma}(t)\,|x(t)|^{\lambda}\left|x^{\triangle}(t)\right|^{\gamma}\triangle t\\ &\leqslant \left(\frac{\gamma}{\lambda+\gamma}\right)\left(\int_{\alpha}^{\tau}(f(t))^{-\gamma}\triangle t\right)^{\frac{\lambda(\gamma-1)}{\gamma}} z^{\frac{(\lambda+\gamma)}{\gamma}}(\tau)\\ &= \left(\frac{\gamma}{\lambda+\gamma}\right)\left(\int_{\alpha}^{\tau}(f(t))^{-\gamma}\triangle t\right)^{\lambda(\gamma-1)/\gamma}\left(\int_{\alpha}^{\tau}(f(t))^{\gamma(\gamma-1)}(g(t))^{\frac{\gamma^2}{(\lambda+\gamma)}}\left|x^{\triangle}(t)\right|^{\gamma}\triangle t\right)^{\frac{(\lambda+\gamma)}{\gamma}}\\ &= \left(\frac{\gamma}{\lambda+\gamma}\right)\left(\int_{\alpha}^{\tau}(f(t))^{-\gamma}\triangle t\right)^{\lambda(\gamma-1)/\gamma}\\ &\quad\times\left(\int_{\alpha}^{\tau}(f(t))^{\frac{-\lambda\gamma}{(\lambda+\gamma)}}(f(t))^{\frac{\gamma^2(\lambda+\gamma-1)}{(\lambda+\gamma)}}(g(t))^{\frac{\gamma^2}{(\lambda+\gamma)}}\left|x^{\triangle}(t)\right|^{\gamma}\triangle t\right)^{\frac{(\lambda+\gamma)}{\gamma}}.\end{aligned}$$

Again applying Hölder's inequality (2.4) with indices $(\lambda+\gamma)/\lambda$ and $(\lambda+\gamma)/\gamma,$ we have

$$\begin{aligned}&\int_{\alpha}^{\tau}(f(t))^{\gamma(\gamma-1)}g^{\gamma}(t)\,|x(t)|^{\lambda}\left|x^{\triangle}(t)\right|^{\gamma}\triangle t\\ &\leqslant \left(\frac{\gamma}{\lambda+\gamma}\right)\left(\int_{\alpha}^{\tau}(f(t))^{-\gamma}\triangle t\right)^{\lambda(\gamma-1)/\gamma}\\ &\quad\times\left(\int_{\alpha}^{\tau}(f(t))^{-\gamma}\triangle t\right)^{\lambda/\gamma}\int_{\alpha}^{\tau}(f(t))^{\gamma(\lambda+\gamma-1)}g^{\gamma}(t)\left|x^{\triangle}(t)\right|^{\lambda+\gamma}\triangle t\\ &= \left(\frac{\gamma}{\lambda+\gamma}\right)\left(\int_{\alpha}^{\tau}(f(t))^{-\gamma}\triangle t\right)^{\lambda}\int_{\alpha}^{\tau}(f(t))^{\gamma(\lambda+\gamma-1)}g^{\gamma}(t)\left|x^{\triangle}(t)\right|^{\lambda+\gamma}\triangle t,\end{aligned}$$

which is the desired inequality (3.24). The proof is complete. □

REMARK 3.5. As a special case when $f(t) = 1$ and $g^{\gamma}(t) = q(t),$ inequality (3.24) becomes inequality (1.5).

Here, we only state the following theorem, since its proof is similar to that of Theorem 3.3 with $[\alpha,\ \tau]_{\mathbb{T}}$ replaced by $[\tau,\ \beta]_{\mathbb{T}}$.

THEOREM 3.4. *Assume that $\mathbb{T}$ be a time scale with $\tau,\ \beta \in \mathbb{T}$, $\lambda \geqslant 0$ and $\gamma \geqslant 1$. Let $f(t)$ be positive on $[\tau,\beta]_{\mathbb{T}}$ with $\int_\tau^\beta (f(t))^{-\gamma}\triangle t < \infty$, and let $g(t)$ be positive non-decreasing and rd-continuous function on $[\tau,\beta]_{\mathbb{T}}$. If $x : [\tau,\beta]_{\mathbb{T}} \longrightarrow \mathbb{R}$ is delta differentiable with $x(\beta) = 0$, then*

$$\int_\tau^\beta (f(t))^{\gamma(\gamma-1)} g^\gamma(t)\,|x(t)|^\lambda \left|x^\triangle(t)\right|^\gamma \triangle t \tag{3.32}$$
$$\leqslant \left(\frac{\gamma}{\lambda+\gamma}\right)\left(\int_\tau^\beta (f(t))^{-\gamma}\triangle t\right)^\lambda \int_\tau^\beta (f(t))^{\gamma(\lambda+\gamma-1)} g^\gamma(t) \left|x^\triangle(t)\right|^{\lambda+\gamma} \triangle t.$$

Now, we prove some new integral inequalities involving two unknowns g_1 and g_2 with two weight functions p and q.

