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A HILBERT–TYPE INEQUALITY

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Abstract. There are many versions of inequalities due to Hilbert and Hardy. Many extensions to nontrivial cases are now available. We present and derive a new Hilbert-type inequality for both sums and integrals. In our setup, the matrix elements associated with the quadratic form are ratios of differences of powers of an increasing function. This inequality contains some known results as special cases.

1. Introduction

In this note we present a Hilbert-type inequality for sums and integrals. There is a vast literature on Hilbert-Hardy inequalitiies, and their various forms and extensions. An excellent introduction can be found in the book by Hardy, Littlewood and Polya (1988). For recent developments the reader may consult Handley (2006), Yang (2009, 2011), Krnić et. al. (2012) and the references therein.

The basic inequality for sums is of the form

$$\left|\sum_{1\leqslant j,k\leqslant m} c_{1j}c_{2k}d_{jk}\right| \leqslant \pi(\sum c_{1j}^2)^{1/2}(\sum c_{2j}^2)^{1/2}, \tag{1}$$

where $d_{jk}=1/(j-k)$, $j\neq k$ (with $d_{jj}=0$) or $d_{jk}=1/(j+k)$. Here $\{c_{1j}\}$ and $\{c_{2j}\}$ are assumed to be real valued. The inequality is valid when $m=\infty$. Integral versions of these results mirror the discrete ones. The cases where d_{jk} are of the form $1/(\lambda_j-\lambda_k)$ or $1/(\lambda_j+\lambda_k)$ were developed by Montgomery and Vaughan (1974). There are other extensions where the kernel d_{jk} is homogeneous or nonhomogeneous satisfying certain properties so that sums can be approximated by integrals (Yang (2011), Krnić (2012)).

We consider here the case when d_{jk} is of the form

$$d_{jk}=[(\lambda_j\lambda_k)^{(1-\alpha)/2}\alpha^{-1}(\lambda_j^\alpha-\lambda_k^\alpha)/(\lambda_j-\lambda_k)][(\dot{\lambda}_j\dot{\lambda}_k)^{1/2}a_{jk}], |\alpha|<2, \tag{2}$$

where $a_{jk}=1/(\lambda_j+\lambda_k+f)$, or more generally of the form

$$d_{jk}=L(\lambda_j,\lambda_k)[(\dot{\lambda}_j\dot{\lambda}_k)^{1/2}a_{jk}], \tag{3}$$

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where f is a nonnegative constant, $\lambda_j = \lambda(j)$, $\dot{\lambda}_j = \dot{\lambda}(j)$, λ is a nonnegative strictly increasing function with derivative $\dot{\lambda}$. Note that (2) is a special case of (3) with

$$L(x,y) = (xy)^{(1-\alpha)/2}\alpha^{-1}[(x^\alpha - y^\alpha)/(x-y)], |\alpha| < 2. \tag{4}$$

and we will show that inequality (1) holds for this form when $|\alpha| \leqslant 1$. The inequality also holds for the case when $1 < |\alpha| < 2$, but the constant on the right hand side is then $\pi R(|\alpha|)$, where R is defined in Theorem 1, instead of π. We note in passing that $R(1) = 1$.

We also address the issue of near singularity of d_{jk}. If $\lambda(x_0) = 0$ for some $0 < x_0 \leqslant 1/2$, then under reasonable condition an inequality of the type in (1) may hold (Theorem 1 in Section 2). But, the case for $1/2 < x_0 < 1$ is different. If x_0 is close to 1 with $f = 0$, then d_{11} can be quite large. The constant obtainable from Hilbert-type inequality in such a case should depend on x_0, whereby the constant is π when $x_0 = 1/2$, but approaches infinity as x_0 approaches 1. We obtain an expression for such a constant in Theorem 4.

We first obtain a Hilbert-type inequality for d_{jk} under conditions on L and apply this result to the specific case when L is of the form (4). We write down the integral versions of the discrete case. We also deal with the issue of near singularity as discussed in the last paragraph. Section 2 states the main results and the necessary auxiliary lemmas. Section 3 contains the proofs.

