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GENERALIZED ROLEWICZ THEOREM FOR CONVEXITY OF HIGHER ORDER

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Abstract. In this paper we prove that if a real function f satisfies the inequality $[x_0,x_1,\ldots,x_{n+1};f] + \varphi(x_{n+1}-x_0) \geqslant 0$ for all $x_0 < x_1 < \ldots < x_n < x_{n+1}$ with some fixed positive integer n and non-negative function φ fulfilling $\lim_{h\to 0+}\varphi(h) = 0$, then f is convex of order n, i.e., f satisfies the former inequality with $\varphi = 0$ as well.

1. Introduction

In 1979 S. Rolewicz [16, Lemma 4] proved that every absolutely continuous function $f:\mathbb{R}\to\mathbb{R}$ which satisfies the inequality

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y)+C|x-y|^{2+p} \tag{1}$$

for every $x,y\in\mathbb{R}$, $t\in[0,1]$, with a fixed non-negative real number C and a fixed positive real number p, has to be convex.

Later, in a series of papers (see [17] and the references therein), S. Rolewicz investigated continuous real functions f satisfying the functional inequality

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y)+Ct(1-t)\alpha(|x-y|) \tag{2}$$

or

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y)+C\alpha(|x-y|) \tag{3}$$

for every $x,y\in\mathbb{R}$, $t\in[0,1]$, with a non-negative constant C and a nondecreasing function $\alpha\colon [0,+\infty[\to[0,+\infty]$ fulfilling $\lim_{t\to 0+}(\alpha(t)/t) = 0$. In this generality, inequality (3) may have solutions that do not satisfy any inequality of the form (2) (even if we allow different C and α) (cf. also [18]). As a generalization of his aforementioned theorem, S. Rolewicz proved that under the additional assumption

$$\lim_{t\to 0+}\frac{\alpha(t)}{t^2} = 0, \tag{4}$$

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every continuous solution f of inequality (3) or (2) is convex.

In this paper we deal with the analogue of (2) for convex functions of higher order, on open intervals, without regularity assumptions. In particular, our method can be applied to prove Rolewicz's theorem without any regularity condition.

Let $I \subset \mathbb{R}$ be an interval, $n \in \mathbb{N}$ and $x_0, x_1, \ldots, x_n, x_{n+1}$ be distinct points in I. Denote by $[x_0, x_1, \ldots, x_n, x_{n+1}; f]$ the divided difference of f at $x_0, x_1, \ldots, x_n, x_{n+1}$ defined by the recurrence

$$[x_0; f] = f(x_0),$$
$$[x_0, x_1, \ldots, x_{n+1}; f] = \frac{[x_1, x_2, \ldots, x_n, x_{n+1}; f] - [x_0, x_1, \ldots, x_n; f]}{x_{n+1} - x_0}, \quad n \in \mathbb{N}.$$

Following Hopf [9] and Popoviciu [14], a function $f : I \to \mathbb{R}$ is called convex of order n if

$$[x_0, x_1, \ldots, x_{n+1}; f] \geqslant 0$$

for all $x_0 < x_1 < \ldots < x_n < x_{n+1}$ in I. It is well known (and easy to verify) that convexity of order 1 coincides with the ordinary convexity. Several results on convex functions of order n are collected, among others, in [8, 10, 12, 14, 15, 19, 20].

As one may easily verify, introducing $x_0 = x$, $x_2 = y$ and $x_1 = tx + (1-t)y$, we obtain $t = \frac{x_2 - x_1}{x_2 - x_0}$, $1 - t = \frac{x_1 - x_0}{x_2 - x_0}$, while the inequality (2) can be rewritten as

$$0 \leqslant [x_0, x_1, x_2; f] + C\frac{\alpha(|x_2 - x_0|)}{(x_2 - x_0)^2}.$$

The purpose of this paper is to establish the following theorem.