THEOREM 3.5. *Assume that $\mathbb{T}$ be a time scale with $\alpha,\ \tau \in \mathbb{T}$, let $p(t)$ is a positive rd-continuous function on $[\alpha,\tau]_{\mathbb{T}}$ with $\int_\alpha^\tau \frac{\triangle t}{p(t)} < \infty$, and $q(t)$ be a positive, non-increasing and rd-continuous function on $[\alpha,\tau]_{\mathbb{T}}$. If $g_1,\ g_2 : [\alpha,\tau]_{\mathbb{T}} \longrightarrow \mathbb{R}$ are delta differentiable with $g_1(\alpha) = g_2(\alpha) = 0$, then*

$$\int_\alpha^\tau q^\sigma(t)\left[\left|(g_1(t)+g_1^\sigma(t))g_2^\triangle(t)\right| + \left|(g_2(t)+g_2^\sigma(t))g_1^\triangle(t)\right|\right]\triangle t \tag{3.33}$$
$$\leqslant \int_\alpha^\tau \frac{\triangle t}{p(t)} \int_\alpha^\tau p(t)q(t)\left[\left|g_1^\triangle(t)\right|^2 + \left|g_2^\triangle(t)\right|^2\right]\triangle t.$$

Proof. Let $f_i(t) := \int_\alpha^t \sqrt{q^\sigma(s)}\left|g_i^\triangle(s)\right|\triangle s$, for $t \in [\alpha,\ \tau]_{\mathbb{T}}$ and $i = 1,\ 2$. Note that $f_i^\triangle(t) = \sqrt{q^\sigma(t)}\left|g_i^\triangle(t)\right| > 0$ and note that for $s < t$ we have $q^\sigma(s) \geqslant q(t)$. Thus

$$|g_i(t)| \leqslant \int_\alpha^t \left|g_i^\triangle(s)\right|\triangle s \leqslant \int_\alpha^t \sqrt{\frac{q^\sigma(s)}{q(t)}}\left|g_i^\triangle(s)\right|\triangle s = \frac{f_i(t)}{\sqrt{q(t)}}.$$

Hence, (note that $t \leqslant \sigma(t)$ which implies $q(t) \geqslant q^\sigma(t)$)

$$\begin{aligned}
&\int_\alpha^\tau q^\sigma(t)\left[\left|(g_1(t)+g_1^\sigma(t))g_2^\triangle(t)\right| + \left|(g_2(t)+g_2^\sigma(t))g_1^\triangle(t)\right|\right]\triangle t\\
&\leqslant \int_\alpha^\tau q^\sigma(t)\left[\left(\frac{f_1(t)}{\sqrt{q(t)}}+\frac{f_1^\sigma(t)}{\sqrt{q^\sigma(t)}}\right)\frac{f_2^\triangle(t)}{\sqrt{q^\sigma(t)}} + \left(\frac{f_2(t)}{\sqrt{q(t)}}+\frac{f_2^\sigma(t)}{\sqrt{q^\sigma(t)}}\right)\frac{f_1^\triangle(t)}{\sqrt{q^\sigma(t)}}\right]\triangle t\\
&\leqslant \int_\alpha^\tau \left[(f_1(t)+f_1^\sigma(t))\,f_2^\triangle(t) + (f_2(t)+f_2^\sigma(t))\,f_1^\triangle(t)\right]\triangle t\\
&= \int_\alpha^\tau (f_1 f_2^\triangle + f_1^\sigma f_2^\triangle + f_2 f_1^\triangle + f_2^\sigma f_1^\triangle)\triangle t\\
&= 2\int_\alpha^\tau (f_1(t)f_2(t))^\triangle(t)\triangle t = 2f_1(\tau)f_2(\tau) \leqslant (f_1(\tau))^2 + (f_2(\tau))^2,
\end{aligned}$$

where we have applied the inequality

$$2\alpha\beta \leqslant (\alpha^2+\beta^2),\ \text{for } \alpha,\ \beta \text{ reals.} \tag{3.34}$$