2. The main results

Let $\lambda(x)$ be a strictly increasing nonnegative function on $[0,\infty)$ or on $[x_0,\infty)$ with $0 < x_0 < 1$. In the latter case we assume $\lambda(x_0) = 0$. For both cases, we assume that $\lambda(\infty) = \infty$. Let D be a $m \times m$ matrix whose element (j,k) is as given in (3). We will assume that for some $1 \leqslant \beta < 2$, the kernel L satisfies the following

$$|L(x_1,x_2)| \leqslant \max(x_1,x_2)^{\beta-1}/(x_1x_2)^{(\beta-1)/2}. \tag{5}$$

There are general results for obtaining Hilbert-type inequality where the "kernel" of the quadratic form satisfies certain conditions (Yang (2011), Krnić (2012)). However, such conditions are not necessarily true when d_{jk} is as given in (3) or more specifically in (4). However, upper bounds of the kernel such as the one given in (5) may have desirable properties whereby sums can be approximated by integrals (Theorem 1) in order to get a Hilbert-type inequality. When L is of the form given in (4), we need to obtain an upper bound for it and then apply Theorem 1 in order to get the desired result. It turns out that the upper bounds are of different type for the cases $|\alpha| \leqslant 1$ and $1 < |\alpha| < 2$. This will be discussed below in detail.

Our main result holds under one of the conditions given below. Either condition 1 or condition 2 is sufficient to give us an inequality similar to (1) except that the constant now is $\pi R(\beta)$.

CONDITION 1. *For any positive constant h, the functions $\psi_1(x) = \lambda(x)^{-\beta/2}(h + \lambda(x))^{-1}\dot{\lambda}(x)$ and $\psi_2(x) = \lambda(x)^{\beta/2-1}(h+\lambda(x))^{-1}\dot{\lambda}(x)$ are non-increasing when $x > 0$.*

CONDITION 2. *The functions* ψ_1 *and* ψ_2 *are convex when* $x > x_0$ *and* $0 < x_0 \leqslant 1/2$.

REMARK 1. When $\beta = 1$, $\psi_1 = \psi_2 = \lambda^{-1/2}(h+\lambda)^{-1}\dot{\lambda} = \psi_0$, say. In this case Conditions 1 and 2 become simpler since we need to check montonicity or convexity of only one function ψ_0.

In the results below, the sequences of real numbers $\{c_{1j} : j = 1,...,m\}$ *and* $\{c_{2j} : j = 1,...,m\}$, *where* m *can be infinity, are assumed to be of unit length, i.e.,* $\sum c_{1j}^2 = \sum c_{2j}^2 = 1$.

THEOREM 1. *Assume that* λ *is a strictly increasing nonnegative function on* $[x_0,\infty)$. *If* $x_0 \leqslant 0$, *then assume that condition 1 holds. If* $0 < x_0 \leqslant 1/2$, *then assume that condition 2 holds. Let* d_{jk} *be as given in (3) with the kernel* L *satisfying inequality (5). Define* $R(\beta) = (2/\pi)\int_0^{1/2} z^{-\beta/2}(1-z)^{\beta/2-1}dz$. *Then*

$$\left|\sum_{1\leqslant j,k\leqslant m} c_{1j}c_{2k}d_{jk}\right| \leqslant \pi R(\beta) \tag{6}$$

It is fairly easy to see that R, which is proportional to the incomplete Beta function, is increasing in β with $R(1) = 1$ and it approaches ∞ as $\beta \to 2$. The result stated here is supposed to be true for any positive integer m and for $m = \infty$. However, when $\beta = 2$ and m finite, it is of course possible to get a finite bound that depend on m. We will now briefly discuss the case for integral inequalities for which the proofs are similar as in the discrete case, but a bit easier since, unlike in the discrete case, no effort is needed to approximate sums by integrals. We will assume without loss of generality that λ is nonnegative and increasing on the interval $[0,n]$, where n can be a finite real number or infinity. Let

$$d(x,y) = L(\lambda(x),\lambda(y))[(\dot{\lambda}(x)\dot{\lambda}(y))^{1/2}/(\lambda(x)+\lambda(y)+f)],$$

where L can be bounded as in (5).