THEOREM 1. *Let $I \subset \mathbb{R}$ be an open interval, $\nu(I)$ denote the length of the interval I, $n \in \mathbb{N}$, and $J_I =]0, \nu(I)[$. Let the function $\varphi : J_I \to [0, +\infty[$ satisfy $\lim_{t\to 0+} \varphi(t) = 0$. If a function $f : I \to \mathbb{R}$ satisfies the inequality*

$$[x_0, x_1, \ldots, x_n, x_{n+1}; f] + \varphi(x_{n+1} - x_0) \geqslant 0 \tag{5}$$

for all $x_j \in I$ $(j = 0, 1, \ldots, n, n+1)$ such that $x_0 < x_1 < \ldots < x_n < x_{n+1}$, then f is convex of order n.

The proof of this theorem is elaborated in Section 3. Some related concepts and results are presented in Section 2. These tools are incorporated in the main argument.

2. Tools and related statements

In order to prove our main result we need the following notions.

We use the concept of difference operators Δ_h^{n+1} defined by the following recursion:

$$\Delta_h^1 f(x) = f(x+h) - f(x) \quad \text{for } x \in I,\ h \in \mathbb{R} \text{ such that } x + h \in I,$$
$$\Delta_h^{n+1} f(x) = \Delta_h^1 \Delta_h^n f(x) \quad \text{for } x \in I,\ h \in \mathbb{R} \text{ such that } x + (n+1)h \in I.$$

The notion of higher-order Jensen-convexity is due to T. Popoviciu ([13, 14]): A function $f\colon I \to \mathbb{R}$ is called Jensen-convex of order n (where $n \in \mathbb{N}$), if

$$\Delta_h^{n+1} f(x) \geqslant 0 \ \text{ for all } x \in I,\ h \geqslant 0 \text{ such that } x + (n+1)h \in I.$$

For properties of functions satisfying the above inequality, see e.g. [1, 2, 14], [10, Chapter XV], [15, VIII.83], and the references therein. Generalizations of Jensen-convexity of order n to higher-dimensional domains were investigated by R. Ger [4, 5].

It is clear that differences can be expressed in terms of divided differences, thus convex functions of order n are also Jensen-convex of order n. It was observed by T. Popoviciu [13] that the converse of this implication also holds for continuous functions. Z. Ciesielski [3, Theorem 1] proved that the continuity of a Jensen-convex function of order n (on an open interval) follows from its boundedness on any set of positive measure. Combining these results we can establish the following corollary.

PROPOSITION 1. *Let I be an open interval, $n \in \mathbb{N}$, and suppose that $f : I \to \mathbb{R}$ is Jensen-convex of order n. If f is bounded on a set $E \subset I$ of positive measure, then f is convex of order n.*

A local characterization of Jensen-convexity of higher order was elaborated by A. Gilányi and Zs. Páles in a somewhat more general context. Let $T = (t_1, \ldots, t_{n+1})$, where $t_1, \ldots, t_{n+1}$ are fixed positive numbers. For $f : I \to \mathbb{R}$, $x \in I$ and $h > 0$ such that $x + (t_1 + \ldots + t_{n+1})h \in I$, let

$$\Delta_h^T f(x) := \Delta_{t_1 h} \ldots \Delta_{t_{n+1} h} f(x).$$

We say that $f : I \to \mathbb{R}$ is T-convex if $\Delta_h^T f(x) \geqslant 0$ for all $x \in I$, $h > 0$ such that $x + (t_1 + \ldots + t_{n+1})h \in I$. Clearly, T-convexity and cT-convexity are equivalent for $c > 0$. In the case $t_1 = \ldots = t_{n+1} = 1$ the notion of T-convexity is obviously the same as Jensen-convexity of order n.