Applying Hölder's inequality (2.4), we get

$$\begin{aligned}
&\int_\alpha^\tau q^\sigma(t)\left[\left|(g_1(t)+g_1^\sigma(t))g_2^\triangle(t)\right|+\left|(g_2(t)+g_2^\sigma(t))g_1^\triangle(t)\right|\right]\triangle t\\
&\leqslant\left(\int_\alpha^\tau \frac{1}{\sqrt{p(t)}}\sqrt{p(t)q^\sigma(t)}\left|g_1^\triangle(t)\right|\triangle t\right)^2+\left(\int_\alpha^\tau \frac{1}{\sqrt{p(t)}}\sqrt{p(t)q^\sigma(t)}\left|g_2^\triangle(t)\right|\triangle t\right)^2\\
&\leqslant\int_\alpha^\tau \frac{\triangle t}{p(t)}\int_\alpha^\tau p(t)q^\sigma(t)\left|g_1^\triangle(t)\right|^2\triangle t+\int_\alpha^\tau \frac{\triangle t}{p(t)}\int_\alpha^\tau p(t)q^\sigma(t)\left|g_2^\triangle(t)\right|^2\triangle t\\
&\leqslant\int_\alpha^\tau \frac{\triangle t}{p(t)}\int_\alpha^\tau p(t)q(t)\left[\left|g_1^\triangle(t)\right|^2+\left|g_2^\triangle(t)\right|^2\right]\triangle t,
\end{aligned}$$

which is the desired inequality (3.33). The proof is complete. □

REMARK 3.6. Note that when $g_1(t)=g_2(t)=g(t)$, then inequality (3.33) becomes the Opial inequality (1.2).

The proof of the following theorem is similar to the proof of Theorem 3.5.

THEOREM 3.6. *Assume that $\mathbb{T}$ be a time scale with $\tau,\ \beta\in\mathbb{T}$, let $p(t)$ be positive and rd-continuous function on $[\tau,\beta]_\mathbb{T}$ with $\int_\tau^\beta \frac{\triangle t}{p(t)}<\infty$, and $q(t)$ be positive, non-decreasing and rd-continuous function on $[\tau,\beta]_\mathbb{T}$. If $g_1,\ g_2:[\tau,\beta]_\mathbb{T}\longrightarrow\mathbb{R}$ are delta differentiable with $g_1(\beta)=g_2(\beta)=0$, then*

$$\begin{aligned}
&\int_\tau^\beta q^\sigma(t)\left[\left|(g_1(t)+g_1^\sigma(t))g_2^\triangle(t)\right|+\left|(g_2(t)+g_2^\sigma(t))g_1^\triangle(t)\right|\right]\triangle t \qquad (3.35)\\
&\leqslant\int_\tau^\beta \frac{\triangle t}{p(t)}\int_\tau^\beta p(t)q(t)\left[\left|g_1^\triangle(t)\right|^2+\left|g_2^\triangle(t)\right|^2\right]\triangle t.
\end{aligned}$$

THEOREM 3.7. *Assume that $\mathbb{T}$ be a time scale with $\alpha,\ \tau\in\mathbb{T}$, $\lambda\geqslant 0$ and $\gamma\geqslant 1$. Let $p(t)$ is a positive rd-continuous function on $[\alpha,\tau]_\mathbb{T}$ with $\int_\alpha^\tau \frac{\triangle t}{p(t)}<\infty$, and $q(t)$ be positive, non-increasing and rd-continuous function on $[\alpha,\tau]_\mathbb{T}$. If $g_1,\ g_2:[\alpha,\tau]_\mathbb{T}\longrightarrow\mathbb{R}$ are delta differentiable with $g_1(\alpha)=g_2(\alpha)=0$, then*

$$\begin{aligned}
&\int_\alpha^\tau q^\sigma(t)\left|g_1(t)g_2(t)\right|^\lambda\left[\left|g_1(t)+g_1^\sigma(t)\right|^\gamma\left|g_2^\triangle(t)\right|^\gamma+\left|g_2(t)+g_2^\sigma(t)\right|^\gamma\left|g_1^\triangle(t)\right|^\gamma\right]\triangle t\\
&\leqslant 2^{\gamma-1}\left(\frac{\gamma}{\lambda+\gamma}\right)(\tau-\alpha)^{2\lambda+\gamma-1}\\
&\times\int_\alpha^\tau \frac{\triangle t}{p(t)}\int_\alpha^\tau p(t)q(t)\left[\left|g_1^\triangle(t)\right|^{2(\lambda+\gamma)}+\left|g_2^\triangle(t)\right|^{2(\lambda+\gamma)}\right]\triangle t. \qquad (3.36)
\end{aligned}$$