THEOREM 2. *Let* f_1 *and* f_2 *be square integrable on the interval* $[0,n]$ *with* $\int f_1^2 = \int f_2^2 = 1$, *where* n *could be infinity. Assume that Condition 1 holds. Then we have*

$$\left|\int\int f_1(x)f_2(y)d(x,y)dxdy\right| \leqslant \pi R(\beta).$$

For the remainder of the Section we will look at $\{d_{jk}\}$ as given in (2). For this case, the kernel L is of the form as stated in (4). Note that when $\alpha = 0$, $L(x,y) = (xy)^{1/2}|\log(x/y)|$ and L is identically equal to 1 when $|\alpha| = 1$. Lemma 2 given below points out that inequality (5) is true the kernel L. More specifically we have

$$L(\lambda_j,\lambda_k) \leqslant \begin{cases} 1 & |\alpha| \leqslant 1 \\ \max(\lambda_j,\lambda_k)^{|\alpha|-1}/(\lambda_j\lambda_k)^{(|\alpha|-1)/2} & 1 < |\alpha| < 2. \end{cases}$$

In other words, the case for $|\alpha| \leqslant 1$ corresponds to $\beta = 1$ in (5). Whereas the case for $1 < |\alpha| < 2$ corresponds to $\beta = |\alpha|$.

REMARK 2. When $|\alpha| \leqslant 1$, we have $\psi_1 = \psi_2$. In such a case, Conditions 1 and 2 become simpler since we need to check montonicity or convexity of only one function ψ_0 as stated in Remark 1.

THEOREM 3. *Assume that λ is a strictly increasing nonnegative function on $[x_0,\infty)$. If $x_0 \leqslant 0$, then assume that either Condition 1 or Condition 2 holds. If $0 < x_0 \leqslant 1/2$, then assume that Condition 2 holds. Let d_{jk} be as given in (3) with the kernel L of the form in (4). Then*

a) inequality (6) holds with the constant $\pi R(1) = \pi$ for any $|\alpha| \leqslant 1$,

b) inequality (6) holds with the constant $\pi R(|\alpha|)$, for any $1 < |\alpha| < 2$.

We will state now an immediate consequence of Theorem 3 when $\lambda(x) = (x-x_0)^\theta$, $\theta > 0$. When $|\alpha| \leqslant 1$, we can take $\beta = 1$ in inequality (5) and, in this case, the functions ψ_0 is non-increasing and convex on (x_0,∞) when $\theta \leqslant 2$. When $1 < |\alpha| < 2$, we can take $\beta = |\alpha|$, and ψ_1 and ψ_2 are non-increasing and convex if $\theta \leqslant 2/|\alpha|$.

COROLLARY 1. *Let $\lambda(x) = (x-x_0)^\theta$, $\theta > 0$, $x_0 \leqslant 1/2$. Let d_{jk} be as given in (3) with the kernel L of the form in (4). Then*

a) inequality (6) holds with the constant $\pi R(1) = \pi$ for any $|\alpha| \leqslant 1$, as long as $\theta \leqslant 2$,

b) inequality (6) holds with the constant $\pi R(|\alpha|)$, $1 < |\alpha| < 2$, as long as $\theta \leqslant 2/|\alpha|$.

Theorem 1 states that when $\lambda(x_0) = 0$ for $0 < x_0 < 1$, convexity of ψ_1 and ψ_2 is enough to get a Hilbert-type inequality if $x_0 \leqslant 1/2$ and $|\alpha| \leqslant 1$. For instance, if d_{jk} is of the form $1/(j+k-2x_0)$ with $x_0 \leqslant 1/2$, we know that Hilbert inequality holds. What happens when $1/2 < x_0 < 1$? We will address this type of issue for when d_{jk} is of the form in (3) with L as in (4) for the case $|\alpha| \leqslant 1$.

THEOREM 4. *Let $\lambda(x)$ be a strictly increasing nonnegative function and that the function $\psi_0 = \lambda^{-1/2}(h+\lambda)^{-1}\dot{\lambda}$ is convex on (x_0,∞) for any positive constant h. Moreover assume that $(1-x_0)\dot{\lambda}_1/\lambda_1 \leqslant \pi/4$. Let d_{jk} be as given in (3) with the kernel L of the form in (4). Then for any $|\alpha| \leqslant 1$, we have*

$$\left| \sum_{1 \leqslant j,k \leqslant m} c_{1j}c_{2k}d_{jk} \right| \leqslant \pi[1 + (1/2)(x_0 - 1/2)(1-x_0)^{-1}].$$

REMARK 3. Note that when $x_0 = 1/2$, the constant on the right hand side equals π. However, the constant grows to infinity as x_0 approaches 1. When $\lambda(x) = (x-x_0)^\theta$, the function ψ_0 stated in Remark 1 is convex on (x_0,∞) as long as $\theta \leqslant 2$ and $|\alpha| \leqslant 1$. In this case, $(1-x_0)\dot{\lambda}_1/\lambda_1 \leqslant \pi/4$ if $\theta \leqslant \pi/2$.