The lower T-Dinghas interval derivative of $f : I \to \mathbb{R}$ at $\xi \in I$ is defined by

$$\underline{D}^T f(\xi) := \liminf_{\substack{(x,h)\to(\xi,0)\\ x\leqslant\xi\leqslant x+(t_1+\ldots+t_{n+1})h}} \frac{\Delta_h^T f(x)}{(t_1 h)\ldots(t_{n+1} h)}.$$

Accordingly, if n denotes a positive integer, the n-th order lower Dinghas interval derivative of $f : I \to \mathbb{R}$ at $\xi \in I$ is defined by

$$\underline{D}^n f(\xi) := \liminf_{\substack{(x,h)\to(\xi,0)\\ x\leqslant\xi\leqslant x+nh}} \frac{\Delta_h^n f(x)}{h^n}.$$

Gilányi and Páles [7, Corollary 1] proved a strong connection between the above two concepts. Namely, they established that a function $f : I \to \mathbb{R}$ is T-convex if, and only if, $\underline{D}^T f(\xi) \geqslant 0$ for every $\xi \in I$. Considering the particular case when $t_1 = \ldots = t_{n+1} = 1$, one obtains the following statement:

PROPOSITION 2. *A function $f : I \to \mathbb{R}$ is Jensen-convex of order n if, and only if, $\underline{D}^{n+1} f(\xi) \geqslant 0$ for every $\xi \in I$.*

3. Proof of the main theorem

We prove Theorem 1 in this section.

Proof. Let us consider $x \in I$ and a positive real number h such that $x+(n+1)h \in I$. Applying our assumption (5) for

$$x_i = x + i \cdot h \qquad (i = 0, 1, \ldots, n, n+1),$$

we have

$$[x, x+h, \ldots, x+(n+1)h; f] + \varphi((n+1)h) \geqslant 0. \tag{6}$$

According to a well-known identity [10, Lemma 15.2.5], the first term above satisfies

$$[x, x+h, \ldots, x+(n+1)h; f] = \frac{\Delta_h^{n+1} f(x)}{(n+1)!\, h^{n+1}}.$$

Taking this representation into consideration, inequality (6) can be written as

$$\frac{\Delta_h^{n+1} f(x)}{h^{n+1}} \geqslant -(n+1)!\varphi((n+1)h). \tag{7}$$

Now let $\xi \in I$ be arbitrary and let us take the $\liminf$ on both sides of (7) as h tends to 0 and x tends to ξ such that $x \leqslant \xi \leqslant x+(n+1)h$. We obtain

$$\underline{D}^{n+1} f(\xi) \geqslant 0.$$

Applying Proposition 2, we get that the function f is Jensen-convex of order n.

On the other hand, we can show that f is locally bounded. Let $\delta > 0$ such that $\varphi(t) < 1$ for $0 < t < \delta$. Let $y_0 \in I$ be arbitrarily fixed and $r > 0$ such that $2r < \delta$ and $I_0 :=]y_0 - r, y_0 + r[\subset I$. Then, for arbitrary $x_j \in I_0$ $(j = 0, 1, \ldots, n, n+1)$ fulfilling $x_0 < x_1 < \ldots < x_n < x_{n+1}$, from (5) we get

$$[x_0, \ldots, x_{n+1}; f] \geqslant -\varphi(x_{n+1} - x_0) > -1.$$

Defining the function $g : I_0 \to \mathbb{R}$ as $g(x) = f(x) + x^{n+1}$ and using the linearity of the divided differences (cf. [6, Lemma 2]), we get that

$$[x_0, \ldots, x_{n+1}; g] = [x_0, \ldots, x_{n+1}; f] + \left[x_0, \ldots, x_{n+1}; x^{n+1}\right].$$

Moreover it is easy to show (cf. [6, Lemma 3]) that $[x_0, \ldots, x_n; x^n] = 1$ for all $n \in \mathbb{N}$ and distinct points $x_0, \ldots, x_n \in \mathbb{R}$, i.e.

$$[x_0, \ldots, x_{n+1}; g] \geqslant -1 + 1 = 0.$$

Thus $g : I_0 \to \mathbb{R}$ is convex of order n.