Proof. Let $f_i(t)=\int_\alpha^t (q(s))^{\gamma/2(\lambda+\gamma)}\left|g_i^\triangle(s)\right|^\gamma\triangle s,\ t\in[\alpha,\ \tau]_\mathbb{T}$ for $i=1,2$. Thus

$$f_i^\triangle(t)=(q(t))^{\gamma/2(\lambda+\gamma)}\left|g_i^\triangle(t)\right|^\gamma>0, \qquad (3.37)$$

and since $g_i(\alpha)=0$, we have

$$|g_i(t)| \leqslant \int_\alpha^t \left|g_i^\triangle(s)\right| \triangle s = \int_\alpha^t \frac{1}{(q(s))^{1/2(\lambda+\gamma)}} (q(s))^{1/2(\lambda+\gamma)} \left|g_i^\triangle(s)\right| \triangle s.$$

Applying Hölder's inequality (2.4) with indices γ and $\gamma/(\gamma-1)$, we obtain

$$\begin{aligned} |g_i(t)| &\leqslant \left(\int_\alpha^t \frac{\triangle s}{(q(s))^{\gamma/2(\lambda+\gamma)(\gamma-1)}}\right)^{\gamma-1/\gamma} \left(\int_\alpha^t (q(s))^{\gamma/2(\lambda+\gamma)} \left|g_i^\triangle(s)\right|^\gamma \triangle s\right)^{1/\gamma} \\ &\leqslant (q(t))^{-1/2(\lambda+\gamma)} (\tau-\alpha)^{(\gamma-1)/\gamma} (f_i(t))^{1/\gamma}. \end{aligned} \tag{3.38}$$

Applying the inequality (3.1) and (3.38), we get (note that $\sigma(t) \geqslant t$ which implies $q^\sigma(t) \leqslant q(t)$), that

$$\begin{aligned} |g_i(t)+g_i^\sigma(t)|^\gamma &\leqslant 2^{\gamma-1}\left[|g_i(t)|^\gamma + |g_i^\sigma(t)|^\gamma\right] \\ &\leqslant 2^{\gamma-1}(\tau-\alpha)^{\gamma-1}(q^\sigma(t))^{-\gamma/2(\lambda+\gamma)}\left[f_i(t)+f_i^\sigma(t)\right]. \end{aligned} \tag{3.39}$$

Hence

$$\begin{aligned} &\int_\alpha^\tau q^\sigma(t)\,|g_1(t)g_2(t)|^\lambda \left[|g_1(t)+g_1^\sigma(t)|^\gamma \left|g_2^\triangle(t)\right|^\gamma + |g_2(t)+g_2^\sigma(t)|^\gamma \left|g_1^\triangle(t)\right|^\gamma\right] \triangle t \\ &\leqslant 2^{\gamma-1}(\tau-\alpha)^{(\gamma-1)(2\lambda+\gamma)/\gamma} \\ &\quad \times \int_\alpha^\tau (f_1(t)f_2(t))^{\lambda/\gamma}\left[(f_1(t)+f_1^\sigma(t))f_2^\triangle(t) + (f_2(t)+f_2^\sigma(t))f_1^\triangle(t)\right]\triangle t \\ &= 2^\gamma(\tau-\alpha)^{(\gamma-1)(2\lambda+\gamma)/\gamma}\int_\alpha^\tau (f_1(t)f_2(t))^{\lambda/\gamma}(f_1(t)f_2(t))^\triangle \triangle t. \end{aligned} \tag{3.40}$$

Applying the chain rule (2.2), we see that

$$\begin{aligned} \left((f_1f_2)^{(\lambda+\gamma)/\gamma}\right)^\triangle &= \frac{\lambda+\gamma}{\gamma}\int_0^1 \left[h(f_1f_2)^\sigma + (1-h)(f_1f_2)\right]^{\lambda/\gamma} dh\,(f_1f_2)^\triangle \\ &\geqslant \frac{\lambda+\gamma}{\gamma}(f_1f_2)^{\lambda/\gamma}(f_1f_2)^\triangle. \end{aligned} \tag{3.41}$$

Substituting (3.41) into (3.40), we obtain

$$\begin{aligned} &\int_\alpha^\tau q^\sigma(t)\,|g_1(t)g_2(t)|^\lambda \left[|g_1(t)+g_1^\sigma(t)|^\gamma \left|g_2^\triangle(t)\right|^\gamma + |g_2(t)+g_2^\sigma(t)|^\gamma \left|g_1^\triangle(t)\right|^\gamma\right] \triangle t \\ &\leqslant 2^\gamma\left(\frac{\gamma}{\lambda+\gamma}\right)(\tau-\alpha)^{(\gamma-1)(2\lambda+\gamma)/\gamma}(f_1(\tau)f_2(\tau))^{(\lambda+\gamma)/\gamma}. \end{aligned} \tag{3.42}$$