The proof of Theorem 3 depends crucially on the following two Lemmas.

LEMMA 1. *For any $0 < x \leqslant 1$ the following holds*

$$\alpha^{-1}(1-x^{\alpha})/(1-x) \leqslant \begin{cases} x^{-(1-\alpha)/2} & 0 \leqslant \alpha \leqslant 1 \\ 1 & 1 < \alpha < 2 \end{cases}$$

For notational convenience let us denote

$$L_0(x_1,x_2;\alpha) = (x_1x_2)^{(1-\alpha)/2}\alpha^{-1}(x_1^{\alpha}-x_2^{\alpha})/(x_1-x_2).$$

Clearly L_0 is nonnegative with $x_1,x_2 \geqslant 0$. As a consequence of the last Lemma we get the following.

LEMMA 2. *For any $x_1 > 0$, $x_2 > 0$, we have*

$$L_0(x_1,x_2;\alpha) \leqslant \begin{cases} 1 & |\alpha| \leqslant 1 \\ \max(x_1,x_2)^{|\alpha|-1}/(x_1x_2)^{(|\alpha|-1)/2} & 1 < |\alpha| < 2 \end{cases}$$

3. The proofs

Proof of Theorem 1. We will follow the usual arguments used in proving the Hilbert inequality. Let $g_{jk} = a_{jk}\max(\lambda_j,\lambda_k)^{\beta-1}/(\lambda_j\lambda_k)^{(\beta-1)/2}$, where $a_{jk} = (\lambda_j + \lambda_k + f)^{-1}$. We apply the Cauchy-Schwartz inequality for the double sum

$$\begin{aligned} \left|\sum c_{1j}c_{2k}d_{jk}\right|^2 &\leqslant \left|\sum(|c_{1j}|\lambda_j^{1/4}g_{jk}^{1/2}\dot{\lambda}_k^{1/2}\lambda_k^{-1/4})(|c_{2k}|\lambda_k^{1/4}g_{jk}^{1/2}\dot{\lambda}_j^{1/2}\lambda_j^{-1/4})\right|^2 \\ &\leqslant \left(\sum_j c_{1j}^2\lambda_j^{1/2}\sum_k g_{jk}\dot{\lambda}_k\lambda_k^{-1/2}\right)\left(\sum_k c_{2k}^2\lambda_k^{1/2}\sum_j g_{jk}\dot{\lambda}_j\lambda_j^{-1/2}\right). \end{aligned}$$

Since $g_{jk} = g_{kj}$, it is enough to obtain approximation for any of the two inner sums in the last expression and show that $\sum_k g_{jk}\dot{\lambda}_k\lambda_k^{-1/2} \leqslant \lambda_j^{-1/2}\pi R(\beta)$. Note that

$$\begin{aligned} \sum_k g_{jk}\dot{\lambda}_k\lambda_k^{-1/2} &\leqslant \lambda_j^{(\beta-1)/2}\sum_{k\leqslant j-1}\lambda_k^{-\beta/2}\dot{\lambda}_k(\lambda_j+\lambda_k+f)^{-1} \\ &\quad + \lambda_j^{-(\beta-1)/2}\sum_{k\geqslant j}\lambda_k^{\beta/2-1}\dot{\lambda}_k(\lambda_j+\lambda_k+f)^{-1} \\ &= J_1 + J_2, \text{ say.} \end{aligned}$$

If Condition 1 holds then

$$\begin{aligned} J_1 &\leqslant \lambda_j^{(\beta-1)/2}\int_0^{j-1}\lambda(x)^{-\beta/2}\dot{\lambda}(x)(\lambda_j+\lambda(x)+f)^{-1}dx \\ &= \lambda_j^{(\beta-1)/2}\int_{\lambda(0)}^{\lambda(j-1)}u^{-\beta/2}(\lambda_j+u+f)^{-1}du \\ &\leqslant \lambda_j^{-1/2}\int_0^T u^{-\beta/2}(1+u)^{-1}du, \end{aligned}$$

where $T = \lambda(j-1)/(\lambda(j)+f)$. Now

$$\begin{aligned} J_2 &\leqslant \lambda_j^{-(\beta-1)/2} \int_{j-1}^{\infty} \lambda(x)^{\beta/2-1} \dot{\lambda}(x)(\lambda_j + \lambda(x) + f)^{-1} dx \\ &= \lambda_j^{-(\beta-1)/2} \int_{\lambda(j-1)}^{\infty} u^{\beta/2-1} (\lambda_j + u + f)^{-1} du \\ &\leqslant \lambda_j^{-1/2} \int_T^{\infty} u^{\beta/2-1} (1+u)^{-1} du \\ &= \lambda_j^{-1/2} \int_0^{1/T} u^{-\beta/2} (1+u)^{-1} du, \end{aligned}$$