Using [10, Theorem 15.8.1] we can state that g is continuous (it is even continuously differentiable if $n > 1$), thus g and therefore also f is bounded on any closed subinterval of I_0. Applying Proposition 1, we obtain that f is convex of order n. □

4. Convex functions of higher order on vector spaces

In this section $\mathbb{R}^m_+$ denotes the set of those $x \in \mathbb{R}^m$ whose first non-zero coordinate is positive:

$$\mathbb{R}^m_+ = \bigcup_{i=1}^{m} \{x = (x_1, \dots, x_m) \in \mathbb{R}^m \,|\, x_1 = \dots = x_{i-1} = 0, x_i > 0\}.$$

Then $\mathbb{R}^m = \mathbb{R}^m_+ \cup \left(-\mathbb{R}^m_+\right) \cup \{0\}$. For $x, y \in \mathbb{R}^m$ we write $x > y$ if $x - y \in \mathbb{R}^m_+$ and $x < y$ if $y > x$. Then for any $x, y \in \mathbb{R}^m$ we have either $x = y$, or $x < y$, or $x > y$. If $x > y$ and $\alpha \in \mathbb{R}$ is positive, then $\alpha x > \alpha y$, and if $\alpha < 0$, then $\alpha x < \alpha y$. We write

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Let $D \subset \mathbb{R}^m$ be a convex set and $f : D \to \mathbb{R}$ be a function. Let $x_0, x_1, \dots, x_n, x_{n+1} \in D$ be distinct collinear points. Put

$$h = \frac{\operatorname{sgn}(x_{n+1} - x_0)}{\|x_{n+1} - x_0\|} \, (x_{n+1} - x_0),$$

thus $h > 0$. Since $x_0, x_1, \dots, x_n, x_{n+1}$ are collinear, they may be represented in the form

$$x_i = x_0 + \lambda_i h, \qquad i = 0, 1, \dots, n, n+1$$

(obviously, with $\lambda_0 = 0$). The divided difference $[x_0, x_1, \dots, x_n, x_{n+1}; f]$ of f at the points $x_0, x_1, \dots, x_n, x_{n+1}$ is defined by the recurrence (see [11], [13], or [4]):

$$\begin{aligned} [x_0; f] &= f(x_0), \\ [x_0, x_1, \dots, x_n, x_{n+1}; f] &= \frac{[x_1, x_2, \dots, x_{n+1}; f] - [x_0, x_1, \dots, x_n; f]}{\lambda_{n+1} - \lambda_0}, \quad n \in \mathbb{N}. \end{aligned}$$

This shows that the divided difference $[x_0, x_1, \dots, x_{n+1}; f]$ depends on the differences of the λ's rather than on the λ's themselves.

Let $n, m \in \mathbb{N}$ and $D \subset \mathbb{R}^m$ be an open convex set. We say that a function $f : D \to \mathbb{R}$ is convex of order n if f satisfies the inequality

$$[x_0, x_1, \dots, x_n, x_{n+1}; f] \geqslant 0$$

for all $x_j \in D$ $(j = 0, 1, \dots, n, n+1)$ such that $x_0 < x_1 < \dots < x_n < x_{n+1}$ are collinear points.

Using this generalization we can state the following theorem:

THEOREM 2. *Let $D \subset \mathbb{R}^m$ be an open and convex set, $\nu(D)$ denote the diameter of the set D, $n, m \in \mathbb{N}$, $J_D =]0, \nu(D)[$. Let the function $\varphi : J_D \to [0, +\infty[$ satisfy $\lim_{t \to 0+} \varphi(t) = 0$. If a function $f : D \to \mathbb{R}$ satisfies the inequality*

$$[x_0, x_1, \dots, x_n, x_{n+1}; f] + \varphi(\|x_{n+1} - x_0\|) \geqslant 0 \tag{8}$$

for all $x_j \in D$ $(j=0,1,\ldots,n,n+1)$ *such that* $x_0 < x_1 < \ldots < x_n < x_{n+1}$ *are collinear points, then* f *is convex of order* n.