Applying inequalities (3.34) and (3.1) yield

$$\begin{aligned} (f_1f_2)^{(\lambda+\gamma)/\gamma} &\leqslant \left(\frac{1}{2}\right)^{(\lambda+\gamma)/\gamma}(f_1^2+f_2^2)^{(\lambda+\gamma)/\gamma} \\ &\leqslant \left(\frac{1}{2}\right)^{(\lambda+\gamma)/\gamma} 2^{\lambda/\gamma}\left[f_1^{2(\lambda+\gamma)/\gamma} + f_2^{2(\lambda+\gamma)/\gamma}\right] \\ &= \frac{1}{2}\left[f_1^{2(\lambda+\gamma)/\gamma} + f_2^{2(\lambda+\gamma)/\gamma}\right]. \end{aligned} \tag{3.43}$$

Substituting (3.43) into (3.42), we get

$$\int_{\alpha}^{\tau} q^{\sigma}(t)\left|g_1(t)g_2(t)\right|^{\lambda}\left[\left|g_1(t)+g_1^{\sigma}(t)\right|^{\gamma}\left|g_2^{\triangle}(t)\right|^{\gamma}+\left|g_2(t)+g_2^{\sigma}(t)\right|^{\gamma}\left|g_1^{\triangle}(t)\right|^{\gamma}\right]\triangle t$$
$$\leqslant 2^{\gamma-1}\left(\frac{\gamma}{\lambda+\gamma}\right)(\tau-\alpha)^{(\gamma-1)(2\lambda+\gamma)/\gamma}\left[(f_1(\tau))^{2(\lambda+\gamma)/\gamma}+(f_2(\tau))^{2(\lambda+\gamma)/\gamma}\right]. \quad (3.44)$$

On the other hand applying Hölder's inequality (2.4) with indices $2(\lambda+\gamma)/\gamma, 2(\lambda+\gamma)/(2\lambda+\gamma)$, we have

$$\begin{aligned}(f_i(\tau))^{2(\lambda+\gamma)/\gamma} &= \left(\int_{\alpha}^{\tau}\frac{(q(t)p(t))^{\gamma/2(\lambda+\gamma)}}{(p(t))^{\gamma/2(\lambda+\gamma)}}\left|g_i^{\triangle}(t)\right|^{\gamma}\triangle t\right)^{2(\lambda+\gamma)/\gamma}\\ &\leqslant \left(\int_{\alpha}^{\tau}\frac{\triangle t}{(p(t))^{\gamma/2\lambda+\gamma}}\right)^{2\lambda+\gamma/\gamma}\int_{\alpha}^{\tau}q(t)p(t)\left|g_i^{\triangle}(t)\right|^{2(\lambda+\gamma)}\triangle t.\end{aligned}$$

Again applying Hölder's inequality (2.4) with indices $(2\lambda+\gamma)/\gamma,\ (2\lambda+\gamma)/2\lambda,$ we get

$$\begin{aligned}(f_i(\tau))^{2(\lambda+\gamma)/\gamma} &\leqslant \left(\int_{\alpha}^{\tau}\triangle t\right)^{2\lambda/\gamma}\int_{\alpha}^{\tau}\frac{\triangle t}{p(t)}\int_{\alpha}^{\tau}q(t)p(t)\left|g_i^{\triangle}(t)\right|^{2(\lambda+\gamma)}\triangle t\\ &= (\tau-\alpha)^{2\lambda/\gamma}\int_{\alpha}^{\tau}\frac{\triangle t}{p(t)}\int_{\alpha}^{\tau}q(t)p(t)\left|g_i^{\triangle}(t)\right|^{2(\lambda+\gamma)}\triangle t. \quad (3.45)\end{aligned}$$

Substituting (3.45) into (3.44), we get the desired inequality (3.36). The proof is complete. □

THEOREM 3.8. *Assume that $\mathbb{T}$ be a time scale with $\tau,\ \beta\in\mathbb{T},\ \lambda\geqslant 0$ and $\gamma\geqslant 1$. Let $p(t)$ is a positive rd-continuous function on $[\tau,\beta]_{\mathbb{T}}$ with $\int_{\tau}^{\beta}\frac{\triangle t}{p(t)}<\infty$, and $q(t)$ be a positive, non-decreasing and rd-continuous function on $[\tau,\beta]_{\mathbb{T}}$. If $g_1,\ g_2:[\tau,\beta]_{\mathbb{T}}\longrightarrow\mathbb{R}$ are delta differentiable with $g_1(\beta)=g_2(\beta)=0$, then*