Hence

$$\begin{aligned} J_1 + J_2 &\leqslant \lambda_j^{-1/2} \int_0^T u^{-\beta/2} (1+u)^{-1} du + \lambda_j^{-1/2} \int_0^{1/T} u^{-\beta/2} (1+u)^{-1} du \\ &= \lambda_j^{-1/2} \gamma(T), \end{aligned}$$

with

$$\gamma(T) = \int_0^T u^{-\beta/2} (1+u)^{-1} du + \int_0^{1/T} u^{-\beta/2} (1+u)^{-1} du$$

Now the function $\gamma(T)$, $0 < T \leqslant 1$, is increasing in T. Hence,

$$\begin{aligned} J_1 + J_2 \leqslant \lambda_j^{-1/2} \gamma(T) \leqslant \lambda_j^{-1/2} \gamma(1) &= \lambda_j^{-1/2} 2 \int_0^1 u^{-\beta/2} (1+u)^{-1} \\ &= \lambda_j^{-1/2} 2 \int_0^{1/2} z^{-\beta/2} (1-z)^{\beta/2-1} dz \\ &= \lambda_j^{-1/2} \pi R(\beta). \end{aligned}$$

When Condition (2) holds, the arguments are basically the same except that we can now use the convexity of ψ_1 and ψ_2 to bound J_1 by an integral over $[.5, j-.5]$ and J_2 by an integral over $[j-.5, \infty)$. □

Proof of Theorem 2. The proof is omitted since it basically follows the arguments given in the proof of Theorem 1. □

Proofs of Theorem 3 and Corollary 1. The proofs are omitted since they are immediate consequences of Theorem 1 and Lemma 2. □

Proof of Theorem 4. The proof is very similar to that in Theorem 1. The only difference is how the terms J_1 and J_2 are approximated. Note that due to Lemma 2, for the case for $|\alpha| \leqslant 1$, the upper bound of $L(x,y)$ is equal to 1, i.e., we can take $\beta = 1$ in J_1 and J_2 in the proof of Theorem 1. First note that the function $\gamma(x) =$

$\lambda(x)^{-1/2}\dot{\lambda}(x)(\lambda_j+\lambda(x))^{-1}$ is decreasing and convex when $x > x_0$. Let $J = J_1 + J_2$ and denote $x^{-1/2}/(1+x)$ by γ_0. When $j \geqslant 2$, we have

$$\begin{aligned}
J &= \sum_{k\geqslant 1} \lambda_k^{-1/2}\dot{\lambda}_k(\lambda_j+\lambda_k+2f)^{-1} \leqslant \sum_{k\geqslant 1}\gamma(k)\\
&= \gamma(1)+\sum_{k\leqslant 2}\gamma(k) \leqslant (2(1-x_0))^{-1}\int_{x_0}^{2-x_0}\gamma(x)+\int_{1.5}^{\infty}\gamma(x)dx\\
&= \lambda_j^{-1/2}\left[(2(1-x_0))^{-1}\int_0^{\lambda(2-x_0)/\lambda_j}\gamma_0+\int_{\lambda(1.5)/\lambda_j}^{\infty}\gamma_0\right]\\
&\leqslant \lambda_j^{-1/2}\left[(2(1-x_0))^{-1}\int_0^{\lambda(2-x_0)/\lambda_j}\gamma_0-\int_0^{\lambda(2-x_0)/\lambda_j}\gamma_0+\int_0^{\infty}\gamma_0\right]\\
&\leqslant \lambda_j^{-1/2}\left[(x_0-.5)(1-x_0))^{-1}\int_0^{1}\gamma_0+\int_0^{\infty}\gamma_0\right]\\
&= \lambda_j^{-1/2}[(x_0-.5)(1-x_0))^{-1}\pi/2+\pi] = \lambda_j^{-1/2}\pi[(1/2)(x_0-.5)(1-x_0))^{-1}+1].
\end{aligned}$$