Proof. Let $y_0, y_1, \ldots, y_n, y_{n+1} \in D$ be arbitrary collinear points which fulfil the relations $y_0 < y_1 < \ldots < y_n < y_{n+1}$. Now let us consider the unit vector

$$h = \frac{y_{n+1} - y_0}{\|y_{n+1} - y_0\|} \in \mathbb{R}^m$$

and $\bar{\lambda}_0, \bar{\lambda}_1, \ldots, \bar{\lambda}_n, \bar{\lambda}_{n+1} \in \mathbb{R}$ such that

$$y_i = y_0 + \bar{\lambda}_i \cdot h$$

for all $i=0,1,\ldots,n,n+1$. Since the mapping $\Psi\colon \mathbb{R} \to \mathbb{R}^m$ defined by the expression $\Psi(\lambda) = y_0 + \lambda \cdot h$ (for all $\lambda \in \mathbb{R}$) is continuous and preserves convex combinations, there exists an open interval I such that $\bar{\lambda}_i \in I$ for every $i=0,1,\ldots,n,n+1$ and $\Psi(\lambda) \in D$ for every $\lambda \in I$. From what we established above it follows that $0 = \bar{\lambda}_0 < \bar{\lambda}_1 < \ldots < \bar{\lambda}_n < \bar{\lambda}_{n+1}$. Furthermore define the function $g : I \to \mathbb{R}$ as

$$g(\lambda) = f(y_0 + \lambda h).$$

Using induction on the number of the points, we can show that for every $\lambda_0 < \lambda_1 < \ldots < \lambda_n$ with $\lambda_i \in I$ and $x_i = y_0 + \lambda_i h$ $(i = 0,1,\ldots,n)$ the equality

$$[\lambda_0, \lambda_1, \ldots, \lambda_{n-1}, \lambda_n; g] = [x_0, x_1, \ldots, x_{n-1}, x_n; f] \tag{9}$$

holds. Namely, in case $n=0$ (that is, when we have only one point) we have

$$[\lambda_0; g] = g(\lambda_0) = f(y_0 + \lambda_0 h) = f(x_0) = [x_0; f].$$

Suppose that (9) holds for every $n+1$ successive points. Then

$$\begin{aligned}
[\lambda_0, \lambda_1, \ldots, \lambda_n, \lambda_{n+1}; g] &= \frac{[\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_{n+1}; g] - [\lambda_0, \lambda_1, \ldots, \lambda_{n-1}, \lambda_n; g]}{\lambda_{n+1} - \lambda_0} \\
&= \frac{[x_1, x_2, \ldots, x_n, x_{n+1}; f] - [x_0, x_1, \ldots, x_{n-1}, x_n; f]}{\lambda_{n+1} - \lambda_0} \\
&= [x_0, x_1, \ldots, x_n, x_{n+1}; f].
\end{aligned}$$

Now let $\lambda_0 < \lambda_1 < \ldots < \lambda_n < \lambda_{n+1}$ such that $\lambda_i \in I$ for $(i = 0,1,\ldots,n,n+1)$ and let $x_i = y_0 + \lambda_i h$ for $(i = 0,1,\ldots,n,n+1)$. Then $x_j \in D$ $(j=0,1,\ldots,n,n+1)$ such that $x_0 < x_1 < \ldots < x_n < x_{n+1}$ are collinear points. Thus inequality (8) holds. Observing

$$\|x_{n+1} - x_0\| = \|(\lambda_{n+1} - \lambda_0)h\| = (\lambda_{n+1} - \lambda_0) \cdot \|h\| = \lambda_{n+1} - \lambda_0$$

and applying the identity (9), we obtain

$$\begin{aligned}
0 &\leqslant [x_0, x_1, \ldots, x_n, x_{n+1}; f] + \varphi(\|x_{n+1} - x_0\|) \\
&= [\lambda_0, \lambda_1, \ldots, \lambda_n, \lambda_{n+1}; g] + \varphi(\lambda_{n+1} - \lambda_0).
\end{aligned}$$

Hence using Theorem 1 we can see that g is convex of order n. In particular, we have

$$0 \leqslant \left[\bar{\lambda}_0, \bar{\lambda}_1, \ldots, \bar{\lambda}_n, \bar{\lambda}_{n+1}; g\right] = [y_0, y_1, \ldots, y_n, y_{n+1}; f].$$

As $y_0, y_1, \ldots, y_n, y_{n+1} \in D$ were arbitrary collinear points fulfilling the relations $y_0 < y_1 < \ldots < y_n < y_{n+1}$, we have proved that f is convex of order n. □

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