$$\int_{\tau}^{\beta} q^{\sigma}(t)\left|g_1(t)g_2(t)\right|^{\lambda}\left[\left|g_1(t)+g_1^{\sigma}(t)\right|^{\gamma}\left|g_2^{\triangle}(t)\right|^{\gamma}+\left|g_2(t)+g_2^{\sigma}(t)\right|^{\gamma}\left|g_1^{\triangle}(t)\right|^{\gamma}\right]\triangle t$$
$$\leqslant 2^{\gamma-1}\left(\frac{\gamma}{\lambda+\gamma}\right)(\beta-\tau)^{2\lambda+\gamma-1}$$
$$\times\int_{\tau}^{\beta}\frac{\triangle t}{p(t)}\int_{\tau}^{\beta}p(t)q(t)\left[\left|g_1^{\triangle}(t)\right|^{2(\lambda+\gamma)}+\left|g_2^{\triangle}(t)\right|^{2(\lambda+\gamma)}\right]\triangle t. \quad (3.46)$$

For $p(t)=1$ in Theorem 3.7, we obtain the following result.

COROLLARY 3.3. *Assume that $\mathbb{T}$ be a time scale with $\alpha,\ \tau\in\mathbb{T},\ \lambda\geqslant 0$ and $\gamma\geqslant 1$. Let $q(t)$ be positive, non-increasing and rd-continuous function on $[\alpha,\tau]_{\mathbb{T}}$. If*

$g_1,\ g_2 : [\alpha,\tau]_{\mathbb{T}} \longrightarrow \mathbb{R}$ *are delta differentiable with* $g_1(\alpha) = g_2(\alpha) = 0$, *then*

$$\int_\alpha^\tau q^\sigma(t)\,|g_1(t)g_2(t)|^\lambda \left[|g_1(t)+g_1^\sigma(t)|^\gamma \left|g_2^\triangle(t)\right|^\gamma + |g_2(t)+g_2^\sigma(t)|^\gamma \left|g_1^\triangle(t)\right|^\gamma\right] \triangle t$$
$$\leqslant 2^{\gamma-1}\left(\frac{\gamma}{\lambda+\gamma}\right)(\tau-\alpha)^{2\lambda+\gamma}\int_\alpha^\tau q(t)\left[\left|g_1^\triangle(t)\right|^{2(\lambda+\gamma)} + \left|g_2^\triangle(t)\right|^{2(\lambda+\gamma)}\right]\triangle t. \quad (3.47)$$

Also for $q(t) = 1$ and $\gamma = 1$ in inequality (3.47), we have the following result.

COROLLARY 3.4. *Assume that* $\mathbb{T}$ *be a time scale with* $\alpha,\ \tau \in \mathbb{T}$ *and* $\lambda \geqslant 0$. *Let* $g_1,\ g_2 : [\alpha,\tau]_{\mathbb{T}} \longrightarrow \mathbb{R}$ *are delta differentiable, and* $g_1(\alpha) = g_2(\alpha) = 0$, *then*

$$\int_\alpha^\tau |g_1(t)g_2(t)|^\lambda \left[\left|(g_1(t)+g_1^\sigma(t))g_2^\triangle(t)\right| + \left|(g_2(t)+g_2^\sigma(t))g_1^\triangle(t)\right|\right]\triangle t$$
$$\leqslant \frac{(\tau-\alpha)^{2\lambda+1}}{\lambda+1}\int_\alpha^\tau \left[\left|g_1^\triangle(t)\right|^{2(\lambda+1)} + \left|g_2^\triangle(t)\right|^{2(\lambda+1)}\right]\triangle t. \quad (3.48)$$

Also for $p(t) = q(t) = 1$ and $\gamma = 1$ in inequality (3.46), we have the following result.

COROLLARY 3.5. *Assume that* $\mathbb{T}$ *be a time scale with* $\tau,\ \beta \in \mathbb{T}$ *and* $\lambda \geqslant 0$. *Let* $g_1,\ g_2 : [\tau,\beta]_{\mathbb{T}} \longrightarrow \mathbb{R}$ *are delta differentiable, and* $g_1(\beta) = g_2(\beta) = 0$, *then*

$$\int_\tau^\beta |g_1(t)g_2(t)|^\lambda \left[\left|(g_1(t)+g_1^\sigma(t))g_2^\triangle(t)\right| + \left|(g_2(t)+g_2^\sigma(t))g_1^\triangle(t)\right|\right]\triangle t$$
$$\leqslant \frac{(\beta-\tau)^{2\lambda+1}}{\lambda+1}\int_\tau^\beta \left[\left|g_1^\triangle(t)\right|^{2(\lambda+1)} + \left|g_2^\triangle(t)\right|^{2(\lambda+1)}\right]\triangle t. \quad (3.49)$$

Now, a combination of (3.48) and (3.49) with $\tau = \frac{\beta+\alpha}{2}$ immediately gives the following result.