When $j = 1$, the argument is a bit different. Note that

$$\begin{aligned}
J &\leqslant \gamma(1)+\sum_{k\leqslant 2}\gamma(k) \leqslant \gamma(1)+\int_{1.5}^{\infty} \leqslant \gamma(1)-\int_0^{\lambda_1}\gamma+\int_{x_0}^{\infty}\gamma\\
&= \gamma(1)-\lambda_1^{-1/2}\pi/2+\lambda_1^{-1/2}\pi = \lambda_1^{-1/2}[\gamma(1)\lambda_1^{1/2}-\pi/2+\pi].
\end{aligned}$$

Since $\gamma(1)\lambda_1^{1/2} = (1/2)\dot{\lambda}_1/\lambda_1 \leqslant (1-x_0)^{-1}\pi/4$ by assumption, we have

$$\gamma(1)\lambda_1^{1/2}-\pi/2 \leqslant (\pi/4)(1-x_0)^{-1}-\pi/2 = (\pi/2)(x_0-.5)(1-x_0)^{-1}.$$

The result now follows. □

Proof of Lemma 1. Note that the result clearly holds when $x = 0$ or $x = 1$. So we will prove the result when $0 < x < 1$.

Case $0 \leqslant \alpha < 1$: We will show the result for $\alpha > 0$ since the case for $\alpha = 0$ follows by letting α go to zero. Note that

$$\alpha^{-1}(1-x^{\alpha})/(1-x) = \alpha^{-1}\sum_{0\leqslant t<\infty}\binom{\alpha}{t+1}(-1)^t(1-x)^t = \sum_{0\leqslant t<\infty} a_t(1-x)^t, \qquad (7)$$

where

$$a_t = \alpha^{-1}\binom{\alpha}{t+1}(-1)^t.$$

Let $b_t = \binom{-(1-\alpha)/2}{t}(-1)^t$. Note that $a_0 = b_0 = 1$, $a_1 = b_1 = (1-\alpha)/2$ and that both a_t and b_t are positive. Clearly our result will be proved if we can show that $a_t \leqslant b_t$ for all t, since

$$\alpha^{-1}(1-x^{\alpha})/(1-x) \leqslant \sum_{0\leqslant t<\infty} b_t(1-x)^t = x^{-(1-\alpha)/2}.$$

We will show that $a_t \leqslant b_t$, by using the method of induction. Assume that the result is true for all indices up to t and we will show that the result holds for $t+1$. Note that

$$a_{t+1} = a_t(t+1-\alpha)/(t+2), \;\; b_{t+1} = b_t(t+(1-\alpha)/2)/(t+1).$$

Now

$$(t+(1-\alpha)/2)/(t+1) - (t+1-\alpha)/(t+2) = (t/2)(t+1)^{-1}(t+2)^{-1}(1+\alpha) > 0.$$

This shows that $a_{t+1} < b_{t+1}$ and the proof is concluded.

Case $1 < \alpha < 2$: Note that the expansion in (7) is still valid for any $x < 1$ and that $a_0 = 1$ and $a_t < 0$ for all $t \geqslant 1$. Hence the result follows as the right hand side in (7) bounded above by 1. □

Proof of Lemma 2. Since $L_0(x_1,x_2;\alpha) = L_0(x_1,x_2;|\alpha|)$, it is enough to verify the result when α is nonnegative. The result is trivially true when $x_1 = x_2$ and when $\alpha = 1$. Let us assume that $x_1 > x_2$. When $0 < \alpha < 1$, using Lemma 1 we get

$$\begin{aligned} \alpha^{-1}(x_1^{\alpha} - x_2^{\alpha})/(x_1 - x_2) &= x_1^{\alpha-1}(1-(x_2/x_1)^{\alpha})/(1-(x_2/x_1)) \\ &\leqslant x_1^{\alpha-1}\alpha(x_2/x_1)^{-(1-\alpha)/2} = \alpha(x_1x_2)^{-(1-\alpha)/2}. \end{aligned} \tag{8}$$

The case for $\alpha = 0$ follows from the above by letting α go to zero. When $1 < \alpha < 2$, and $x_1 > x_2$, the quantity on the right hand side in (8) can be bounded above by 1 using Lemma 1 and the result follows. □

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