COROLLARY 3.6. *Assume that* $\mathbb{T}$ *be a time scale with* $\alpha,\ \beta \in \mathbb{T}$ *and* $\lambda \geqslant 0$. *Let* $g_1,\ g_2 : [\alpha,\beta]_{\mathbb{T}} \longrightarrow \mathbb{R}$ *are delta differentiable, and* $g_1(\alpha) = g_1(\beta) = g_2(\alpha) = g_2(\beta) = 0$, *then*

$$\int_\alpha^\beta |g_1(t)g_2(t)|^\lambda \left[\left|(g_1(t)+g_1^\sigma(t))g_2^\triangle(t)\right| + \left|(g_2(t)+g_2^\sigma(t))g_1^\triangle(t)\right|\right]\triangle t$$
$$\leqslant \frac{1}{\lambda+1}\left(\frac{\beta-\alpha}{2}\right)^{2\lambda+1}\int_\alpha^\beta \left[\left|g_1^\triangle(t)\right|^{2(\lambda+1)} + \left|g_2^\triangle(t)\right|^{2(\lambda+1)}\right]\triangle t. \quad (3.50)$$

Note that when $\mathbb{T} = \mathbb{R}$, we have $g_i^\sigma = g_i$ for $i = 1,\ 2$. Then from the above inequalities (3.36), (3.48), and (3.49), we obtain the following results due to Lin and Yang [7], and Pachpatte [8].

COROLLARY 3.7. *Let $\lambda \geqslant 0$, $\gamma \geqslant 1$ and let $g_1(t), g_2(t)$ be absolutely continuous functions on $[\alpha, \tau]$ such that $g_1(\alpha) = g_2(\alpha) = 0$, then*

$$\int_\alpha^\tau q(t)\,|g_1(t)g_2(t)|^\lambda \left[\left|g_1(t)g_2'(t)\right|^\gamma + \left|g_1'(t)g_2(t)\right|^\gamma\right] dt$$
$$\leqslant \frac{\gamma}{2(\lambda+\gamma)}(\tau-\alpha)^{2\lambda+\gamma-1}\int_\alpha^\tau \frac{dt}{p(t)}\int_\alpha^\tau p(t)q(t)\left[\left|g_1'(t)\right|^{2(\lambda+\gamma)} + \left|g_2'(t)\right|^{2(\lambda+\gamma)}\right] dt, \tag{3.51}$$

where $p(t)$ is a positive and continuous function with $\int_\alpha^\tau \frac{\triangle t}{p(t)} < \infty$, and $q(t)$ is a positive, bounded, and non-increasing function on $[\alpha, \tau]$.

COROLLARY 3.8. *Let $\lambda \geqslant 0$, and let $g_1(t)$, $g_2(t)$ be absolutely continuous functions on $[\alpha, \tau]$ such that $g_1(\alpha) = g_2(\alpha) = 0$, then*

$$\int_\alpha^\tau |g_1(t)g_2(t)|^\lambda \left[\left|g_1(t)g_2'(t)\right| + \left|g_1'(t)g_2(t)\right|\right] dt$$
$$\leqslant \frac{(\tau-\alpha)^{2\lambda+1}}{2(\lambda+1)}\int_\alpha^\tau \left[\left|g_1'(t)\right|^{2(\lambda+1)} + \left|g_2'(t)\right|^{2(\lambda+1)}\right] dt. \tag{3.52}$$

COROLLARY 3.9. *Let $\lambda \geqslant 0$, and let $g_1(t)$, $g_2(t)$ be absolutely continuous functions on $[\tau, \beta]$ such that $g_1(\beta) = g_2(\beta) = 0$, then*

$$\int_\tau^\beta |g_1(t)g_2(t)|^\lambda \left[\left|g_1(t)g_2'(t)\right| + \left|g_1'(t)g_2(t)\right|\right] dt$$
$$\leqslant \frac{(\beta-\tau)^{2\lambda+1}}{2(\lambda+1)}\int_\tau^\beta \left[\left|g_1'(t)\right|^{2(\lambda+1)} + \left|g_2'(t)\right|^{2(\lambda+1)}\right] dt. \tag{3.53}$$

COROLLARY 3.10. *Let $\lambda \geqslant 0$, and let $g_1(t)$, $g_2(t)$ be absolutely continuous functions on $[\alpha, \beta]$ such that $g_1(\alpha) = g_2(\alpha) = g_1(\beta) = g_2(\beta) = 0$, then*

$$\int_\alpha^\beta |g_1(t)g_2(t)|^\lambda \left[\left|g_1(t)g_2'(t)\right| + \left|g_1'(t)g_2(t)\right|\right] dt$$
$$\leqslant \frac{1}{2(\lambda+1)}\left(\frac{\beta-\alpha}{2}\right)^{2\lambda+1}\int_\tau^\beta \left[\left|g_1'(t)\right|^{2(\lambda+1)} + \left|g_2'(t)\right|^{2(\lambda+1)}\right] dt. \tag{3.54}$$

Note that when $\mathbb{T} = \mathbb{N}$ and $p(n) = 1$, we have from (3.36) the following discrete Opial type inequality.

COROLLARY 3.11. *Assume that λ, γ be positive real numbers such that $\gamma \geqslant 1$, $\{q(i)\}_{0\leqslant i\leqslant N}$ is a non-increasing sequence of non-negative real number. Further, let $\{g_1(i)\}_{0\leqslant i\leqslant N}$ and $\{g_2(i)\}_{0\leqslant i\leqslant N}$ be two sequences of real numbers with $g_1(0) =$*

$g_2(0)=0$. *Then*

$$\sum_{n=1}^{N} q(n+1)\left|g_1(n)g_2(n)\right|^{\lambda} \times\left[\left|g_1(n)+g_1(n+1)\right|^{\gamma}\left|\triangle g_2(n)\right|^{\gamma}+\left|g_2(n)+g_2(n+1)\right|^{\gamma}\left|\triangle g_1(n)\right|^{\gamma}\right] \leqslant 2^{\gamma-1}\left(\frac{\gamma}{\lambda+\gamma}\right)(N-\alpha)^{2\lambda+\gamma}\sum_{n=1}^{N} q(n)\left[\left|\triangle g_1(n)\right|^{2(\lambda+\gamma)}+\left|\triangle g_2(n)\right|^{2(\lambda+\gamma)}\right]. \tag{3.55}$$

Also when $p(n)=q(n)=1$ and $\gamma=1$, we have the following discrete Opial type inequality.

COROLLARY 3.12. *Let* $\{g_1(i)\}_{0\leqslant i\leqslant N}$ *and* $\{g_2(i)\}_{0\leqslant i\leqslant N}$ *be two sequences of real numbers with* $g_1(0)=g_2(0)=0$. *Then for* $\lambda\geqslant 0$

$$\sum_{n=1}^{N}\left|g_1(n)g_2(n)\right|^{\lambda}\left[\left|g_1(n)+g_1(n+1)\right|\left|\triangle g_2(n)\right|+\left|g_2(n)+g_2(n+1)\right|\left|\triangle g_1(n)\right|\right] \leqslant \frac{(N-\alpha)^{2\lambda+1}}{\lambda+1}\sum_{n=1}^{N}\left[\left|\triangle g_1(n)\right|^{2(\lambda+1)}+\left|\triangle g_2(n)\right|^{2(\lambda+1)}\right]. \tag{3.56}$$

REFERENCES

[1] M. BOHNER AND B. KAYMAKÇALAN, *Opial Inequalities on time scales*, Annales Polonici Mathematici, vol. 77, no. 1, pp. 11–20, 2001.

[2] M. BOHNER AND A. PETERSON, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[3] M. BOHNER AND A. PETERSON, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[4] S. HILGER, *Analysis on measure chains – a unified approach to continuous and discrete calculus*, Results Math. **18** (1990) 18–56.

[5] V. KAC AND P. CHEUNG, *Quantum Calculus*, Springer, New York, 2001.

[6] B. KARPUZ, B. KAYMAKÇALAN, AND Ö. ÖCLAN, *A generalization of Opial's inequality and applications to second order dynamic equations*, Differential Equations and Dynamical Systems, vol. 18, no. 1–2, pp. 11–18, 2010.

[7] C. T. LIN, AND G. S. YANG, *On some integrodifferential inequalities*, Tamkang Journal of Mathematics, vol. 16, pp. 123–129, 1985.

[8] B. G. PACHPATTE, *On Opial-type integral inequalities*, Journal of Mathematical Analysis and Applications, vol. 120, pp. 547–556, 1986.

[9] S. H. SAKER, *Some Opial-type inequalities on time scales*, Abstract and Applied Analysis, vol. 2011, Article ID 265316, 19 pages, 2011.

[10] H. M. SRIVASTAVA, K. L. TSENG, S. J. TSENG, AND J. C. LO, *Some weighted Opial-type inequalities on time scale*, Taiwanese Journal of Mathematics, vol. 14, no. 1, pp. 107–122, 2010.

[11] F. H. WONG, W. C. LIAN, S. L. YU, AND C. C. YEH, *Some generalizations of Opial's inequalities on time scales*, Taiwanese Journal of Mathematics, vol. 12, no. 2, pp. 463–471, 2008.

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